A GENERALIZED STERN-BROCOT TREE

Dhroova Aiylam
Massachusetts Institute of Technology, Cambridge, Massachusetts
dhroova@mit.edu

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Abstract

We discuss and prove several properties of the classical Stern-Brocot tree before turning our attention to a natural generalization, in which the tree is allowed to begin with arbitrary rational starting terms. We prove that regardless of the choice of starting terms, the tree will include every rational number between them. Moreover, this is true independently of the way in which fractions are reduced.

1. Introduction

The Stern-Brocot tree was discovered independently by Moritz Stern [1] in 1858 and Achille Brocot [2] in 1861. It was originally used by Brocot to design gear systems with a gear ratio close to some desired value (like the number of seconds in a day) by finding a ratio of smooth numbers (numbers that decompose into small prime factors) near that value. Since smooth numbers factor into small primes, several small gears could be connected in sequence to generate an effective ratio of the product of their teeth, thus creating a relatively small gear train while minimizing its error [7].

The Stern–Brocot tree begins with the values $\frac{0}{1}$ and $\frac{1}{0}$. Subsequent levels of the tree are formed by inserting the mediant fraction $\frac{a+c}{b+d}$ between every pair of neighboring values $\frac{a}{b}$ and $\frac{c}{d}$, and the process is repeated to infinity.

There is a close connection between the Stern-Brocot tree and continued fractions: for instance, both can be used to compute the best smaller-denominator rational approximation to a given fraction [7]. The connection comes from the fact that the mediant of consecutive terms in the Stern-Brocot tree can be expressed as an operation on the continued fraction expansions, whereby continued fractions allow for a precise determination of where in the Stern-Brocot tree a particular fraction will appear [3]. Retracing the tree upward gives a series of progressively worse rational approximations with decreasing denominators.

Comment. Stern and Brocot had inadvertently developed an elegant way to find the best smaller-denominator rational approximation to a given fraction. It was known that continued fractions could be used for the same purpose [7], which suggested a connection between the two. Indeed, it was later discovered that the mediant could be expressed as an operation on continued fraction expansions, whereby continued fractions provided a way to determine where in the Stern-Brocot tree a particular fraction would appear [3]. Retracing the tree upward would then give a series of progressively worse rational approximations with decreasing denominators.

There are many other topics related to the Stern-Brocot tree [4]. For instance, Farey sequences (ordered lists of the rationals between 0 and 1 with denominator smaller than n) can be obtained by discarding fractions with denominator greater than n from the n^{th} row of the left half of the Stern-Brocot tree [3]. In addition, the radii of Ford circles vary inversely with the squares of the corresponding terms in the left half of the Stern-Brocot tree [3]. Variants of the Stern-Brocot tree include the Calkin-Wilf tree [5] (another binary tree generated from a mediant-like procedure, which also enumerates the rationals) and Stern's diatomic series [6], studied more recently in [8]. The Stern-Brocot tree has also been used to provide elementary proofs of such results as Hurwitz' theorem [9].

We begin this paper with a brief discussion of the classical Stern-Brocot tree and give proofs of several of its properties. In Section 2 we cover the symmetry of the tree, certain algebraic relations its elements satisfy, and they way its terms reduce. We then introduce the notion of the cross-difference and analyze its role in the reduction of fractions, en route to a proof of Theorem 1 – that every rational number between 0 and 1 appears in the Stern-Brocot tree.

In Section 3 we present a variant of the original Stern-Brocot tree, to which the bulk of the paper is devoted. We consider starting with terms other than $\frac{0}{1}$ and $\frac{1}{0}$, and ask which properties of the original tree extend to the general setting. In particular, we prove that regardless of the choice of starting terms, every rational number between the two starting terms appears in the tree. We do this first for special values of the cross-difference with Theorems 2, 3, and 4, before establishing the general result as Theorem 7. As part of the proof, we develop the important idea of tree equivalence.

2. The Classical Case – Notation and Definitions

In number theory, the Stern-Brocot tree is an infinite complete binary tree in which the vertices correspond precisely to the positive rational numbers. The Stern-Brocot tree can be defined in terms of Stern-Brocot sequences as follows. The 0^{th} Stern-Brocot sequence, which we denote by SB^0, is $\left(\frac{0}{1}, \frac{1}{0}\right)$. In general, SB^i denotes the i^{th} Stern-Brocot sequence and is formed by copying SB^{i-1}, inserting the mediant $\frac{a+b}{c+d}$ between every pair of consecutive fractions $\frac{a}{b}, \frac{c}{d}$, and reducing all fractions to lowest terms. We have,

$$SB^0 = \frac{0}{1}, \quad \frac{1}{0}$$
$$SB^1 = \frac{0}{1}, \quad \frac{\mathbf{1}}{\mathbf{1}}, \quad \frac{1}{0}$$
$$SB^2 = \frac{0}{1}, \quad \frac{\mathbf{1}}{\mathbf{2}}, \quad \frac{1}{1}, \quad \frac{\mathbf{2}}{\mathbf{1}}, \quad \frac{1}{0}$$
$$SB^3 = \frac{0}{1}, \quad \frac{\mathbf{1}}{\mathbf{3}}, \quad \frac{1}{2}, \quad \frac{\mathbf{2}}{\mathbf{3}}, \quad \frac{1}{1}, \quad \frac{\mathbf{3}}{\mathbf{2}}, \quad \frac{2}{1}, \quad \frac{\mathbf{3}}{\mathbf{1}}, \quad \frac{1}{0}.$$

In bold are the mediant fractions that have been inserted, which are the vertices of the Stern-Brocot tree. Thus the i^{th} level of the Stern-Brocot tree is $SB^i \setminus SB^{i-1}$, the fractions which appear for the first time in SB^i. However, we will abuse the notation slightly and use the term "tree" to refer to the collection of sequences $\bigcup_i SB^i$. The distinction is only semantic, but to be clear we refer to the SB^i as the "rows" of the Stern-Brocot tree (as opposed to levels).

Observe that the rows have reciprocal symmetry about their center; that is, the j^{th} term counted from the left is the reciprocal of the j^{th} term counted from the right [10]. In light of this, we will consider only the left half of the rows, whose values are between 0 and 1 (inclusive) and which we will call Stern-Brocot half-sequences. More formally, the Stern-Brocot half-sequence SB_i is the sequence $\{x \in SB^{i+1} | x \leq 1\}$.

Finally, let the *cross-difference* of two fractions $\frac{a}{b}, \frac{c}{d}$ denote the quantity $(bc - ad)$. The following lemma is well-known [3]; we provide a proof because it illustrates a method used later.

Lemma 1. *For any two consecutive fractions $\frac{a}{b}$ and $\frac{c}{d}$ in SB_i, their cross-difference equals 1.*

Proof. The proof is by induction on the row number. The lemma obviously holds for $SB_0 = \left(\frac{0}{1}, \frac{1}{1}\right)$. Now suppose that it holds for the n^{th} Stern-Brocot half-sequence, and let $\frac{a}{b}$ and $\frac{c}{d}$ be any two consecutive fractions in SB_n. Their mediant is equal to $\frac{a+c}{b+d}$ which can be written as

$$\frac{(a+c)/g}{(b+d)/g}$$

in lowest terms, where $g = \gcd(a+c, b+d)$. Yet we have

$$1 = (bc - ad) = (bc + ba) - (ad + ba) = (a+c)b - (b+d)a$$

where the first step follows from the induction hypothesis and the right-hand side is divisible by g. It follows that $g = 1$, meaning $\frac{a+c}{b+d}$ is in lowest terms. Thus, consecutive terms in SB_{n+1} are either of the form $\left(\frac{a}{b}, \frac{a+c}{b+d}\right)$ or $\left(\frac{a+c}{b+d}, \frac{c}{d}\right)$ for consecutive $\frac{a}{b}, \frac{c}{d}$ in SB_n. We verify:

$$b(a+c) - a(b+d) = (bc - ad) = 1$$

$$(b+d)c-(a+c)d=(bc-ad)=1$$

and so in either case the cross-difference is 1 – the induction is complete. □

As a part of this proof, we have established the following corollary:

Corollary 1. *Suppose the mediant of $\frac{a}{b}, \frac{c}{d}$ is reduced by a factor of g. Then the cross-difference of the mediant with each of $\frac{a}{b}, \frac{c}{d}$ is $(bc-ad)/g$, so that $g|(bc-ad)$. In particular, mediants in the Stern-Brocot tree never need to be reduced.*

Let $SB_i[j]$ denote the j-th element in the i-th half-sequence. The following two lemmas are quite straightforward [4]; their proofs are left as an exercise for the reader.

Lemma 2. *There are exactly 2^i+1 elements in SB_i.*

Lemma 3. *We have $SB[j]+SB[2^i-j]=1$.*

The next lemma will allow us to compute the terms of the Stern-Brocot half-sequences explicitly. We remind the reader that Stern's diatomic series, sometimes called the Stern sequence, is the given by $s(0)=0$, $s(1)=1$, and for $n \geq 2$

$$s(n)=\begin{cases} s(n/2) & \text{if } n \text{ is even} \\ s((n-1)/2)+s((n+1)/2) & \text{if } n \text{ is odd} \end{cases}$$

The Stern sequence satisfies the identities $s(2^i+j)=s(j)+s(2^i-j)=s(2^{i+1}-j)$[6].

Lemma 4. *We have*

$$SB_i[j]=\frac{s(j)}{s(2^i+j)}=\frac{s(j)}{s(j)+s(2^i-j)}=\frac{s(j)}{s(2^{i+1}-j)}.$$

Proof. It is enough to show $SB_i[j]=s(j)/s(2^i+j)$; the other assertions follow immediately from the above identites. We proceed by induction on i. The base case $i=0$ is straightforward, so let us suppose the result holds for $i=n$ and consider $SB_{n+1}[j]$. There are two cases here: either j is even, in which case $SB_{n+1}[j]$ was copied from $SB_n[j]$, or else j is odd, in which case $SB_{n+1}[j]$ is the mediant of $SB_n[\lfloor j/2\rfloor]$ and $SB_n[\lceil j/2\rceil]$.

If j is even, then $j=2j'$ so we can write

$$SB_{n+1}[j]=SB_n[j']=\frac{s(j')}{s(2^n+j')}=\frac{s(j)}{s(2^{n+1}+j)}$$

where the second step follows from the induction hypothesis, and the third step from the fact that $s(2k)=s(k)$.

If j is instead odd, then $j=2j'+1$ so we have

$$SB_{n+1}[j]=\text{mediant}(SB_n[j'],SB_n[j'+1])=\frac{s(j')+s(j'+1)}{s(2^n+j')+s(2^n+j')}=\frac{s(j)}{s(2^{n+1}+j)}.$$

The second step follows from the induction hypothesis, and the third step uses the fact that $s(2k+1) = s(k+1) + s(k)$. The induction is complete, and the lemma follows. □

We can now apply these lemmas together to prove the following result; while the theorem is well-known, the proof is different in spirit from the standard proofs. In essence, it relies on the algorithm for writing a rational number as a finite continued fraction. More about the connection between continued fractions and Stern sequences can be found in [6].

Theorem 1. *All (reduced) fractions* $\frac{p}{q} \in [0,1]$ *appear in some* SB_i.

Proof. Let us induct on the denominator q. The base cases $q = 1$ and $q = 2$ are straightforward – the fractions $\frac{0}{1}$ and $\frac{1}{1}$ appear in SB_0, whereas $\frac{1}{2}$ appears in SB_1. Now suppose the result holds for all $q \leq n$, and let $\frac{m}{n+1}$ be any irreducible fraction with denominator $n+1$.

If $\frac{m}{n+1} < \frac{1}{2}$, then consider the fraction $\frac{m}{n+1-m}$, which has denominator at most n. Since $\gcd(m, n+1-m) = \gcd(m, n+1) = 1$, $\frac{m}{n+1-m}$ appears in SB_i for some i by the induction hypothesis. Equivalently, by Lemma 4, there exist i, j with $s(j) = m$ and $s(2^i - j) = (n+1-m)$. It follows that

$$SB_{i+1}[j] = \frac{s(j)}{s(j) + s(2^i - j)} = \frac{m}{m + (n+1-m)} = \frac{m}{n+1}$$

as desired. If, instead, $\frac{m}{n+1} > \frac{1}{2}$, then by Lemma 3 this fraction appears in SB_i if and only if $\frac{n+1-m}{n+1}$ appears in SB_i. Since $\frac{n+1-m}{n+1} < \frac{1}{2}$, we can reason as we did above to finish. □

By reciprocal symmetry it follows that the Stern-Brocot tree contains all the positive rationals.

3. Arbitrary Starting Terms

One variant of the Stern-Brocot tree that arises quite naturally comes from varying the two starting terms – that is, beginning instead with any pair of rational numbers. The process of inserting mediants is exactly the same: the mediant fraction $\frac{a+c}{b+d}$ is reduced and inserted between the consecutive fractions $\frac{a}{b}$ and $\frac{c}{d}$. Since the cross-difference $(bc - ad)$ is no longer necessarily 1, the generalized tree may contain reducible fractions, which can be reduced in many ways: to lowest terms, not at all, or partially. Moreover, these choices can be made independently for all fractions. An example of a tree with nontrivial reduction is the following, with reduced fractions in bold:

$$\frac{2}{5}, \quad \frac{5}{11}$$
$$\frac{2}{5}, \quad \frac{7}{16}, \quad \frac{5}{11}$$
$$\frac{2}{5}, \quad \frac{\mathbf{3}}{\mathbf{7}}, \quad \frac{7}{16}, \quad \frac{\mathbf{4}}{\mathbf{9}}, \quad \frac{5}{11}$$
$$\frac{2}{5}, \quad \frac{5}{12}, \quad \frac{3}{7}, \quad \frac{10}{23}, \quad \frac{7}{16}, \quad \frac{11}{25}, \quad \frac{4}{9}, \quad \frac{9}{20}, \quad \frac{5}{11}.$$

Let us introduce some notation. We use $S_0(\frac{a}{b}, \frac{c}{d})$ to stand for the sequence $\left(\frac{a}{b}, \frac{c}{d}\right)$, and for $i \geq 1$ let $S_i(\frac{a}{b}, \frac{c}{d})$ denote a sequence formed by inserting mediants between all consecutive pairs of fractions in $S_{i-1}(\frac{a}{b}, \frac{c}{d})$ and reducing fractions somehow. We use the term "tree" to refer to $T(\frac{a}{b}, \frac{c}{d}) = \bigcup_i S_i(\frac{a}{b}, \frac{c}{d})$, and say that S_i is the i^{th} row of $T(\frac{a}{b}, \frac{c}{d})$. For example, the first few rows of $T(\frac{2}{1}, \frac{3}{1})$ are

$$S_0\left(\frac{2}{1}, \frac{3}{1}\right) = \frac{2}{1}, \quad \frac{3}{1}$$

$$S_1\left(\frac{2}{1}, \frac{3}{1}\right) = \frac{2}{1}, \quad \frac{5}{2}, \quad \frac{3}{1}$$

$$S_2\left(\frac{2}{1}, \frac{3}{1}\right) = \frac{2}{1}, \quad \frac{7}{3}, \quad \frac{5}{2}, \quad \frac{8}{3}, \quad \frac{3}{1}$$

We shall investigate how these generalized Stern-Brocot trees behave and whether they exhibit any of the properties of the original. Of particular interest to us is whether every rational number between the two starting terms appears in some row. We are also interested in the cross-difference, as it is critical in determining how values in the tree reduce. We now present three intermediate results, characterized by the value of the cross-difference.

Theorem 2. *Suppose the fractions $\frac{a}{b}$ and $\frac{c}{d}$ satisfy $(bc - ad) = 1$. Then for any tree $T(\frac{a}{b}, \frac{c}{d})$, every rational number in the interval $[\frac{a}{b}, \frac{c}{d}]$ appears in $T(\frac{a}{b}, \frac{c}{d})$.*

Proof. Suppose $\frac{x}{y} \in [\frac{a}{b}, \frac{c}{d}]$. We can write

$$\frac{x}{y} = \frac{(1-\lambda)a + \lambda c}{(1-\lambda)b + \lambda d}$$

where $0 \leq \lambda \leq 1$ by virtue of the fact that $\frac{a}{b} \leq \frac{x}{y} \leq \frac{c}{d}$. In fact, we can solve for λ exactly:

$$\lambda = \frac{(bx - ay)}{(bx - ay) + (cy - dx)}.$$

Since $(bc - ad) = 1$, fractions are never reducible and so we can write down the j^{th} element of the i^{th} row of $T(\frac{a}{b}, \frac{c}{d})$ explicitly as

$$\frac{s(2^i - j)a + s(j)c}{s(2^i - j)b + s(j)d}.$$

This follows by an identical induction to that in the proof of Lemma 4, but is also intuitively clear: without reduction, the expression of $SB_i(\frac{a}{b}, \frac{c}{d})[j]$ as a weighted combination of $\frac{a}{b}$ and $\frac{c}{d}$ is the same as in the original Stern-Brocot tree. Now observe that if we could write $\lambda = s(j)/s(2^i - j)$ for some i, j then we would be done since $SB_i(\frac{a}{b}, \frac{c}{d}) = \frac{x}{y}$ by definition of λ and choice of i, j. Yet this *must* be possible, since $0 \leq \lambda \leq 1$, together with the fact that λ is rational, implies that λ appears in the original Stern-Brocot tree whence it must be of the form $s(j)/s(2^i - j)$ for some i, j by Lemma 4. The result follows. □

Comment. *Proof.* We know the result holds when $\frac{a}{b} = \frac{0}{1}$ and $\frac{c}{d} = \frac{1}{1}$. Every number in this range can be written as $\frac{z}{w+z}$ for some choice of w, z. But this is just

$$\frac{0w + 1z}{1w + 1z}$$

which means we can obtain any combination of 2 weights that describe the contribution of the two starting terms.

If $bc - ad = 1$ for some a, b, c, d, reduction of fractions never takes place, meaning we only need to show that any number $\frac{x}{y}$ with $\frac{a}{b} \leq \frac{x}{y} \leq \frac{c}{d}$ can be written as

$$\frac{aw + cz}{bw + dz}$$

for appropriate choice of w, z, since under the transformation $0 \to a$, $1 \to b$, $1 \to c$, $1 \to d$, we can reduce the problem to the appearance of a particular fraction in $T(\frac{0}{1}, \frac{1}{1})$ which we know happens.

We want

$$\frac{aw + cz}{bw + dz} = \frac{x}{y},$$

or equivalently,

$$\begin{gathered}(ay)w + (cy)z = (bx)w + (dx)z, \\ (bx - ay)w = (cy - dx)z \\ (bx - ay)(w + z) = [(bx - ay) + (cy - dx)]z, \\ \frac{z}{z + w} = \frac{(bx - ay)}{(bx - ay) + (cy - dx)},\end{gathered}$$

meaning we can take $w = cy - dx$ and $z = bx - ay$ which are both positive integers. Then the fraction $\frac{x}{y}$ appears in $T(\frac{a}{b}, \frac{c}{d})$. □

Let us consider a concrete example. Suppose we wanted to prove that $\frac{4}{5} \in [\frac{2}{3}, \frac{1}{1}]$ appears in $T(\frac{2}{3}, \frac{1}{1})$. Keeping the same notation, $\lambda = \frac{2}{3}$ since

$$\frac{4}{5} = \frac{\frac{1}{3}(2) + \frac{2}{3}(1)}{\frac{1}{3}(3) + \frac{2}{3}(1)}.$$

Then $\frac{4}{5}$ should appear in $T(\frac{2}{3}, \frac{1}{1})$ in exactly the same position as $\frac{2}{3}$ appears in $T(\frac{0}{1}, \frac{1}{1})$, which is easy to verify.

The idea of looking at weight combinations to turn questions about the appearance of fractions in one tree into questions about the appearance of fractions in another tree is important. Indeed, this can only be done when the trees are "equivalent", a notion we will make precise later. The main result depends heavily on this principle.

Observe that Theorem 2 deals with the special case when $(bc - ad) = 1$. Since fractions are never reducible in this case, there is a unique tree $T(\frac{a}{b}, \frac{c}{d})$. For other values of the cross-difference, fractions may have non-trivial reduction. This means the expression $T(\frac{a}{b}, \frac{c}{d})$ is ambiguous if we have not specified a reduction scheme. For the next two theorems, we assume we reduce fractions to lowest terms. This will enable us to use Theorem 2 to prove that all rationals between the starting terms appear in the tree.

For a prime p, let $S_i^p(\frac{a}{b}, \frac{c}{d})$ denote the highest power of p which divides any of the cross-differences $x_j y_{j-1} - x_{j-1} y_j$, where $\frac{x_{j-1}}{y_{j-1}}, \frac{x_j}{y_j}$ are consecutive in $S_i(\frac{a}{b}, \frac{c}{d})$. Also let $v_p(x)$ denote the highest power of p which divides x.

Theorem 3. *Suppose $\frac{a}{b}, \frac{c}{d}$ are fractions with $v_2(bc - ad) = m$. Then $S_i^2(\frac{a}{b}, \frac{c}{d}) \leq \max(m - i, 0)$ for all i.*

For instance, consider $T(\frac{1}{9}, \frac{1}{1})$, where $bc - ad = 8 = 2^3$ – after 3 rows, all cross-differences become unity. Again, reduced fractions are indicated in bold:

$$\frac{1}{9} \quad \frac{1}{1}$$

$$\frac{1}{9} \quad \mathbf{\frac{1}{5}} \quad \frac{1}{1}$$

$$\frac{1}{9} \quad \mathbf{\frac{1}{7}} \quad \frac{1}{5} \quad \mathbf{\frac{1}{3}} \quad \frac{1}{1}$$

$$\frac{1}{9} \quad \mathbf{\frac{1}{8}} \quad \frac{1}{7} \quad \mathbf{\frac{1}{6}} \quad \frac{1}{5} \quad \mathbf{\frac{1}{4}} \quad \frac{1}{3} \quad \mathbf{\frac{1}{2}} \quad \frac{1}{1}$$

Proof. We prove this by strong induction on m. When $m = 0$, $(bc - ad)$ is odd; by Corollary 1 all cross-differences in $T(\frac{a}{b}, \frac{c}{d})$ divide $(bc - ad)$ so they are also odd. Now assume the result holds for all $k \leq \ell$, and suppose $v_2(bc-ad) = \ell+1$. Consider the parity of a, b, c, d; since $\frac{a}{b}$ and $\frac{c}{d}$ are in lowest terms, a and b cannot both be even, nor can both c and d. Also, bc and ad must have the same parity since their difference is a non-trivial power of 2. If both are odd, then a, b, c, d must all be odd. If both are even, either a, c are even and b, d are odd, or else a, c are odd and b, d are even. In either case, the numerator and denominator of the mediant fraction $\frac{a+c}{b+d}$ are both even. Thus in the next row, we have

$$S_1\left(\frac{a}{b}, \frac{c}{d}\right) = \frac{a}{b}, \frac{x}{y}, \frac{c}{d}$$

where $x = (a+c)/g$, $y = (b+d)/g$, and $2|g$. By Corollary 1, $(bx-ay) = (cy-dx) = (bc-ad)/g$ so that $S_1^2(\frac{a}{b}, \frac{c}{d}) \leq \ell$. The result now follows by applying the induction hypothesis to the subtrees $S_0(\frac{a}{b}, \frac{x}{y})$ and $S_0(\frac{x}{y}, \frac{c}{d})$. □

In particular, if $(bc-ad) = 2^m$ then every consecutive pair $(\frac{x_{j-1}}{y_{j-1}}, \frac{x_j}{y_j})$ in $S_m(\frac{a}{b}, \frac{c}{d})$ satisfies $x_j y_{j-1} - x_{j-1} y_j = 1$. Applying Theorem 2 to each such pair we have the following.

Corollary 2. *Let $\frac{a}{b}, \frac{c}{d}$ be fractions satisfying $bc - ad = 2^m$. Then the tree $T(\frac{a}{b}, \frac{c}{d})$ obtained by reducing fractions to lowest terms contains every rational number in $[\frac{a}{b}, \frac{c}{d}]$.*

Replacing the prime $p = 2$ with $p = 3$ yields the following analog of Theorem 3.

Theorem 4. *Suppose $\frac{a}{b}, \frac{c}{d}$ are fractions with $v_3(bc - ad) = n$. Then $S_{2i}^3(\frac{a}{b}, \frac{c}{d}) \leq \max(n-i, 0)$ for all i.*

Proof. We prove this by induction on n. When $n = 0$, $(bc - ad)$ is not divisible by 3; by Corollary 1 all cross-differences in $T(\frac{a}{b}, \frac{c}{d})$ divide $(bc - ad)$, so they must also not be divisible by 3. Now assume the result holds for all $k \leq \ell$, and suppose $v_3(bc - ad) = \ell + 1$. We do casework on the values of a, b, c, d modulo 3, using the fact that $3|(bc - ad)$.

If $bc \equiv ad \equiv 0 \pmod 3$, then 3 divides one of b, c and one of a, d. Since $\gcd(a, b) = \gcd(c, d) = 1$, either $3|b, d$ or $3|a, c$. Without loss of generality suppose $3|a, c$. Then $3 \nmid b, d$ meaning either $b \equiv -d \pmod 3$ or $b \equiv -2d \pmod 3$. In the first case, $\frac{a+c}{b+d} \in S_1(\frac{a}{b}, \frac{c}{d})$ is reduced by a factor g where $3|g$; it follows by Corollary 1 that $S_1^3(\frac{a}{b}, \frac{c}{d}) \leq \ell$ so we can finish by induction on the subtrees. In the second case, $\frac{2a+c}{2b+d}, \frac{a+2c}{b+2d} \in S_2(\frac{a}{b}, \frac{c}{d})$ are reduced by factors g_1, g_2 where $3|g_1, g_2$; it follows by Corollary 1 that $S_2^3(\frac{a}{b}, \frac{c}{d}) \leq \ell$ and again we finish by induction on the subtrees.

If $bc \equiv ad \equiv 1 \pmod 3$, then $(b, c), (a, d) \pmod 3 \in \{(1, 1), (2, 2)\}$. If (b, c) and (a, d) are the same modulo 3 – both either $(1, 1)$ or $(2, 2)$ – then $3|2a + c$, $3|2b + d$, $3|a+2c$, and $3|b+2d$. Thus $\frac{2a+c}{2b+d}, \frac{a+2c}{b+2d} \in S_2(\frac{a}{b}, \frac{c}{d})$ are reducible by a factor of 3, and we reason as before. Instead, if one of $(a, b), (c, d)$ is $(1, 1)$ and the other is $(2, 2)$, then $3|a + c$ and $3|b + d$, so the fraction $\frac{a+c}{b+d} \in SB_1(\frac{a}{b}, \frac{c}{d})$ is reducible by a factor of 3. Either way, $S_2^3(\frac{a}{b}, \frac{c}{d}) \leq \ell$ so we can invoke the induction hypothesis as before.

Finally, if $bc \equiv ad \equiv 2 \pmod 3$ then $(b,c),(a,d) \pmod 3 \in \{(2,1),(1,2)\}$. If they are the same modulo 3 – both either $(1,2)$ or $(2,1)$ – then $3|a+c$ and $3|b+d$ so $\frac{a+c}{b+d} \in SB_1(\frac{a}{b},\frac{c}{d})$ is reducible by a factor of 3. If, instead, one of $(a,b),(c,d)$ is $(1,2)$ and the other is $(2,1)$, then $3|2a+c$, $3|2b+d$, $3|a+2c$, and $3|b+2d$ so the fractions $\frac{2a+c}{2b+d}, \frac{a+2c}{b+2d} \in SB_2(\frac{a}{b},\frac{c}{d})$ are reducible by a factor of 3. Once again, $S_2^3(\frac{a}{b},\frac{c}{d}) \le \ell$ so we are done by induction. □

In particular, if $(bc-ad) = 2^m 3^n$ then combining Theorems 3 and 4 we conclude that every consecutive pair $(\frac{x_{j-1}}{y_{j-1}},\frac{x_j}{y_j})$ in $S_{\max(m,2n)}(\frac{a}{b},\frac{c}{d})$ satisfies $x_j y_{j-1} - x_{j-1} y_j = 1$. Applying Theorem 2 to each such pair we see that the following holds.

Corollary 3. *Let $\frac{a}{b},\frac{c}{d}$ be fractions satisfying $bc-ad = 2^m 3^n$. Then the tree $T(\frac{a}{b},\frac{c}{d})$, obtained by reducing fractions to lowest terms, contains every rational number in $[\frac{a}{b},\frac{c}{d}]$.*

Comment. Let the *width* $w_i(T)$ denote the maximum cross-determinant of any consecutive pair in the i-th row of a tree T, we have proven that $w_{i+1}(T)|w_i(T)$ (Corollary 1). Moreover, we've shown that $2|w_i(T)$ implies $w_{i+1}(T) < w_i(T)$ and that $3|w_i(T)$ implies $w_{i+2}(T) < w_i(T)$ when $w_i(T) > 1$.

The primes 2 and 3 are special; in general, non-trivial cross-differences can persist indefinitely in the tree. Take, for instance, $p = 5$, and consider the tree $T(\frac{1}{3},\frac{2}{1})$, displayed below with reduced fractions in bold. Despite many reductions, there is always a consecutive pair of fractions whose cross-difference is divisible by 5. Lemma 5 makes this precise.

$$
\begin{array}{ccccccccc}
 & & & \frac{1}{3} & \frac{2}{1} & & & & \\
 & & & \frac{1}{3} & \frac{3}{4} & \frac{2}{1} & & & \\
 & & \frac{1}{3} & \frac{4}{7} & \frac{3}{4} & \boldsymbol{\frac{1}{1}} & \frac{2}{1} & & \\
\frac{1}{3} & \boldsymbol{\frac{1}{2}} & \frac{4}{7} & \frac{7}{11} & \frac{3}{4} & \frac{4}{5} & \frac{1}{1} & \frac{3}{2} & \frac{2}{1}
\end{array}
$$

Lemma 5. *Let $p > 3$ be a prime. For all i, there exists j so that the cross-differences of the consecutive pairs $S_i\left(\frac{0}{1},\frac{p}{1}\right)[2j], S_i\left(\frac{0}{1},\frac{p}{1}\right)[2j+1]$ and $S_i\left(\frac{0}{1},\frac{p}{1}\right)[2j+1], S_i\left(\frac{0}{1},\frac{p}{1}\right)[2j+2]$ are divisible by p. In particular, $S_i^p(\frac{0}{1},\frac{p}{1}) \ge 1$ for all i.*

Proof. We use induction on i; the base cases $i = 0$ and $i = 1$ follow by construction and from the fact that $p \neq 2$, respectively. Let us now suppose the result holds for $S_n(\frac{0}{1},\frac{p}{1})$, i.e. there are consecutive fractions

$$\frac{u}{v}, \quad \frac{(u+w)/g}{(v+x)/g}, \quad \frac{w}{x}$$

in $S_n(\frac{0}{1}, \frac{p}{1})$ where $\frac{u}{v}$, $\frac{w}{x}$ are consecutive in $S_{n-1}(\frac{0}{1}, \frac{p}{1})$ and its mediant in $S_n(\frac{0}{1}, \frac{p}{1})$ is reduced by some factor g which is coprime to p. In fact, since g divides the cross-difference of the starting terms (Corollary 1), we must have $g = 1$. We will show that either $\frac{u}{v}$ or $\frac{w}{x}$, along with $\frac{u+w}{v+x}$ and their mediant with the former, are the consecutive fractions we seek in $S_{n+1}(\frac{0}{1}, \frac{p}{1})$.

Suppose otherwise, for the sake of contradiction. Then both the mediants $\frac{u+(u+w)}{v+(v+x)}$ and $\frac{(u+w)+w}{(v+x)+x}$ would have to be reduced by a factor of (exactly) p. From $p|(2u+w)$ and $p|(u+2w)$ we conclude that $u \equiv w \equiv 0 \pmod p$ since $p \neq 3$. Similarly, $v \equiv x \equiv 0$ which gives the contradiction: we would have reduced the mediant $\frac{u+w}{v+w}$ by a factor of p in the previous row. Thus the induction is complete, and the result follows. □

We would now like to extend Theorems 2, 3, and 4 to all possible values of the cross-difference and all reduction schemes. That is, we wish to show that regardless of the value of $bc - ad$ and the way in which fractions are reduced, the tree $T(\frac{a}{b}, \frac{c}{d})$ contains all rational numbers in $[\frac{a}{b}, \frac{c}{d}]$. In order to do this, we will require the notions of *corresponding elements* and *equivalent trees.*

Let e_1 be an element of some tree T_1 which occupies position j in row i of this tree. For any other tree T_2, we will call e_1 and $e_2 \in T_2$ *corresponding elements* if and only if e_2 occupies position j in row i of T_2.

Given two trees, we say they are *equivalent* if and only if all pairs of corresponding elements have the same gcd and are reduced by exactly the same factor. We do not insist this factor be the greatest common factor; here and afterwards, we allow trees with any *reduction scheme* — i.e., any pattern of (proper) fraction reduction that may vary from fraction to fraction. Equivalent trees are very closely related in structure. In fact,

Theorem 5. *Let $T_1 = T(\frac{a_1}{b_1}, \frac{c_1}{d_1})$ and $T_2 = T(\frac{a_2}{b_2}, \frac{c_2}{d_2})$ be two equivalent trees. If $e_1 = \frac{p_1}{q_1} \in T_1$ and $e_2 = \frac{p_2}{q_2} \in T_2$ are corresponding elements, then e_1 and e_2 are the same weighted combination of the initial terms in their respective trees. More formally, let (x, y, g) be the unique triple of nonnegative integers satisfying* $\gcd(x, y)$ *= 1, $p_1 = (a_1x + c_1y)/g$, and $q_1 = (b_1x + d_1y)/g$. Then $p_2 = (a_2x + c_2y)/g$ and $q_2 = (b_2x + d_2y)/g$.*

This theorem can be viewed as a generalization of Theorem 2.

Proof. We argue by induction on the row number r. When $r = 0$, the conclusion is obvious. Now let m_1, n_1 be arbitrary consecutive fractions in row r of T_1, and let m_2, n_2 be the corresponding elements in T_2. It suffices to show that the mediants of these pairs are the same weighted combination of the initial terms in their respective trees. We can write

$$m_1 = \frac{(w_m a_1 + z_m c_1)/g_m}{(w_m b_1 + z_m d_1)/g_m}$$

and

$$n_1 = \frac{(w_n a_1 + z_n c_1)/g_n}{(w_n b_1 + z_n d_1)/g_n}$$

for the appropriate $w_m, z_m, w_n, z_n, g_m, g_n$. Taking the mediant, we arrive at

$$s_1 = \frac{([g_n w_m + g_m w_n]a_1 + [g_n z_m + g_m z_n]c_1)/g_m g_n}{([g_n w_m + g_m w_n]b_1 + [g_n z_m + g_m z_n]d_1)/g_n g_m}.$$

Note that the weights are no longer necessarily coprime; however, if we let $g_w = \gcd(g_n w_m + g_m w_n, g_n z_m + g_m z_n)$ then we can write

$$s_1 = \frac{([(g_n w_m + g_m w_n)/g_w]a_1 + [(g_n z_m + g_m z_n)/g_w]c_1)/(g_m g_n/g_w)}{([(g_n w_m + g_m w_n)/g_w]b_1 + [(g_n z_m + g_m z_n)/g_w]d_1)/(g_m g_n/g_w)}$$

where the weights are now relatively prime. Additionally, it is possible we reduce s_1. If g is the common factor we cancel, then s_1 is a weighted combination of the starting terms $\frac{a_1}{b_1}, \frac{c_1}{d_1}$ precisely as follows:

$$s_1 = \frac{([(g_n w_m + g_m w_n)/g_w]a_1 + [(g_n z_m + g_m z_n)/g_w]c_1)/(g_m g_n g/g_w)}{([(g_n w_m + g_m w_n)/g_w]b_1 + [(g_n z_m + g_m z_n)/g_w]d_1)/(g_m g_n g/g_w)}.$$

Now consider T_2. By the induction hypothesis, m_2, n_2 are the same weighted combination of the starting terms as m_1, n_1. We can therefore compute the mediant of m_2, n_2 via analogous algebra:

$$s_2 = \frac{([(g_n w_m + g_m w_n)/g_w]a_2 + [(g_n z_m + g_m z_n)/g_w]c_2)/(g_m g_n/g_w)}{([(g_n w_m + g_m w_n)/g_w]b_2 + [(g_n z_m + g_m z_n)/g_w]d_2)/(g_m g_n/g_w)}.$$

Once more, we must divide to account for the fact that the weights are not necessarily coprime. Yet T_1 and T_2 are equivalent, so the factor by which T_2 is reduced is the same factor by which T_1 was reduced – namely, g. Hence s_2 is a weighted combination of the starting terms $\frac{a_2}{b_2}, \frac{c_2}{d_2}$ precisely as follows:

$$s_2 = \frac{([(g_n w_m + g_m w_n)/g_w]a_2 + [(g_n z_m + g_m z_n)/g_w]c_2)/(g_m g_n g/g_w)}{([(g_n w_m + g_m w_n)/g_w]b_2 + [(g_n z_m + g_m z_n)/g_w]d_2)/(g_m g_n g/g_w)}.$$

Thus the mediants s_1 and s_2 are the same weighted combination of the initial terms in their respective trees, and so the result follows from induction. □

Immediately we have the following lemma:

Lemma 6. *If $T_1 = T(\frac{a_1}{b_1}, \frac{c_1}{d_1})$ and $T_2 = T(\frac{a_2}{b_2}, \frac{c_2}{d_2})$ are equivalent trees and T_2 contains all rational numbers in the interval $[\frac{a_2}{b_2}, \frac{c_2}{d_2}]$, then T_1 contains all rational numbers in the interval $[\frac{a_1}{b_1}, \frac{c_1}{d_1}]$.*

Proof. The tree T_2 contains all rational numbers in the interval $[\frac{a_2}{b_2}, \frac{c_2}{d_2}]$ if and only if all possible weights (x, y) are attainable. Indeed, every rational number in this interval can be written uniquely as a weighted combination of $\frac{a}{b}, \frac{c}{d}$, which means if some weight is not attainable, the corresponding fraction does not appear. But since T_1 and T_2 are equivalent, the set of weights attainable in T_1 is exactly the set of weights attainable in T_2. Since all possible weights are attainable in T_2, they are all attainable in T_1 and so T_1 contains all rational numbers in $[\frac{a_1}{b_1}, \frac{c_1}{d_1}]$. □

Now that we can indirectly show that a tree $T(\frac{a}{b}, \frac{c}{d})$ contains all rational numbers in the interval $[\frac{a}{b}, \frac{c}{d}]$, we are motivated to establish equivalence between a general tree $T(\frac{a}{b}, \frac{c}{d})$ and some particularly malleable one.

Theorem 6. *For any tree $T(\frac{a}{b}, \frac{c}{d})$, there exists a positive integer V such that $T(\frac{a}{b}, \frac{c}{d})$ is equivalent to a tree of the form $T(\frac{0}{1}, \frac{D}{V})$, where $D = (bc - ad)$ is the cross-difference of the pair $\frac{a}{b}$, $\frac{c}{d}$.*

Proof. Suppose there existed a positive integer V such that $\gcd(ax+cy, bx+dy) = \gcd(Dy, x+Vy)$ for all x, y (although we consider only coprime x, y since clearly both sides are linear). We claim it would follow that $T_1 = T(\frac{a}{b}, \frac{c}{d})$ and $T_2 = T(\frac{0}{1}, \frac{D}{V})$ are equivalent when T_2 is reduced according to the same reduction scheme as T_1. Indeed, if this condition is met then for any fraction in T_1 which is reduced, it is possible to reduce the corresponding fraction in T_2 by the same factor so that the reduction schemes can be kept consistent.

It remains only to show that some such V exists. Let the prime factorization of D be $p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$. By Corollary 1, if a fraction in a tree with cross-difference D reduces by some factor g, then $g|D$. Hence it is enough to show that there exists some V such that for all p_i, $\min(v_{p_i}(ax+cy), v_{p_i}(bx+dy)) = \min(v_{p_i}(Dy), v_{p_i}(x+Vy))$. We remind the reader that $v_p(x)$ denotes the p-adic valuation of x, i.e. the highest power to which p divides it.

From the fact that $p_i|D$ and $\gcd(a,b) = \gcd(c,d) = 1$, it follows that p_i divides exactly zero, one, or two of a, b, c, d. However, p_i cannot divide just one of a, b, c, d lest it divide exactly one of ad and bc, contradicting the fact that it divides their difference D. We now consider the two cases separately.

`Case 1`: $p_i \nmid a, b, c, d$.
We claim we can take $V \equiv a^{-1}c \equiv b^{-1}d \pmod{p_i^{e_i}}$. Let $m_i = \min(v_{p_i}(ax+cy), v_{p_i}(bx+dy))$ and let $n_i = \min(v_{p_i}(Dy), v_{p_i}(x+Vy))$. Since $p_i^{m_i}$ divides any linear combination of $(ax+cy)$ and $(bx+dy)$, in particular we have

$$p_i^{m_i} | b(ax+cy) - a(bx+dy) = (bc-ad)y = Dy \implies v_{p_i}(Dy) \geq m_i.$$

Suppose $v_{p_i}(x+Vy) < e_i$. Since a is not divisible by p_i,

$$v_{p_i}(x+Vy) = v_{p_i}(ax+aVy) = v_{p_i}(ax+cy) \geq m_i.$$

Of course, we can also reason in the opposite direction:

$$v_{p_i}(ax+cy) = v_{p_i}(ax+aVy) = v_{p_i}(x+Vy) \geq n_i$$

and

$$p_i^{n_i}|(ab(x+Vy)-Dy) = a(bx+dy) \implies v_{p_i}(bx+dy) \geq n_i.$$

Hence, unless $v_{p_i}(x+Vy) \geq e_i$, we have $n_i \geq m_i$ and $m_i \geq n_i$ whence $m_i = n_i$ as desired. If, instead, $v_{p_i}(x+Vy) \geq e_i$, we will show $m_i = n_i = e_i$ so that either way the claim holds. Indeed, $v_{p_i}(Dy) = e_i$ since $p_i|y$ together with $p_i|(x+Vy)$ would imply $p_i|x$ which contradicts the fact that x, y are relatively prime. By hypothesis $v_{p_i}(x+Vy) \geq e_i$, and so $n_i = \min(v_{p_i}(Dy), v_{p_i}(x+Vy)) = e_i$. Next observe $v_{p_i}(ax+aVy) \geq e_i$ and $v_{p_i}((aV-c)y) \geq e_i$ so that $v_{p_i}(ax+cy) \geq e_i$. Moreover,

$$a(bx+dy) = b(ax+cy) - Dy$$

whence $v_{p_i}(bx+dy) = v_{p_i}(a(bx+dy)) \geq e_i$. However, Dy has p_i-adic valuation exactly e_i, and is a linear combination of $(ax+cy)$ and $(bx+dy)$; it follows that $m_i = \min(v_{p_i}(ax+cy), v_{p_i}(bx+dy)) = e_i$, and so we are done with Case 1.

`Case 2`: Exactly two of a, b, c, d are divisible by p_i.
Note that p_i cannot divide a and b simultaneously, lest $\frac{a}{b}$ would be reducible. Similarly, p_i cannot divide c and d simultaneously. It is also not possible that $p_i|a, d$ or $p_i|b, c$; in this case, exactly one of bc, ad would be divisible by p_i which contradicts the fact that their difference D is divisible by p_i. Thus either p_i divides a and c, or else p_i divides b and d.

Without loss of generality suppose $p_i|a, c$. Then in particular, b is invertible modulo p^{e_i} and so we can choose $V \equiv b^{-1}d \pmod{p_i^{e_i}}$. If the choice of $a^{-1}c$ as opposed to $b^{-1}d$ in Case 1 seemed arbitrary, it is because $a^{-1}c \equiv b^{-1}d \pmod{p_i^{e_i}}$ when they are both well-defined. Now suppose $v_{p_i}(bx+dy) < e_i$. Since b is not divisible by p_i we have

$$v_{p_i}(bx+dy) = v_{p_i}(b^{-1}(bx+dy)) = v_{p_i}(x+Vy).$$

In addition, $v_{p_i}(ax+cy) = v_{p_i}(b(ax+cy))$ where we can write

$$b(ax+cy) = a(bx+dy) + Dy.$$

It follows that $v_{p_i}(ax+cy) = v_{p_i}(bx+dy)$. Finally, from $v_{p_i}(Dy) \geq e_i > v_{p_i}(bx+dy)$ we conclude that

$$m_i = v_{p_i}(bx+dy) = v_{p_i}(x+Vy) = n_i.$$

The only case left to consider is when $v_{p_i}(bx+dy) \geq e_i$. Notice that both Dx and Dy are linear combinations of $(ax+cy)$ and $(bx+dy)$, as we have

$$Dx = c(bx+dy) - d(ax+cy)$$

and

$$Dy = b(ax + cy) - a(bx + dy).$$

Thus $v_{p_i}(Dx), v_{p_i}(Dy) \geq m_i$. However, since x and y are relatively prime,

$$\min(v_{p_i}(Dx), v_{p_i}(Dy)) = e_i$$

so that $m_i \leq e_i$. From the fact that $v_{p_i}(Dx), v_{p_i}(Dy) \geq e_i$ it is apparent that equality must hold, i.e. $m_i = e_i$.

Finally, we show $n_i = e_i$ as well. First, $v_{p_i}(x + Vy) = v_{p_i}(bx + dy) \geq e_i$. If y is not divisible by p_i, then $v_{p_i}(Dy) = e_i$ and so $n_i = \min(v_{p_i}(Dy), v_{p_i}(x + Vy)) = e_i$ as desired. Yet if p_i were to divide y, then $p_i|(x + Vy)$ would force $p_i|x$ as well, contradicting the fact that x, y are coprime. At last, we are finished with the proof of Theorem 6. □

We can take advantage of the linearity of the equivalent tree $T(\frac{0}{1}, \frac{D}{V})$ to prove the following:

Theorem 7. *If $\frac{a}{b}, \frac{c}{d}$ are any two rational numbers, $T(\frac{a}{b}, \frac{c}{d})$ contains all rational numbers in the interval $[\frac{a}{b}, \frac{c}{d}]$.*

Proof. We use strong induction on the value of the cross-difference $D = (bc - ad)$. When $D = 1$, the claim is simply the statement of Theorem 2. Now suppose that, for some n, the claim holds for all values $D \leq n$. To show that it holds for $D = n + 1$, we will show that for all positive integers V and reduction schemes, the tree $T(\frac{0}{1}, \frac{n+1}{V})$ contains all rational numbers in the interval $[\frac{0}{1}, \frac{n+1}{V}]$. By Lemma 6 and Theorem 6, this is sufficient.

Let $x \in [\frac{0}{1}, \frac{n+1}{V}]$ be an arbitrary rational. The value $\frac{x}{n+1}$ appears in the tree $T(\frac{0}{1}, \frac{1}{V})$ since this tree has cross-difference 1. Suppose this value appears for the first time in row k. For $0 \leq i < k$, define L_i to be the greatest fraction less than $\frac{x}{n+1}$ in row i of $T(\frac{0}{1}, \frac{1}{V})$. Similarly, let R_i be the least fraction greater than $\frac{x}{n+1}$ in row i of $T(\frac{0}{1}, \frac{1}{V})$.

Suppose for the sake of contradiction that x does not appear in $T(\frac{0}{1}, \frac{n+1}{V})$. Let us analogously define l_i to be the greatest fraction less than x in row i of $T(\frac{0}{1}, \frac{n+1}{V})$ and r_i to be the least fraction greater than x.

At first, $l_0 = \frac{0}{1}$ and $r_0 = \frac{n+1}{V}$ while $L_0 = \frac{0}{1}$ and $R_0 = \frac{1}{V}$. We also know that

$$(l_{i+1}, r_{i+1}) \in \{(l_i, \text{mediant}(l_i, r_i)), (\text{mediant}(l_i, r_i), r_i)\}.$$

If $\text{mediant}(l_i, r_i)$ is ever reduced, the cross-difference of l_{i+1} and r_{i+1} is reduced by the same factor (Corollary 1) meaning it is strictly less than $n + 1$. But since $x \in [l_{i+1}, r_{i+1}]$ the inductive hypothesis implies $x \in T(l_i, r_i)$ with the induced reduction scheme, which is in turn contained in $T(\frac{0}{1}, \frac{n+1}{V})$.

Thus we can assume the mediant of l_i and r_i never needs to be reduced for any i. By the linearity of addition, a simple (finite) induction gives $l_i = (n + 1)L_i$ and

$r_i = (n+1)R_i$. Since the mediant of L_{k-1} and R_{k-1} is $\frac{x}{n+1}$ by hypothesis, and because the mediant of l_{k-1} and r_{k-1} does not have to be reduced, the mediant fraction formed from l_{k-1} and r_{k-1} must be $(n+1)(\frac{x}{n+1}) = x$, contradicting the fact that x does not appear in $T(\frac{0}{1}, \frac{n+1}{V})$.

It follows by induction that every rational number in $[\frac{a}{b}, \frac{c}{d}]$ appears in $T(\frac{a}{b}, \frac{c}{d})$, regardless of the choice of a, b, c, d and independently of the reduction scheme. $\square$

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