SUM OF TWO REPDIGITS A SQUARE

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Abstract

A *repdigit* is a natural number greater than 10 which has all of its base-10 digits the same. In this paper we find all examples of two repdigits adding to a square. The proofs lead to interesting questions about consecutive quadratic residues and non-residues, and provide an elementary application of elliptic curves.

1. Preliminaries and Notation

The fictional trilogy *Lord of the Rings* by J. R. R. Tolkein [4] opens with the joint birthday party of the "hobbits," Bilbo and Frodo. The head table seats 144 which, we are told, is the sum of "eleventy-one" and thirty-three, the respective ages of the two hobbits. A *repdigit* is a positive integer all of whose (base-10) digits are the same. Prof. Tolkein has provided us with an example of two repdigits adding to a perfect square: $111 + 33 = 144$. In this paper we find all examples of two repdigits summing to a square.

While The Online Encyclopedia of Integer Sequences at https://oeis.org (A010785) allows 1-digit numbers to be repdigits, in this paper we will not (since nothing is *rep*eated.) In this way, we avoid examples like $2 + 2 = 4$ and $7 + 9 = 16$. Also, if we add a repdigit consisting of an even number of 9's to 1 (a 1-digit repdigit) the result is an even power of 10, which is a perfect square. We prefer to avoid these cases and assume all repdigits are greater than 10.

For the sake of definiteness, if a is a decimal digit, the repdigit $aa \dots a$, consisting of m a's, will be denoted a_m. In this notation, the example above reads $1_3 + 3_2 = 12^2$. A little algebra yields the formula:

$$a_m = aa\dots a = a(11\dots 1) = a\left(\frac{10^m-1}{9}\right).$$

The main result of the present paper is a complete classification of perfect squares that are the sum of two repdigits.

Theorem 1. *The only perfect squares that are sums of two repdigits are the following:*

$$\begin{aligned} 11^2 &= 99+22 = 88+33 = 77+44 = 66+55 \\ 12^2 &= 111+33 \\ 38^2 &= 1111+333 \\ 211^2 &= 44444+77. \end{aligned}$$

In Sections 2 and 3 we use congruence conditions to show that if $a_m + b_n$ is a perfect square then one of m or n is less than 6 and eliminate as many cases as possible. In Section 4, we handle the remaining cases by using results about integer points on elliptic curves and conclude the proof of Theorem 1. In Section 5 we raise some further questions.

2. A Couple of Pleasant Properties of Base-10

Of course, once our result is proven in base-10, we would want to generalize to other bases. However, $10-1$ is a perfect square, a property of 10 not true of most bases, so the problem may be much harder in non-decimal bases. Another happy accident is that there is a useful string of consecutive quadratic non-residues modulo 10^6, which is not to be expected using other bases. Both of these properties are used in the proof of our first theorem, where we show that in the case of two repdigits adding to a square, at least one of the repdigits has fewer than 6 digits:

Theorem 2. *Let a and b be non-zero base-10 digits. If $a_m + b_n$ is a perfect square, with $m \geq n \geq 2$, then $n < 6$.*

Proof. Suppose that $a_m + b_n = k^2$ for some integer k, and that m and n are both at least 6. Then using our formula above, we have

$$a\left(\frac{10^m-1}{9}\right) + b\left(\frac{10^n-1}{9}\right) = k^2$$

which simplifies to

$$a10^m + b10^n - (a+b) = (3k)^2. \tag{1}$$

Since m and $n \geq 6$, we can reduce this last equation modulo 10^6 to get

$$-(a+b) \equiv (3k)^2 \pmod{10^6}.$$

Because a and b are non-zero base-10 digits, $-(a+b)$ must be an integer between -2 and -18, inclusive. The congruence above insists that $-(a+b)$ is a quadratic residue modulo 10^6. But inspection of Table A in the Appendix shows that this is impossible. That is, the last column in the table consists of all X's, which means that none of the numbers $-2, -3, -4, \ldots, -18$ is a quadratic residue modulo 10^6. We conclude that $n < 6$. □

3. Congruence Considerations

The rest of this paper is devoted to showing that m must also be less than 6. So we now assume that $m \geq 6$ and start eliminating cases. Our definition of *repdigit* implies that m and n are both at least 2, so reducing Equation (1) modulo 10^2 gives us

$$-(a+b) \equiv (3k)^2 \pmod{10^2}.$$

Inspection of the second column of Table A shows us that $-(a+b) \equiv -4, -11$, or $-16 \pmod{10^2}$, which divides our work into three cases.

`Case 1:` $(a+b) = 4$. In this case the only possibilities are $(a, b) = (3, 1), (2, 2),$ or $(1, 3)$. Also note that, from Table A, -4 is a quadratic non-residue modulo 10^4, so we must have $n < 4$ in this case. (Else, we reduce Equation (1) modulo 10^4 and get a contradiction.) Thus, the only possibilities are:

$$3_m + 11, 2_m + 22, 1_m + 33, 3_m + 111, 2_m + 222, \text{ and } 1_m + 333.$$

Four of these six subcases can be eliminated by congruence considerations. If we put the case $3_m + 11$ into Equation (1) and reduce modulo 10^6, we have

$$3 \cdot 10^m - 3 + 10^2 - 1 \equiv 96 \equiv (3k)^2 \pmod{10^6}.$$

A solution to this congruence implies $96 + d10^6 = x^2$ for some integers d and x. Factoring 2^5 from the left side gives us $2^5(3 + 2d \cdot 5^6) = x^2$. The quantity in parentheses is odd, and so x^2 must be exactly divisible by 2^5, which is impossible. In other words, 96 is not a quadratic residue of 10^6, so there is no solution. In the future, we will just use Maple to compute whether a number is a quadratic residue.

With a similar calculation, the three subcases $1_m + 33$, $2_m + 222$ and $1_m + 333$ require 296, 1996 and 2996, respectively, to be quadratic residues modulo 10^6. Maple outputs that all three of them are quadratic nonresidues, so we have eliminated these three subcases. We will handle the remaining cases $2_m + 22$ and $3_m + 111$ later.

`Case 2:` $(a+b) = 11$. In this case the possibilities are $(a, b) = (2, 9)$, $(3, 8)$, $(4, 7)$, $(5, 6)$, $(6, 5)$, $(7, 4)$, $(8, 3)$, or $(9, 2)$. Note that if $n \geq 3$, then modulo 10^3, Equation (1) reduces to $-11 \equiv (3k)^2 \pmod{10^3}$, and that -11 is a quadratic nonresidue of 10^3. Therefore $n = 2$ is the only possibility. This leaves us with 8 subcases: $2_m + 99$, $3_m + 88$, $4_m + 77$, $5_m + 66$, $6_m + 55$, $7_m + 44$, $8_m + 33$, and $9_m + 22$.

As we did in Case 1, we put each of these subcases in Equation (1) and reduce modulo 10^6. Respectively, these congruences require 889, 789, 689, 589, 489, 389, 289 and 189 to be quadratic residues modulo 10^6. Maple outputs that 789, 589, 389 and 189 are quadratic non-residues module 10^6, so we have eliminated the subcases $3_m + 88$, $5_m + 66$, $7_m + 44$ and $9_m + 22$. The other four subcases we save for Section 4.

`Case 3:` $(a+b) = 16$. In this case the possibilities are $(a, b) = (7, 9)$, $(8, 8)$ or $(9, 7)$. The $(8, 8)$ subcase would have $8_m + 8_n = k^2$, in which case k would be even and we could divide the equation by 4 to get $2_m + 2_n = (k/2)^2$, which is one of our leftover subcases from Case 1. When $2_m + 22$ is eliminated in Section 4, it will take this subcase with it.

From Theorem 1, $n \leq 5$, so the possibilites are $9_m + 77$, $9_m + 777$, $9_m + 7777$, $9_m + 77777$, $7_m + 99$, $7_m + 999$, $7_m + 9999$, and $7_m + 99999$. We again put each subcase into Equation (1) and reduce modulo 10^6. Respectively, $700-16$, $7000-16$, $70000-16$, $700000-16$, $900-16$, $9000-16$, $90000-16$ and $900000-16$ are required to be quadratic residues modulo 10^6. But only $700000-16$, $90000-16$ and $900000-16$ are. So we are left with only $9_m + 77777$, $7_m + 9999$, and $7_m + 99999$ as possibilities in this case.

It turns out that $90000 - 16$ and $700000 - 16$ are not quadratic residues modulo 7, so we can also eliminate $9_m + 77777$ and $7_m + 9999$ from the list.

4. Elliptic Curve Considerations

We have eliminated all cases except this short list: $2_m + 22$, $3_m + 111$, $2_m + 99$, $4_m + 77$, $6_m + 55$, $8_m + 33$ and $7_m + 99999$. Each of these cases will be broken into three subcases depending on the character of m modulo 3. In most cases, this will eliminate the case. In the rest, we will discover the known solutions to the problem. We will rely on SAGE to find all the integer points on the elliptic curves which arise.

We start with $8_m + 33$. Suppose $m = 3l$ for some integer l. Then Equation (1) becomes $8 \cdot 10^{3l} + 300 - (8 + 3) = (3k)^2$. If we set $y = 3k$ and $x = 2 \cdot 10^l$, the equation is $y^2 = x^3 + 289$. According to Siegel's famous theorem [3], this curve can have only finitely many integral points. SAGE 6.7 implements an algorithm due to Peth, Zimmer, Gebel, and Herrmann [2], which will find all integral points on an elliptic curve. SAGE outputs that the only integer points on this curve are $(-4, 15)$, $(0, 17)$ and $(68, 561)$. We can observe that none of the x-values are of the form $2 \cdot 10^l$, which eliminates this possibility.

Next suppose that $m = 3l+1$ for some integer l. Equation (1), for this case, reads $8 \cdot 10^{3l+1} + 289 = (3k)^2$. Multiply through by 100 to get $8 \cdot 10^{3l+3} + 28900 = (30k)^2$. Set $x = 2 \cdot 10^{l+1}$ and $y = 30k$ and we have $y^2 = x^3 + 28900$. SAGE outputs that the only integer point on this curve is $(0, 170)$ and since $0 \neq 2 \cdot 10^{l+1}$, we have eliminated this case.

Next suppose that $m = 3l + 2$ for some integer l. Equation (1), for this case reads $8 \cdot 10^{3l+2} + 289 = (3k)^2$. Multiply through by 10^4 to get $8 \cdot 10^{3l+6} + 289 \cdot 100^2 = (300k)^2$. Set $y = 300k$ and $x = 2 \cdot 10^{l+2}$ and we have $y^2 = x^3 + 2890000$. SAGE outputs that the only integer points on this curve are $(-136, 612)$, $(0, 1700)$, $(200, 3300)$ and $(425, 8925)$. Looking at the x-values, we see one that fits: $200 = 2 \cdot 10^{l+2}$. So $l = 0$ and $m = 2$ and we are led to the solution $88 + 33 = 121$.

The other cases yield similarly. If $m = 3l + r$, with $r = 0, 1$, or 2, then we multiply Equation (1) by $a^2 10^r$ to get

$$a^3 10^{3(l+r)} + a^2 10^r (b10^n - (a + b)) = (3a \cdot 10^r k)^2.$$

Then we set $x = a10^{l+r}, y = 3a \cdot 10^r k$ and $N = a^2 10^r (b10^n - (a + b))$ to get the elliptic curve

$$y^2 = x^3 + N.$$

Then we use SAGE to compute all the integer points on the curve. If any of those points have x-coordinate of the form $a \cdot 10^p$, then we might have a solution (and in each case, it will turn out that $m < 6$.) If not, the case is eliminated. In the case $a = 8$, which is already a cube, it is sufficient to multiply Equation (1) by 10^r, as we did above. Table B in the Appendix summarizes the calculations, where r is the least non-negative residue of m modulo 3. Whenever the x-coordinate of an integer point matches the form $a10^{l+r}$, we put that number in bold in the last column of the table. In each such case, $l = 0$ or 1, which means that $m < 6$.

Proof of Theorem 1. Therefore every case of the sum of two repdigits equaling a square must have repdigits of 5 or fewer digits. There are only 45 repdigits of 2, 3, 4 or 5 digits, so that leaves us $45 \cdot 46/2 = 1035$ cases to check. This is easily done and we find the complete list of solutions as in the statement. □

5. Further Observations and Questions

The proof of Theorem 2 relies entirely on the fact that there is a convenient string of consecutive quadratic non-residues modulo 10^6 (due mostly to the fact that 10 is even.) Finding strings of consecutive quadratic residues and non-residues has been studied. (See [1] and [5].) Certainly base-10 is special, but there are some observations in other bases:

We noted above that there is a family of solutions $22+99 = 33+88 = 44+77 = 55+66 = 11^2$ in base 10. In an arbitrary base $c \geq 2$, we have $(c+1)^2 = (c-k)(c+1) + (k+1)(c+1)$ for $1 \leq k \leq c-1$. This shows that there is a similar family regardless of the choice of base.

Similarly, $(c+2)^2 = c^2+4c+4 = 1(c^2+c+1)+3(c+1)$ shows that $111+33 =$ is a square in every base. Also $(2c^2+c+1)^2 = 4c^4+4c^3+5c^2+2c+1 = 4(c^4+c^3+c^2+c+1)+(c-3)(c+1)$, so $44444+77$ is just the base 10 manifestation of this identiy.

In the more special case that $c = m^2-1$, we have the identity $c^5+c^4+4c^3+4c^2+4c+4 = (c+1)(c^2+2)^2 = m^2(c^2+2)^2$, which is to say $(111111)_c+(3333)_c$ is a square.

The most interesting solutions to our equation seem to come in groups. It seems $111+33 = 12^2$ and $1111+333 = 38^2$ are related, as are $4+77 = 9^2$, $44+77 = 11^2$ and $44444+77 = 221^2$. We also have the infinite number of solutions $9_{2m}+1 = (10^m)^2$, if we allow 1-digit solutions. As far as we know, this is accidental, but perhaps there is a useful reason behind this. If so, we wonder whether there is a solution to the problem that does not rely on SAGE computing on elliptic curves.

As mentioned earlier, the case of base 10 is easier than some others. Of particular note is base 7, where there are a number of examples, the most striking of which is

$$48060^2 = 2309763600 = (11111111111)_7 + (3333333)_7.$$

It seems difficult to use modular arithmetic to put a bound on m and n subject to

$$a \cdot \frac{7^m-1}{6} + b \cdot \frac{7^n-1}{6} = x^2.$$

The solutions to this equation are a subset of those to

$$Au^5 + Bv^5 + Cx^5 = 6w^2x^3.$$

This defines a (possibly singular) surface of general type, and it is conjectured by Bombieri and Lang that the *rational* points on a surface of general type lie on a finite union of curves. This supports our suspicion that there should only be finitely many solutions, but a proof eludes us.

References

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Appendix: Tables

Table A: Quadratic residues modulo 10^n. In the following table, generated with Maple, the entry is O if $-(a+b)$ is a quadratic residue modulo 10^k and X otherwise.

$-(a+b)$	10^2	10^3	10^4	10^5	10^6
-2	X	X	X	X	X
-3	X	X	X	X	X
-4	O	O	X	X	X
-5	X	X	X	X	X
-6	X	X	X	X	X
-7	X	X	X	X	X
-8	X	X	X	X	X
-9	X	X	X	X	X
-10	X	X	X	X	X
-11	O	X	X	X	X
-12	X	X	X	X	X
-13	X	X	X	X	X
-14	X	X	X	X	X
-15	X	X	X	X	X
-16	O	O	O	O	X
-17	X	X	X	X	X
-18	X	X	X	X	X

Table B: Summary of Calculations

Case	r	x	y	N	x-coords
$8_m + 33$	0	$2 \cdot 10^l$	$3k$	289	$-4, 0, 68$
$8_m + 33$	1	$2 \cdot 10^{l+1}$	$30k$	28900	0
$8_m + 33$	2	$2 \cdot 10^{l+2}$	$300k$	2890000	$-136, 0, \mathbf{200}, 425$
$7_m + 99999$	0	$7 \cdot 10^l$	$21k$	44099216	10577
$7_m + 99999$	1	$7 \cdot 10^{l+1}$	$210k$	4409921600	-1064
$7_m + 99999$	2	$7 \cdot 10^{l+2}$	$2100k$	440992160000	$-5936, -5900, -5516, 2800,$ $20825, 21056, 721364000$
$6_m + 55$	0	$6 \cdot 10^l$	$18k$	17604	$-20, -12, 816$
$6_m + 55$	1	$6 \cdot 10^{l+1}$	$180k$	1760400	$-120, 24, 160, 14640$
$6_m + 55$	2	$6 \cdot 10^{l+2}$	$1800k$	176040000	$\mathbf{600}$
$4_m + 77$	0	$4 \cdot 10^l$	$12k$	11024	1
$4_m + 77$	1	$4 \cdot 10^{l+1}$	$120k$	1102400	$-100, -95, -16, \mathbf{40}, 160,$ $584, 1420, 26764$
$4_m + 77$	2	$4 \cdot 10^{l+2}$	$1200k$	110240000	$-464, -400, 64, \mathbf{400},$ $425, 625, 1076,$ $\mathbf{4000}, 1154800$
$3_m + 111$	0	$3 \cdot 10^l$	$9k$	8964	21
$3_m + 111$	1	$3 \cdot 10^{l+1}$	$90k$	896400	$-96, -80, -15, 25,$ $40, 49, 120, 256,$ $280, 1200, 16576$
$3_m + 111$	2	$3 \cdot 10^{l+2}$	$900k$	89640000	$-375, 124, \mathbf{300},$ $700, 5241$
$2_m + 99$	0	$2 \cdot 10^l$	$6k$	3556	none
$2_m + 99$	1	$2 \cdot 10^{l+1}$	$60k$	355600	$-55, -40, 144, 4320$
$2_m + 99$	2	$2 \cdot 10^{l+2}$	$600k$	35560000	$-216, \mathbf{200}, 704, 800,$ $1500, 1800, 7400$ 199200
$2_m + 22$	0	$2 \cdot 10^l$	$6k$	784	$-7, 0, 8, 56$
$2_m + 22$	1	$2 \cdot 10^{l+1}$	$60k$	78400	$-40, 0, 56, 140, 480$
$2_m + 22$	2	$2 \cdot 10^{l+2}$	$600k$	7840000	$-175, 0, 224, 800$