2-ADIC VALUATIONS OF GENERALIZED FIBONACCI NUMBERS OF ODD ORDER

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Abstract

Let T_n denote the generalized Fibonacci number of order k defined by the recurrence $T_n = T_{n-1} + T_{n-2} + \cdots + T_{n-k}$ for $n \geq k$, with initial conditions $T_0 = 0$ and $T_i = 1$ for $1 \leq i < k$. In this paper we establish the 2-adic valuation of T_n in almost all cases when k is odd. Our results settle some conjectures of Lengyel and Marques.

1. Introduction

Let T_n denote the generalized Fibonacci number of order k defined by the recurrence $T_n = T_{n-1} + T_{n-2} + \cdots + T_{n-k}$ for $n \geqslant k$, with initial conditions $T_0 = 0$ and $T_i = 1$ for $1 \leqslant i < k$. When $k = 2$ this is the usual Fibonacci sequence, whereas for $k = 5$ we have the sequence

$$0, 1, 1, 1, 1, 4, 8, 15, 29, 57, 113, 222, 436, 857, 1685, 3313, 6513, \ldots. \tag{1.1}$$

The 2-adic valuation $\nu_2(T_n)$ has been a topic of recent interest, having been determined when $k = 3$ [4], $k = 4$ [3], $k = 5$ in almost all cases [3], and k even [7]. Motivated by the formulas and conjectures in [3], in the present article we focus primarily on the case where $k \geqslant 5$ is odd, and answer those conjectures (one affirmatively, one negatively) by considering 2-adic analytic functions which interpolate subsequences of (T_n) in residue classes modulo $2k + 2$. The main result is the following:

Theorem 1. *If $k \geqslant 5$ is odd, then for all integers n we have*

$$\nu_2(T_n) = \begin{cases} 0, & \text{if } n \not\equiv 0, k \pmod{k+1}, \\ \nu_2(k-1), & \text{if } n \equiv k \pmod{2k+2}, \\ \nu_2(k-3), & \text{if } n \equiv -1 \pmod{2k+2}, \\ \nu_2(n-k-1), & \text{if } n \equiv k+1 \pmod{2k+2} \quad \text{and} \\ & \quad \nu_2(n-k-1) < \nu_2(k^2-1), \\ \nu_2(n-2)+1, & \text{if } n \equiv k+1 \pmod{2k+2} \quad \text{and} \\ & \quad \nu_2(n-k-1) > \nu_2(k^2-1), \\ \nu_2(n) - \nu_2(k+1) + 1, & \text{if } n \equiv 0 \pmod{2k+2}, \end{cases}$$

We remark that in the case $k = 5$, the above Theorem 1 is equivalent to Theorem 2 of [3]. The above theorem also implies the odd k case of Conjecture 2 of [3], which hypothesizes that for integers r, k, s with $r \geqslant 1$, $k \geqslant 2$, and s odd, the 2-adic valuation of the subsequence $(T_{s(k+1)2^r})$ has the form

$$\nu_2(T_{s(k+1)2^r}) = r + c(k) \tag{1.2}$$

where $c(2) = 2$, and otherwise $c(k) = \nu_2(k-2) + 1$. The even k case of this conjectured formula (1.2) was recently proved by Sobolewski [7], using a different method than the present paper.

For odd $k \geqslant 5$, Theorem 1 gives the exact valuation $\nu_2(T_n)$ in all cases except when $n \equiv k+1 \pmod{2k+2}$ and $\nu_2(n-k-1) = \nu_2(k^2-1)$. In the case $k = 5$, Lengyel and Marques ([3], Conjecture 1) conjectured a formula for $\nu_2(T_n)$ when $n = 12m + 6$ and $\nu_2(n-6) = 3$. Although their formula is correct for positive integers n less than three million, we will show in the last section that it is not correct in general. However, the conjectured formula is correct in spirit; in fact, we have the following:

Theorem 2. *Suppose $k \geqslant 5$ is odd and let $a = \nu_2(k-1)$. Then there exists a 2-adic integer $z \in \mathbb{Z}_2$ with $\nu_2(z) = a - 1$ and*

$$z \equiv \frac{k-1}{4-2k} \pmod{2^{3a-1}\mathbb{Z}_2}$$

such that $\nu_2(T_n) = \nu_2(m-z) + 2$ when n is of the form $n = (2k+2)m + k + 1$.

When n is not a multiple of $k+1$ the above Theorem 1 can be established by simplifying the recurrence for (T_n), as we show in the next section. To handle the cases where n is a multiple of $k+1$, we will rely on the following theorem which may be proved using elementary 2-adic analysis.

Theorem 3. *Write $k+1 = 2^e l$ with l odd. Then for each $j \in \mathbb{Z}$ there exists a continuous function $f_j : \mathbb{Z}_2 \to \mathbb{Z}_2$ such that $f_j(n) = T_{ln+j}$ for all $n \in \mathbb{Z}$. Furthermore, for each $j \in \mathbb{Z}$ there exists a function g_j which is analytic on $D = \{x \in \mathbb{C}_2 : \nu_2(x) > -1\}$ such that $g_j(n) = T_{2(k+1)n+j}$ for all $n \in \mathbb{Z}$.*

The continuous functions f_j described above will not play a computational role in the present paper, but they do illustrate an interesting property of the sequences (T_n); for example, when $k = 7$ the sequence (T_n) extends to a continuous function of n on $\mathbb{Z}_2$, but for $k = 5$ it does not. The analytic functions g_j will be of much greater use in establishing the valuations $\nu_2(T_n)$.

2. Generalized Fibonacci Numbers

The characteristic polynomial of the recurrence for (T_n) is

$$p(x) = x^k - x^{k-1} - x^{k-2} - \cdots - x - 1 = \frac{x^{k+1} - 2x^k + 1}{x-1} = \frac{q(x)}{x-1}. \tag{2.1}$$

Therefore the order k recurrence

$$T_n = T_{n-1} + \cdots + T_{n-k} \tag{2.2}$$

is equivalent to the order $k+1$ recurrence

$$T_{n+1} = 2T_n - T_{n-k}. \tag{2.3}$$

It is then easily seen that (T_n) is periodic modulo 2 with period $k+1$. Moreover, considering the initial conditions, for even k we have T_n even if and only if $n \equiv 0 \pmod{k+1}$, whereas for odd k we have T_n even if and only if $n \equiv 0, -1 \pmod{k+1}$. From this recurrence one can easily compute T_n for n near zero, giving

$$\begin{aligned}
T_{-2k-2} &= 4k-8 \quad (\text{if} \quad k \geqslant 3)\\
T_{-2k-1} &= 13-4k\\
T_{-2k} &= -3\\
T_{-k-i} &= 1 \quad \text{for} \quad 3 \leqslant i \leqslant k-1\\
T_{-k-2} &= k-1 \quad (\text{if} \quad k \geqslant 3)\\
T_{-k-1} &= 6-2k\\
T_{-k} &= -1\\
T_{-i} &= 1 \quad \text{for} \quad 2 \leqslant i \leqslant k-1\\
T_{-1} &= 3-k\\
T_0 &= 0\\
T_i &= 1 \quad \text{for} \quad 1 \leqslant i \leqslant k-1\\
T_k &= k-1\\
T_{k+i} &= 2^{i-1}(2k-3)+1 \quad \text{for} \quad 1 \leqslant i \leqslant k\\
T_{2k+1} &= 2^k(2k-3)-k+3\\
T_{2k+2} &= 2^{k+1}(2k-3)-4k+8.
\end{aligned}$$

From these initial values, it is easy to establish the following proposition by induction on r.

Proposition 1. *For all nonnegative integers r we have*

$$T_{r(k+1)+i} \equiv 1 \pmod{2^i}, \quad 1 \leqslant i \leqslant k-1,$$

$$T_{r(k+1)+k} \equiv \begin{cases} k-1, & r \text{ even}, \\ 3-k, & r \text{ odd}, \end{cases} \pmod{2^k},$$

$$T_{r(k+1)} \equiv \begin{cases} 4r-2rk, & r \text{ even}, \\ 2rk-4r+2, & r \text{ odd}. \end{cases} \pmod{2^{k+1}}.$$

Remark. Once we show that the functions $T_{(2k+2)m+j}$ are 2-adically continuous functions of m in the next section, it will follow that the above Proposition 1 is valid for negative integers r as well.

3. Construction of Interpolating Functions

Let $\mathbb{Z}_2$ denote the ring of 2-adic integers, $\mathbb{Q}_2$ the field of 2-adic numbers, and $\mathbb{C}_2$ the completion of an algebraic closure of $\mathbb{Q}_2$. The 2-adic valuation $\nu_2(n)$ of an integer n is equal to the highest power of 2 which divides n, with the convention that $\nu_2(0) = +\infty$. This valuation extends uniquely to $\mathbb{C}_2$, on which it takes rational values (for example, $\nu_2(\sqrt{6}) = 1/2$).

For a polynomial $f(x) = \sum_{i=0}^{n} a_i(x-\alpha)^i \in \mathbb{C}_2[x]$, the *Newton polygon* of f at α is the upper convex hull of the set of points $\{(i, \nu_2(a_i)) : 0 \leqslant i \leqslant n\}$. A basic property ([1], Ch. IV.3, Lemma 4; [5], Theorem 9.1) is that the Newton polygon of f at α has a side of slope m and horizontal run l if and only if f has l zeros (counted with multiplicity) $\alpha_i \in \mathbb{C}_2$ with $\nu_2(\alpha_i - \alpha) = -m$.

In order to 2-adically interpolate the sequence (T_n), we first use the theory of Newton polygons to locate the roots of $p(x)$ in $\mathbb{C}_2$. If $k+1 = 2^e l$ with l odd, then the roots are partitioned into l subsets, each of which lie close to an l-th root of unity in $\mathbb{C}_2$. If $\zeta^l = 1$ and α is a root of $p(x)$ with $\nu_2(\alpha - \zeta) > 0$, we will say α *corresponds to* ζ; this means that they have the same image in the residue class field of $\mathbb{C}_2$.

Proposition 2. *Write $k+1 = 2^e l$ with l odd. Corresponding to each nontrivial solution $\zeta \in \mathbb{C}_2$ to $\zeta^l = 1$ there are 2^e roots α of $p(x)$ which satisfy $\nu_2(\alpha - \zeta) = 2^{-e}$. Corresponding to the trivial solution $\zeta = 1$, there are $2^e - 1$ roots α of $p(x)$ which satisfy $\nu_2(\alpha - 1) > 0$. When $e = 1$, this root satisfies $\nu_2(\alpha - 1) = \nu_2(k-1)$; when $e > 1$ these $2^e - 1$ roots all satisfy $\nu_2(\alpha - 1) = (2^e - 1)^{-1}$. If α_i, α_j are two roots of $p(x)$ which correspond to the same ζ, then $\nu_2(\alpha_i - \alpha_j) = (2^e - 1)^{-1}$; otherwise $\nu_2(\alpha_i - \alpha_j) = 0$ for roots α_i, α_j of $p(x)$ corresponding to distinct solutions to $\zeta^l = 1$.*

Proof. For any $\alpha \in \mathbb{C}_2$ with $\nu_2(\alpha) = 0$ we have

$$q(x) = (\alpha + x - \alpha)^{k+1} - 2(\alpha + x - \alpha)^k + 1 = \sum_{i=0}^{k+1} r_i (x-\alpha)^i \tag{3.1}$$

where

$$r_i = \binom{k+1}{i}\alpha^{k+1-i} - 2\binom{k}{i}\alpha^{k-i} + \delta_{i,0}. \tag{3.2}$$

Since $\nu_2(\binom{n}{m})$ equals the number of carries in the binary addition $m + (n-m) = n$, we see that $i = 2^e$ is the least positive index for which $\nu_2(r_i) = 0$. We also have $r_1 = (k+1)\alpha^k - 2k\alpha^{k-1} = \alpha^{k-1}((k+1)\alpha - 2k)$. If $e = 0$ we thus have $\nu_2(r_1) = 0$. If $e > 1$ we have $\nu_2(r_1) = 1$, and if $e = 1$ we have $\nu_2(r_1) = 1$ unless α corresponds to 1.

First take $\alpha = \zeta$ in (3.1), (3.2), where $\zeta^l = 1$. In this case $r_0 = q(\zeta) = 2(1-\zeta^{-1})$ has positive 2-adic valuation; this valuation is 1 unless $\zeta = 1$, in which case it is $+\infty$. Therefore the vertices of the Newton polygon of q at ζ are $(0,1)$, $(2^e,0)$, and $(k+1,0)$ when $\zeta \neq 1$. This establishes $\nu_2(\alpha - \zeta) = 2^{-e}$ for each of the 2^e roots α of $p(x)$ corresponding to each nontrivial l-th root of unity ζ. At $\zeta = 1$ the vertices are $(0,+\infty)$, $(1,1)$, $(2^e,0)$, and $(k+1,0)$ when $e > 1$, and $(0,+\infty)$, $(1,\nu_2(k-1))$, $(2,0)$, and $(k+1,0)$ when $e = 1$. This establishes the stated valuations $\nu_2(\alpha - 1)$ for the roots of $p(x)$ corresponding to 1. (Recall that 1 is a root of q but not of p.)

Now assume $e > 0$, let α_i be any root of $p(x)$, and assume that $e > 1$ if α_i corresponds to 1. Under these assumptions, from (3.1), (3.2), the vertices of the Newton polygon of q at α_i are $(0,+\infty)$, $(1,1)$, $(2^e,0)$, and $(k+1,0)$. This shows that each root α_i of $p(x)$ has either 2^e-2 or 2^e-1 other roots α_j of $p(x)$ (according to whether α_i corresponds to 1) which satisfy $\nu_2(\alpha_i - \alpha_j) = (2^e-1)^{-1}$. This completes the proof. □

Having determined the location of the roots of the characteristic polynomial $p(x)$ in $\mathbb{C}_2$, we may now give the proof of Theorem 3. The required functions are constructed as linear combinations of power functions of the form

$$(1+z)^x := \sum_{m=0}^{\infty} \binom{x}{m} z^m \tag{3.3}$$

which are continuous functions of $x \in \mathbb{Z}_2$ when $\nu_2(z) > 0$ ([6], Theorem 51.1) and analytic functions of $x \in \mathbb{Z}_2$ when $\nu_2(z) > 1$ ([6], Theorem 54.4).

Proof of Theorem 3. According to Proposition 2, the roots of the characteristic polynomial $p(x)$ are distinct, so the sequence T_n may be expressed in Binet form $T_n = \sum_{i=1}^k c_i\alpha_i^n$, where $\alpha_1, ..., \alpha_k$ are the roots of $p(x)$ in $\mathbb{C}_2$ and $c_i \in \mathbb{C}_2$. Moreover, each root α_i may be expressed in the form $\alpha_i = \zeta_i(1+\varepsilon_i)$, where $\zeta_i^l = 1$ and $\nu_2(\varepsilon_i) \geqslant 2^{-e}$. Given $j \in \mathbb{Z}$, define the function $f_j : \mathbb{Z}_2 \to \mathbb{Z}_2$ by

$$f_j(x) := \sum_{i=1}^{k} c_i\alpha_i^j(1+\gamma_i)^x = \sum_{i=1}^{k} c_i\alpha_i^j \sum_{m=0}^{\infty} \binom{x}{m}\gamma_i^m \tag{3.4}$$

where $(1+\varepsilon_i)^l = 1+\gamma_i$ with $\nu_2(\gamma_i) \geqslant 2^{-e}$. This gives an expansion of $f_j(x)$ in terms of the basis $\{\binom{x}{m}\}$ with coefficients $\sum_i c_i\alpha_i^j\gamma_i^m$ tending to 0 as $m \to \infty$. By Mahler's Theorem ([6], Theorem 51.1), f_j is a continuous function on $\mathbb{Z}_2$. For any integer n we have

$$\begin{aligned} T_{ln+j} &= \sum_{i=1}^{k} c_i\alpha_i^{ln+j} = \sum_{i=1}^{k} c_i\alpha_i^j(\zeta_i(1+\varepsilon_i))^{ln} \\ &= \sum_{i=1}^{k} c_i\alpha_i^j(1+\gamma_i)^n = f_j(n). \end{aligned} \tag{3.5}$$

Since f_j is continuous on $\mathbb{Z}_2$ and maps $\mathbb{Z}$ to $\mathbb{Z}$, it maps $\mathbb{Z}_2$ to $\mathbb{Z}_2$. Therefore the existence of the required continuous functions f_j has been established.

We now construct the analytic functions g_j, paying attention to the coefficients. If $\nu_2(\varepsilon) = r \in (0,1]$, then $(1+\varepsilon)^{2l} = 1+\eta$ with $\nu_2(\eta) \geqslant 2r$. By induction it follows that if $\nu_2(\varepsilon_i) \geqslant 2^{-e}$, then $(1+\varepsilon_i)^{2(k+1)} = 1+\eta_i$ with $\nu_2(\eta_i) \geqslant 2$. Since $\nu_2(\eta_i) \geqslant 2$, we have

$$\log_2(1+\eta_i) := \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m}\eta_i^m = \lambda_i \tag{3.6}$$

with $\nu_2(\lambda_i) = \nu_2(\eta_i) \geqslant 2$. Since $\nu_2(\lambda_i) \geqslant 2$ we have

$$\exp_2(x\lambda_i) := \sum_{m=0}^{\infty} \frac{\lambda_i^m}{m!}x^m = \sum_{m=0}^{\infty} b_{i,m}x^m \tag{3.7}$$

with $\nu_2(b_{i,m}) \geqslant 2m - \nu_2(m!) = m + S_2(m)$, where $S_2(m)$ denotes the binary digit sum of m. Using the fact that

$$(1+\eta_i)^x = \exp_2(x\log_2(1+\eta_i)) \tag{3.8}$$

when $x \in \mathbb{Z}_2$ and $\nu_2(\eta_i) > 1$ ([6], Theorem 47.10), we define

$$\begin{aligned} g_j(x) &:= \sum_{i=1}^{k} c_i\alpha_i^j(1+\eta_i)^x \\ &= \sum_{i=1}^{k} c_i\alpha_i^j\left(\sum_{m=0}^{\infty} b_{i,m}x^m\right) = \sum_{m=0}^{\infty} a_m x^m \end{aligned} \tag{3.9}$$

with coefficients $a_m = \sum_i c_i\alpha_i^j b_{i,m}$. Since $\nu_2(b_{i,m}) \geqslant m$, the series (3.9) converges on $D = \{x \in \mathbb{C}_2 : \nu_2(x) > -1\}$ to the function g_j, which is therefore analytic on this disc. Finally, for any integer n we compute

$$\begin{aligned} T_{2(k+1)n+j} &= \sum_{i=1}^{k} c_i \alpha_i^{2(k+1)n+j} = \sum_{i=1}^{k} c_i \alpha_i^{j} (\zeta_i (1+\varepsilon_i))^{2(k+1)n} \\ &= \sum_{i=1}^{k} c_i \alpha_i^{j} (1+\eta_i)^n = g_j(n). \end{aligned} \tag{3.10}$$

Thus the existence of the required analytic functions g_j has been established.

□

Remark. Although the functions $f_j(x)$ and $g_j(x)$ may be evaluated at rational arguments $x \in \mathbb{Z}_2$, we caution that the values obtained do not correspond to values of T_n when $x \notin \mathbb{Z}$. For example, when $k = 5$ the function $g_0(x)$ interpolates the values $\{T_{12x}\}$ when $x \in \mathbb{Z}$ and converges at $x = 1/3 \in \mathbb{Z}_2$, but $g_0(1/3)$ does not equal T_4. We will see in the next section that $\nu_2(g_0(1/3)) = 2$, while of course $T_4 = 1$. The reason for this is that $(\alpha^n)^{1/n}$ does not equal α in general.

Corollary 1. *The sequence (T_n) may be extended to a continuous function of $n \in \mathbb{Z}_2$ if and only if k is of the form $k = 2^e - 1$.*

Proof. If $k = 2^e - 1$ then $l = 1$ and the function f_0 constructed above provides the required extension. If k is not of the form $2^e - 1$, then $k + 1$ has an odd prime factor, so for any positive integer n, $2^n \not\equiv 0 \pmod{k+1}$ if k is even, and $2^n \not\equiv 0, -1 \pmod{k+1}$ if k is odd. So the sequence (2^n) converges to 0 in $\mathbb{Z}_2$, but the sequence (T_{2^n}) consists only of odd integers, and therefore cannot converge to $T_0 = 0$ in $\mathbb{Z}_2$. Therefore (T_n) cannot be extended to a continuous function on $\mathbb{Z}_2$. □

4. Coefficients of the Analytic Functions $g_j(x)$

Proposition 3. *The sequence T_n may be expressed in Binet form $T_n = \sum_{i=1}^{k} c_i \alpha_i^n$, where $\alpha_1, ..., \alpha_k$ are the roots of $p(x)$ in $\mathbb{C}_2$ and $c_i \in \mathbb{C}_2$. If k is even, then $\nu_2(c_i) \geqslant 0$ for all i, and if k is odd then $\nu_2(c_i) \geqslant -1$ for all i.*

Proof. Let $e = \nu_2(k + 1)$ and $a = \nu_2(k - 1)$. The initial conditions on T_n for $n = 0, 1, ..., k - 1$ determine the constants c_i according to the equation $AC = T$, where $C = [c_1\ c_2 \cdots c_k]^T$, $T = [0\ 1 \cdots 1]^T$, and

$$A = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \alpha_1 & \alpha_2 & \cdots & \alpha_k \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_1^{k-1} & \alpha_2^{k-1} & \cdots & \alpha_k^{k-1} \end{bmatrix} = V_k(\alpha_1, \alpha_2, ..., \alpha_k) \tag{4.1}$$

is the $k \times k$ Vandermonde matrix with parameters $\alpha_1, \alpha_2, ..., \alpha_k$. By Cramer's Rule, the solution to this system is given by $c_i = \det(A_i)/\det(A)$, where A_i is the matrix obtained by replacing the i-th column of A with T. We have

$$\det(A) = \prod_{1 \leqslant i < j \leqslant k} (\alpha_i - \alpha_j), \tag{4.2}$$

and therefore by Proposition 2 on the distances between the roots we have $\nu_2(\det(A)) = 0$ if $e = 0$. Since $C = A^{-1}T$ the theorem is therefore proved in that case. If $e > 0$ we use Proposition 2 and (4.2) to compute

$$\begin{aligned} \nu_2(\det(A)) &= (l-1)\binom{2^e}{2}(2^e-1)^{-1} + \binom{2^e-1}{2}(2^e-1)^{-1} \\ &= (l-1)2^{e-1} + (2^{e-1}-1) = (k-1)/2. \end{aligned} \tag{4.3}$$

The matrix A_i is formed by replacing the i-th column of A with $T = [0\ 1 \cdots 1]^T$. We then calculate $\det(A_i)$ by cofactor expansion along the first row. This expresses $\det(A_i)$ as a sum of $k-1$ nonzero $(k-1) \times (k-1)$ cofactors of the form

$$\pm \frac{\alpha_1 \cdots \alpha_k}{\alpha_i \alpha_j} V_{k-1}(\alpha_1, \cdots, 1_i, \cdots, \hat{\alpha}_j, \cdots, \alpha_k) \tag{4.4}$$

where the symbol $\hat{\alpha}_j$ means that α_j is omitted from the parameter list, and 1_i means that α_i is replaced with 1. We think of each of these V_{k-1} Vandermonde matrices in (4.4) as being obtained from the V_k matrix (4.1) by inserting 1 among the parameters to obtain $V_{k+1}(\alpha_1, ..., \alpha_k, 1)$ (up to permutation of columns), and then removing two parameters α_i and α_j. Suppose that $e > 1$. Then from Proposition 2 and (4.2) we see that including 1 increases the valuation of the determinant by $(2^e-1)(2^e-1)^{-1} = 1$, while removing each of α_i and α_j decreases the valuation of the determinant by at most 1. Since $\det(A_i)$ is a sum of unit multiples of $k-1$ such cofactors, we have $\nu_2(\det(A_i)) \geqslant \nu_2(\det(A))+1-2$, which implies that $\nu_2(c_i) \geqslant -1$. In the case $e = 1$, including 1 increases the valuation by a, and removing α_i and α_j decreases the valuation by at most $a+1$, so $\nu_2(\det(A_i)) \geqslant \nu_2(\det(A)) - 1$, which implies that $\nu_2(c_i) \geqslant -1$ in that case as well. □

We now consider in detail the coefficients a_m of the analytic functions $g_j(x) = \sum_m a_m x^m$. From (3.9) and Proposition 3 we see that *a priori*

$$\nu_2(a_m) \geqslant \begin{cases} m + S_2(m) - 1, & k \text{ odd}, \\ m + S_2(m), & k \text{ even}. \end{cases} \tag{4.5}$$

We will primarily focus on the case where k is odd, since the even k case is similar. It is immediate that $T_j = g_j(0) = a_0$. In general one may approximate the coefficients a_m by computing $g_j(n)$ for several integers n and solving a system of linear equations. For example, for any exponent r, considering

$$g_j(2^r) - g_j(-2^r) = 2^{r+1}a_1 + 2^{3r+1}a_3 + 2^{5r+1}a_5 + \cdots \tag{4.6}$$

leads to the determination

$$a_1 \equiv \frac{g_j(2^r) - g_j(-2^r)}{2^{r+1}} \pmod{2^{2r+4}\mathbb{Z}_2}, \tag{4.7}$$

and similarly

$$a_2 \equiv \frac{g_j(2^r) + g_j(-2^r) - 2g_j(0)}{2^{2r+1}} \pmod{2^{2r+4}\mathbb{Z}_2}. \tag{4.8}$$

As an example, in the case $j = 0$, taking $r = \lfloor k/3 \rfloor - 1$ in (4.7) and observing from Proposition 1 that $g_0(n) \equiv (8 - 4k)n \pmod{2^{k+1}}$ for all integers n yields

$$a_1 \equiv 8 - 4k \pmod{2^{\lfloor (2k+4)/3 \rfloor}\mathbb{Z}_2} \quad (\text{for } j = 0), \tag{4.9}$$

and taking $r = \lfloor k/4 \rfloor - 1$ in (4.8) gives

$$a_2 \equiv 0 \pmod{2^{2\lfloor k/4 \rfloor + 2}\mathbb{Z}_2} \quad (\text{for } j = 0). \tag{4.10}$$

Although simple congruences such as these are sufficient for our purposes here, we remark that one may obtain stronger congruences by solving larger systems of equations. For example, if A is the $k \times k$ Vandermonde submatrix whose (i, j) entry is j^i, and $\vec{b}$ denotes the first column of A^{-1}, then one may compute the i-th entry $b_i = (-1)^{i+1}\binom{k}{i}/i$. It follows that

$$\sum_{i=1}^{k} b_i g_j(i) - \left(\sum_{i=1}^{k} b_i\right) g_j(0) = a_1 + k! a_{k+1} + \cdots . \tag{4.11}$$

In the case $j = 0$ we may conclude from Proposition 1 the stronger congruence

$$a_1 \equiv 8 - 4k \pmod{2^{k+1-\lfloor \log_2 k \rfloor}\mathbb{Z}_2} \quad (\text{for } j = 0). \tag{4.12}$$

The following table summarizes a few congruences for the coefficients a_1 relevant to the valuation of T_n.

Coefficients of g_j, k odd

j	a_0	a_1
0	0	$8 - 4k \pmod{2^{k+1-\lfloor \log_2 k \rfloor}\mathbb{Z}_2}$
$1 \leqslant i \leqslant k-1$	1	
k	$k-1$	$0 \pmod{2^{k-\lfloor \log_2 k \rfloor}\mathbb{Z}_2}$
$k+1$	$2k-2$	$4k - 8 \pmod{2^{k+1-\lfloor \log_2 k \rfloor}\mathbb{Z}_2}$
-1	$3-k$	$0 \pmod{2^{k-\lfloor \log_2 k \rfloor}\mathbb{Z}_2}$
$1-k \leqslant i \leqslant -2$	1	
$-k$	-1	

5. 2-adic Valuation of T_n

Proof of Theorem 1. All cases of Theorem 1 except the $n \equiv 0 \pmod{k+1}$ cases follow directly from Proposition 1. Suppose that $n \equiv 0 \pmod{2k+2}$, and write $n = (2k+2)m$ with $m \in \mathbb{Z}$. Then we have

$$T_n = g_0(m) = a_1 m + a_2 m^2 + a_3 m^3 + \cdots . \tag{5.1}$$

Since (4.9), (4.10) imply that $\nu_2(a_1) = 2$ and $\nu_2(a_i) \geqslant 4$ for $i \geqslant 2$, we have $\nu_2(T_n) = 2 + \nu_2(m)$, proving the result in that case.

Now suppose that $n = k + 1 + (2k+2)m$ with $m \in \mathbb{Z}$. Then

$$T_n = g_{k+1}(m) = 2k - 2 + a_1 m + a_2 m^2 + a_3 m^3 + \cdots , \tag{5.2}$$

with $\nu_2(a_1) = 2$ and $\nu_2(a_i) \geqslant 3$ for $i \geqslant 2$. Let $a = \nu_2(k-1)$. First consider the case where $\nu_2(m) > a - 1$. In this case we have $\nu_2(T_n) = \nu_2(2k-2) = a + 1$ from (5.2). Also in this case, $\nu_2((2k+2)m) > a$, so $\nu_2(n-2) = \nu_2(k-1) = a$. Since $\nu_2(m) = \nu_2(n-k-1) - 1 - \nu_2(k+1)$, the condition $\nu_2(m) < \nu_2(k-1) - 1$ is therefore equivalent to $\nu_2(n-k-1) < \nu_2(k^2-1)$.

Finally suppose that $n = k+1+(2k+2)m$ with $\nu_2(m) < a-1$. For this case to hold, we must have $a \geqslant 2$; since $k-1$ is a multiple of 4 we then have $\nu_2(k+1) = 1$, which implies $\nu_2(n-k-1) = \nu_2(m)+2$, so that $\nu_2(n-k-1) < a+1 = \nu_2(k^2-1)$. In this case, we have from (5.2) that $\nu_2(T_n) = \nu_2(a_1 m) = \nu_2(m) + 2$. Therefore $\nu_2(T_n) = \nu_2(n-k-1)$ as claimed, completing the proof. $\square$

It appears that the determination of $\nu_2(T_n)$ in the case where $n \equiv k+1 \pmod{2k+2}$ and $\nu_2(n-k-1) = \nu_2(k^2-1)$ requires more delicate analysis. We now examine the formula conjectured in ([3], Conjecture 1) in the case $k = 5$ and $n \equiv 6 \pmod{12}$.

Theorem 4. *In the case $k = 5$, the formula*

$$\nu_2(T_n) = \begin{cases} \nu_2(n+2), & \text{if } n \equiv 6 \pmod{12} \text{ and } \nu_2(n+2) < 8, \\ \nu_2(n+43266), & \text{if } n \equiv 6 \pmod{12} \text{ and } \nu_2(n+2) \geqslant 8 \end{cases}$$

conjectured in [3] is correct when $\nu_2(n+2) \neq 8$, but is not correct in general.

Proof. For the affirmative part, it will suffice to compute T_{12m+6} modulo 2^9. We consider the analytic function $g_6(m)$ which interpolates the values T_{12m+6}, and write $g_6(m) = \sum_i a_i m^i$. We use the recurrence to compute the values $g_6(0) = 8$, $g_6(1) = 25172$, $g_6(2) = 83904288$, $g_6(-1) = -4$, $g_6(-2) = -16$. As in (4.7) with $r = 0$, we have

$$12588 = a_1 + a_3 + a_5 + \cdots \tag{5.3}$$

and with $r = 1$ we get

$$20976076 = a_1 + 4a_3 + 16a_5 + \cdots . \tag{5.4}$$

Since $\nu_2(a_3) \geqslant 4$, we initially get $a_1 \equiv 12 \pmod{2^6}$ from (5.4). Substituting this into (5.3) then gives $a_3 \equiv 32 \pmod{2^6}$ since $\nu_2(a_5) \geqslant 6$. Substituting $a_3 = 32 + 64y$ back into (5.4) then shows $a_1 \equiv 76 \pmod{2^8}$. A similar argument from (4.8) with $r = 0$ and $r = 1$ reveals that $a_2 \equiv 224 \pmod{2^9}$ and $a_4 \equiv 0 \pmod{2^7}$. If $n = 12m + 6$ with $\nu_2(m) = 0$ then

$$T_n = g_6(m) = 8 + a_1 m + \cdots \equiv 8 + 12m \pmod{2^5} \tag{5.5}$$

and therefore $\nu_2(T_n) = \nu_2(8 + 12m) = 2 = \nu_2(n + 2)$, proving the theorem in the case $\nu_2(m) = 0$. If $\nu_2(m) \geqslant 2$ then (5.5) shows that $\nu_2(T_n) = 3 = \nu_2(n+2)$, proving the theorem when $\nu_2(m) \geqslant 2$.

Finally, we consider the case where $\nu_2(m) = 1$, and write $m = 2u$ with u odd. Then modulo 2^9 we compute

$$\begin{aligned} T_n = g_6(m) \quad &\equiv 8 + a_1(2u) + a_2(4u^2) + a_3(8u^3) \\ &\equiv 8 + 152u + 896u^2 + 256u^3 \\ &\equiv (8 + 24u) + 128u + 896u^2 + 768u^3 \\ &= (n + 2) + 128u(1 + u)(1 + 6u) \pmod{2^9} \end{aligned} \tag{5.6}$$

Since $1 + u$ is even, we see that $g_6(m) \equiv (n+2) \pmod{2^8}$. It follows that $\nu_2(T_n) = \nu_2(n+2)$ as long as $\nu_2(n+2) < 8$. Suppose that $\nu_2(n+2) > 8$. Since $\nu_2(8+24u) > 8$, we have $\nu_2(1 + 3u) > 5$. Since u is odd, this implies that $\nu_2(1 + u) = 1$, so that the factor $128u(1 + u)(1 + 6u)$ has valuation exactly 8. From (5.6) we conclude that $\nu_2(T_n) = 8$. But since $\nu_2(43264) = 8$, we also have $\nu_2(n + 43266) = 8$, proving the theorem in the case $\nu_2(n + 2) > 8$.

Numerical calculation shows that the conjectured formula $\nu_2(T_n) = \nu_2(n+43266)$ is correct for positive integers $n = 12m + 6$ less than three million, however the formula is not correct in general. Assuming the formula were correct for positive integers n, it would necessarily also hold for negative integers n by the continuity of the analytic function $g_6(m)$. However, the formula fails for $n = -43266$, as $\nu_2(T_n) = 20$ while $\nu_2(n + 43266) = +\infty$. $\square$

Remark. The above argument indicates how one may find actual positive integer counterexamples to the conjectured formula, using the continuity of the analytic function $g_6(m)$. However, this requires more extensive computation. The first two positive integer counterexamples are $n = 3 \cdot 2^{20} - 43266$, for which $\nu_2(T_n) = 22$ while $\nu_2(n + 43266) = 20$; and $n = 3 \cdot 2^{21} - 43266$, for which $\nu_2(T_n) = 20$ while $\nu_2(n + 43266) = 21$. From these calculations and Theorem 2, we can state the correct formula for the case $n \equiv 6 \pmod{12}$ in the form $\nu_2(T_n) = \nu_2(n - y)$, where $y \equiv 3 \cdot 2^{20} - 43266 \pmod{2^{22}\mathbb{Z}_2}$; here $y = 12z + 6$ where z is the root of $g_6(x)$ guaranteed by Theorem 2.

Proof of Theorem 2. We consider the Newton polygon of the power series $g_{k+1}(x) = \sum_m a_m x^m$. As was done in (4.9), (4.10) in the case $j = 0$, we compute from Proposition 1 that

$$a_1 \equiv 4k - 8 \pmod{2^{\lfloor (2k+4)/3 \rfloor}\mathbb{Z}_2} \quad (\text{for } j = k+1) \tag{5.7}$$

and

$$a_2 \equiv 0 \pmod{2^{2\lfloor k/4 \rfloor + 2}\mathbb{Z}_2} \quad (\text{for } j = k+1). \tag{5.8}$$

Since $g(0) = a_0 = 2k - 2$, we have $(0, a+1)$ and $(1, 2)$ as vertices of the Newton polygon for the power series $g(x)$ at 0, with all other points $(i, \nu_2(a_i))$ lying on or above the diagonal line through the origin with slope 1. All sides of the Newton polygon beyond the first therefore have slope at least 1. Since the first side has horizontal run 1 and slope $1 - a$, the power series g has precisely one root $z \in \mathbb{C}_2$ with $\nu_2(z) = a - 1$, and no other roots with valuation less than -1.

Consider the power series $h(x) = g_{k+1}(x)/4$, which has coefficients in $\mathbb{Z}_2$. If $x_0 = (k-1)/(4-2k)$, then

$$g_{k+1}(x_0) = (2k-2) + a_1 x_0 + a_2 x_0^2 + a_3 x_0^3 + \cdots \equiv 0 \pmod{2^{3a+1}\mathbb{Z}_2}. \tag{5.9}$$

Since $h(x_0) \equiv 0 \pmod{2\mathbb{Z}_2}$ and $h'(x_0) \equiv k \not\equiv 0 \pmod{2\mathbb{Z}_2}$, by Hensel's Lemma ([1], Theorem 3) there exists $z \in \mathbb{Z}_2$ with $z \equiv x_0 \pmod{2\mathbb{Z}_2}$ and $h(z) = 0$. This root is therefore the root z described in the preceding paragraph, and thus lies in $\mathbb{Z}_2$. We then have

$$(2k-2) + a_1 z \equiv 0 \pmod{2^{3a+1}\mathbb{Z}_2}, \tag{5.10}$$

and dividing by a_1, which has valuation 2, gives the congruence of the Theorem.

Now write $g_{k+1}(x) = \sum_m a_m x^m = (x - z)\sum_m b_m x^m$. Then $b_0 = -a_0/z$ has $\nu_2(b_0) = 2$, and $b_m = -(a_0 + a_1 z + \cdots + a_m z^m)/z^{m+1}$ for $m \geqslant 1$. Since $g_{k+1}(z) = 0$ we have $\nu_2(b_1) \geqslant 4$ and $\nu_2(b_m) \geqslant m + 1$ for all $m > 0$. Therefore $\sum_m b_m x^m$ also converges on $D = \{x \in \mathbb{C}_2 : \nu_2(x) > -1\}$. Since $\nu_2(b_m) > 2$ for all $m > 0$, we have $\nu_2(\sum_m b_m x^m) = 2$ for all $x \in \mathbb{Z}_2$. It follows that $\nu_2(g_{k+1}(x)) = \nu_2(x - z) + 2$ for all $x \in \mathbb{Z}_2$, completing the proof. $\square$

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