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PATTERN CONTAINMENT IN CIRCULAR PERMUTATIONS

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Abstract

Pattern containment in permutations, as opposed to pattern avoidance, involves two aspects. The first is to contain every pattern at least once from a given set, known as superpatterns; while the second is to contain some given pattern as many times as possible, known as pattern packing. In this note we explore these two questions in circular permutations and present some interesting observations. We also raise some questions and propose directions for future study.

1. Introduction

In the study of permutations, pattern containment describes how one permutation can 'contain' another permutation. In a sense, these patterns are subpermutations of the original permutation.

Definition 1.1. Let π and τ be permutations of lengths n and k, respectively, with $n \geq k$. We say that π *contains* τ *as a pattern* if there is a subsequence of entries of π, $(\pi_{i_1}, \pi_{i_2}, \pi_{i_3}, \ldots, \pi_{i_k})$, which is order isomorphic to τ, i.e. $\pi_{i_s} \leq \pi_{i_t}$ if and only if $\tau_s \leq \tau_t$. Such a subsequence is called an *occurrence* of τ in π. If no occurrence of τ is present in π, we say that π *avoids* τ.

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Example 1.2. If $\pi = 316452$ and $\tau = 132$, then the subsequence 364 of π is order isomorphic to τ and hence π contains τ as a pattern.

The systematic study of pattern containment was first proposed by H. Wilf in his 1992 address to the SIAM meeting on Discrete Mathematics [11]. Until then, most research regarding patterns concerned pattern avoidance. For an excellent introduction into these results, consult [4].

In the study of pattern containment, there are two natural questions one might ask. First, what is the smallest positive integer n such that there exists a permutation of length n which contains every element in some subset of permutations? Second, and almost dual in nature to the first, for a given pattern τ and a given positive integer n, what permutation of length n contains the most number of occurrences of τ. We define the first concept below.

Definition 1.3. Let P be a subset of permutations. We say that a permutation π is a P-superpattern if it contains at least one occurrence of every $\tau \in P$. Further, define $sp(P)$ to be the length of the shortest P-superpattern, i.e.

$$sp(P) = \min\{n : \text{there is a } P\text{-superpattern of length } n\}.$$

In the case that P consists of all permutations of length k, we employ the traditional notations k-superpattern and $sp(k)$.

As an illustration, suppose we wish to contain all permutations of length 3, i.e. we want a 3-superpattern. Then, $\pi = 25314$ accomplishes this; further, there is no permutation of length 4 that could possibly contain all permutations of length 3 since it would be impossible for such a permutation to contain both 123 and 321. Hence, $sp(3) = 5$.

In 1999, R. Arratia was the first to publish bounds for $sp(k)$. He found that $\frac{k^2}{e^2} \leq sp(k) \leq k^2$ [2]. This lower bound is still the best known. In 2007, H. Eriksson et al. improved Arratia's upper bound with $sp(k) \leq \frac{2k^2}{3} + O(k^{3/2}(\log(k))^{1/2})$ [8]. Alison Miller [13] then showed, in 2010, that $sp(k) \leq \frac{k(k+1)}{2}$, which strongly supports the conjecture by Eriksson et al that $sp(k) \sim \frac{k^2}{2}$.

Many results on $sp(P)$ for various sets of permutations P have also been found. As a brief list, bounds have been found for $sp(P)$ when P has been the set of layered permutations [9], the set of 321-avoiding permutations [3], the set of m-colored permutations [10], the set of compositions [12], and the set of words [5]. For these last two items, the concept of pattern containment extends naturally to m-colored permutations and to the set of words.

Now, we proceed to our second question, which concerns the number of occurrences of a particular pattern τ. Here, we seek to 'pack' as many occurrences of τ as possible into a permutation of length n.

Definition 1.4. For permutations π and τ (of length k), we let $f(\pi, \tau)$ be the number of occurences of τ in π, and we define

$$g(n, \tau) = \max\{f(\sigma, \tau) : \sigma \text{ is a permutation of length } n\}.$$

If π is of length n and $f(\pi, \tau) = g(n, \tau)$, then we say that π is τ-*optimal*. The *packing density* of τ is defined by

$$\delta(\tau) = \lim_{n\to\infty} \frac{g(n, \tau)}{\binom{n}{k}}.$$

As a simple example, for the pattern $\tau = 123$ and the permutation $\pi = 123456$, it is not hard to see that π is τ-optimal and

$$g(6, \tau) = f(\pi, \tau) = \binom{6}{3}.$$

In general, if π is the monotone increasing permutation of length n, then π is τ-optimal. Hence, the packing density of τ is

$$\delta(\tau) = \lim_{n\to\infty} \frac{g(n, \tau)}{\binom{n}{3}} = \lim_{n\to\infty} \frac{\binom{n}{3}}{\binom{n}{3}} = 1.$$

In 1993, W. Stromquist found the packing density of 132 to be $2\sqrt{3} - 3$ [15]. Along with the packing density of 123 being trivially 1, this closed the length-3 case by symmetries (reversals, inverses, and complements of permutations preserve pattern containment). In 1997, A. Price found the packing densities of 1432, 2143, and 1324 [14]. Since there are 7 equivalence classes of length-4 patterns, this left 4 unsolved packing densities. In 2002, M. H. Albert et al. found the packing density of 1243 [1]. The remaining 3 are still unsolved.

In this note we generalize the study of pattern containment to circular permutations. We first introduce the related concepts in Section 2. In Section 3 we discuss circular superpatterns and R-rev superpatterns in general. We then study packing density in circular permutations and its R-rev analogue in Section 4. We briefly comment on our findings and raise some questions in Section 5.

2. Circular Permutations and Number of Revolutions

In order to define the circular analogue of a permutation, we first introduce the circular shift of a permutation.

Definition 2.1. Let $\pi = \pi_1\pi_2\pi_3 \cdots \pi_n$ be a permutation of length n. The *circular shift of* π, denoted $S(\pi)$, is given by

$$S(\pi) = \pi_n\pi_1\pi_2 \cdots \pi_{n-1}.$$

For instance, with $\pi = 31524$, we have $S(\pi) = 43152$, $S^2(\pi) = 24315$, $S^3(\pi) = 52431$, $S^4(\pi) = 15243$, and $S^5(\pi) = 31524 = \pi$.

Definition 2.2. Let π be a permutation of length n. Then, the *circular permutation of* π, denoted π_c, is the permutation obtained by wrapping π clockwise around a circle in one revolution; so for $2 \le i \le n-1$, we have π_{i-1}, π_i, and π_{i+1} appear in clockwise ordering, with π_n being the counterclockwise neighbor of π_1. We say that two permutations are *equivalent* if one is the circular shift of the other.

Note that when a circular permutation is considered, circular shift does not change the permutation. As an example, Figure 1 shows the circular versions of a permutation and its circular shift, which are identical. This natural interpretation of permutations placed on a circle was used in the study of pattern avoidance in [6] and even earlier in [7] where sorting algorithms are studied.

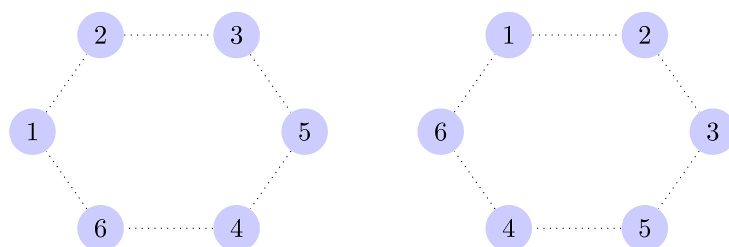

Figure 1: A circular permutation $\pi_c = (123546)_c$ (left) and $(S(\pi))_c = \pi_c$ (right).

Remark 2.3. When considering pattern containment in circular permutations, it is important and convenient to note that an occurrence of τ in π_c is the same as a linear occurrence of $S^i(\tau)$ in π for some i.

Definition 2.4. Let P be a set of permutations. We say that π is a *circular P-superpattern* if there is an occurrence of τ in π_c for every $\tau \in P$. Further, let $sp_c(P)$ be the length of the shortest circular P-superpattern.

In the case that P consists of all (regular) permutations of length k, similar to the traditional notations, we use terms circular k-superpattern and $sp_c(k)$.

Definition 2.5. For permutations π and τ (of length k), we define $f_c(\pi, \tau)$ to be the number of occurrences of τ in π wrapped around a circle, i.e.

$$f_c(\pi, \tau) = f(\pi, \tau) + f(\pi, S(\tau)) + f(\pi, S^2(\tau)) + \cdots + f(\pi, S^{k-1}(\tau)).$$

Likewise,

$$g_c(n, \tau) = \max\{f_c(\sigma, \tau) : \sigma \text{ is a permutation of length } n\}.$$

If π is of length n and $f_c(\pi, \tau) = g_c(n, \tau)$, then we say that π is *circular τ-optimal*.

Definition 2.6. Let τ be a permutation of length k. The *circular packing density* of τ, denoted $\delta_c(\tau)$, is defined by

$$\delta_c(\tau) = \lim_{n\to\infty} \frac{g_c(n,\tau)}{\binom{n}{k}}.$$

A circular permutation allows one to travel clockwise along the permutation with any choice of starting point, stopping our travel before we reach our starting point, i.e. we travel within one revolution of the circle. It is also interesting to explore the scenario when we can 'wrap' around the circle more than once. For instance, the pattern 321 occurs as 654 in the circular permutation in Figure 1, but not the pattern 4321. If we are allowed two revolutions, then we have 6541 as an occurrence of the pattern 4321.

Definition 2.7. Let π be a permutation of length n, and create a word $\pi_{(R)}$ by laying out R copies of π in a line, i.e.

$$\pi_{(R)} = \pi_1\pi_2\cdots\pi_n\ \pi_1\pi_2\cdots\pi_n\ \cdots\pi_1\pi_2\cdots\pi_n.$$

Select a subsequence of distinct entries of $(\pi_{(R)})_c$, i.e. wrap $\pi_{(R)}$ clockwise around a circle and select a subsequence so that no two entries have the same value. Such a subsequence will be called an *R-revolution subsequence* (or just *R-rev* subsequence) of π_c.

Take $\pi = 1234$ for example, $(\pi_{(2)})_c$ is simply 12341234 wrapped around a circle. Then 421 is a 2-rev subsequence of π_c, but is clearly not a 1-rev subsequence of π_c.

Definition 2.8. Let π and τ be permutations of length n and k, respectively, with $n \geq k$, and let R be a positive integer. We say that there is an *R-rev occurrence* of τ in π if there is an R-rev subsequence of entries of π_c which is order isomorphic to τ. We may then analogously define *R-rev P-superpattern*, *R-rev τ-optimal*, and *R-rev packing density*. Further, let $sp_R(P)$ be the length of the shortest R-rev P superpattern. It is easy to see that $sp_c = sp_1$, $\delta_c = \delta_1$, etc.

3. Circular Superpatterns

We start with some simple observations which bound $sp_c(k)$.

Theorem 3.1. *Given a positive integer k, we have*

$$sp_c(k) \geq g(k)\frac{k^2}{e^2},$$

where $g(k) \to 1$ as $k \to \infty$.

Proof. Let $n = sp_c(k)$. Note that the circular shift partitions the set of permutations of length k into $(k-1)!$ equivalence classes, each of size k. Any circular k-superpattern must contain at least one representative of each of these equivalence classes. Hence, the number of subsequences of length k in a circular k-superpattern must be at least $(k-1)!$. Then,

$$\binom{n}{k} \geq (k-1)!.$$

Notice that $\frac{n^k}{k!} \geq \binom{n}{k}$ and, by Stirling's Approximation, $k! \geq \sqrt{2\pi k}\frac{k^k}{e^k}$. So,

$$\begin{aligned} n^k &\geq k!(k-1)! \\ n &\geq \sqrt[k]{\frac{(k!)^2}{k}} \\ &\geq \sqrt[k]{2\pi \cdot \frac{k^{2k}}{e^{2k}}} \\ &= (2\pi)^{1/k}\frac{k^2}{e^2}. \end{aligned}$$

Letting $g(k) = (2\pi)^{1/k}$, we clearly see that $g(k) \to 1$ as $k \to \infty$, which proves our result. $\square$

To state our next observation we first recall that the *direct sum* $\pi \oplus \pi'$ of two permutations $\pi = \pi_1 \dots \pi_\ell$ and $\pi' = \pi'_1 \dots \pi'_m$ is defined as

$$\pi_1 \dots \pi_\ell(\pi'_1 + \ell) \dots (\pi'_m + \ell).$$

Theorem 3.2. *For any k-superpattern π, $(\pi \oplus 1)_c$ is a circular $(k+1)$-superpattern. Consequently*

$$sp_c(k+1) \leq sp(k) + 1$$

for any $k \geq 1$.

Proof. Suppose $\pi = \pi_1 \dots \pi_n$, then $\pi \oplus 1 = \pi_1 \dots \pi_n(n+1)$. Given any permutation/pattern τ of length $k+1$, $S^i(\tau)$ is of the form $\tau_1 \dots \tau_k(k+1)$ for some i. Since π is a k-superpattern, some subsequence $\pi_{i_1} \dots \pi_{i_k}$ of π is order isomorphic to $\tau_1 \dots \tau_k$. Then

$$\pi \oplus 1 = \pi_1 \dots \pi_n(n+1)$$

contains the subsequence

$$\pi_{i_1} \dots \pi_{i_k}(n+1)$$

that is order isomorphic to

$$\tau_1 \dots \tau_k(k+1) = S^i(\tau).$$

Thus $(\pi \oplus 1)_c$ contains τ as a pattern, implying that $(\pi \oplus 1)_c$ is a circular $(k+1)$-superpattern. $\square$

When allowing extra revolutions, the following result states that every sufficiently long circular permutation, with enough revolutions allowed, contains every pattern.

Theorem 3.3. *Given any $k \geq 2$ and $R \geq k-1$, any permutation of length at least k is an R-rev k-superpattern.*

Proof. It is sufficient to show the statement for $R = k-1$. We proceed by induction on k. The initial case is trivial.

Assume now, that any permutation of length at least k is a $(k-1)$-rev k-superpattern and let π be a permutation of length at least $k+1$.

Given any pattern τ of length $k+1$, we show that there is a k-rev occurrence of some circular shift of τ in π. Note that some circular shift $S^i(\tau)$ is of the form $\tau_1 \dots \tau_k(k+1)$. Let π' be the permutation of length at least k obtained by removing the largest entry n of π. Then π' is a $(k-1)$-rev k-superpattern by induction hypothesis. That is, some subsequence of distinct entries of

$$\underbrace{\pi' \dots \pi'}_{k-1 \text{ times}}$$

is order isomorphic to $\tau_1 \dots \tau_k$. This subsequence, with n appended at the end, is order isomorphic to $\tau_1 \dots \tau_k(k+1) = S^i(\tau)$. Such a sequence is a subsequence of distinct entries of

$$\underbrace{\pi \dots \pi}_{k \text{ times}},$$

which is to say that there is a k-rev occurrence of $S^i(\tau)$ in π.

Hence π is a k-rev $(k+1)$-superpattern. $\square$

Remark 3.4. In some sense Theorem 3.3 is the best possible. For instance, let $\pi = 12\dots k$ and $\tau = k \dots 21$, indeed it requires $k-1$ revolutions for τ to occur as a subsequence of π_c.

Since a k-superpattern (regardless of the number of revolutions allowed) has to contain at least k entries, Theorem 3.3 immediately implies the following.

Corollary 3.5. *For any $k \geq 2$ and $R \geq k-1$, $sp_R(k) = k$.*

4. Packing Density in Circular Permutations

For regular permutations, it is easy to see that the monotonic patterns are the only ones with packing density 1, with corresponding optimal permutations being monotonic as well. It is interesting to see that this is also the case for circular permutations. For brevity, we will say that a circular pattern τ_c is monotonic if $S^i(\tau)$ is monotonic for some i.

Theorem 4.1. *The circular packing density of a pattern τ is at most 1, with equality if and only if τ_c is monotonic. In this case the circular τ-optimal permutation must also be monotonic. Further, if τ_c is not monotonic, then $\delta_c(\tau) \leq \frac{2}{3}$.*

Proof. It is obvious that $\delta_c(\tau) \leq 1$ for any pattern τ. In the case that τ is a monotonic pattern (say $12 \dots k$) and π is a monotonic permutation $12 \dots n$ of length n, it is easy to see that

$$f_c(\pi, \tau) = \binom{n}{k} = g_c(n, \tau)$$

and hence $\delta_c(\tau) = 1$. If π_c is not monotonic, then some subsequence of length k is not order isomorphic to τ, hence π_c would not be τ-optimal. The case of τ and π being decreasing is similar.

Suppose, then, that τ_c is not monotonic. In order to show $\delta_c(\tau)$ is strictly less than 1, we will show that the number of subsequences of length k which are not occurrences of τ in any τ-optimal circular permutation is relatively large with respect to n, i.e. the probability that a randomly selected subsequence of length k is not an occurrence of τ is non-zero as $n \to \infty$.

In τ_c, there exists a smallest i such that i and $i-1$ are not neighbors (where 0 is equivalent to k). Without loss of generality, suppose that i is clockwise of $i-1$. Let $n >> k$ and let π be a circular τ-optimal permutation of length n.

Suppose that T is an occurrence of τ in π with T_r and T_s playing the roles of $i-1$ and i, respectively. Notice that if $T_r < \gamma < T_s$, then γ cannot play any role in T. Otherwise, γ would be representing some x in τ such that $i-1 < x < i$, which is impossible.

Since either $T_r = T_s - 1$ or $T_r < T_s - 1$, consider two cases:

- Let P be the set of τ-occurrences, T, where $T_r = T_s - 1$.

  Clearly, $|P| \leq \binom{n}{k-1}$. Indeed, once we select x to play the role of $i-1$ in such a τ-occurrence, the role of i will automatically be assigned to $x+1$.

- Let Q be the set of τ-occurrences, T, where $T_r < T_s - 1$.

  Let $\gamma = \lfloor \frac{T_r + T_s}{2} \rfloor$, and recall that γ plays no role in T. Let T' be obtained from T by inserting γ into T and removing T_s and let T'' be obtained by inserting γ into T and removing T_r. We will show that at most one of T' and T'' is a τ-occurrence.

  To see this, let T' be an occurrence of τ. recall that $T_r < \gamma < T_s$ with T_r playing the role of $i-1$ and T_s playing the role of i in T. Hence, the only possible role that γ can play in T' is the role of i. If T' is indeed an occurrence of τ, we may further say that γ occupies the same position in T' as T_s did in T.

  Hence, the number of entries of T between T_r and T_s (clockwise) is exactly the same as the number of entries between T_r and γ in T', and the number of entries of T between T_s and T_r in T is exactly the same as the number of entries between γ and T_r in T'. Thus, if γ is clockwise (resp. counter-clockwise) of T_s, then there are no entries of T between T_s and γ (resp. γ and T_s). This means in T'', γ and T_s are neighbors. Since γ is now playing the role of $i-1$ (because T_r was removed) and T_s is playing the role of i, and i and $i-1$ are not neighbors in τ_c by assumption, T'' cannot be an occurrence of τ.

Thus, for every occurrence, T, of τ in Q, at least one of T' and T'' is not an occurrence of τ. Going through all T's in Q results in at least $|Q|$ such non-occurrences (with possible repetition). We make the following statement.

Claim: Each of these non-occurrences can occur at most twice.

Hence, the number of distinct subsequences of π which are not occurrences of τ is at least $\frac{1}{2}|Q|$.

Proof of the Claim. Given such an non-occurrence, it is immediately obvious whether the non-occurrence is a T' or a T'' since T' and T'' are identical to T except for the placement of one entry. If the non-occurrence is a T' then T_r and γ are known, and there are only two possible values of T_s (since we defined $\gamma = \lfloor \frac{T_r+T_s}{2} \rfloor$) which will give us the same γ; similarly, if it was a T'' then T_s and γ are known, and there are only two possible values of T_r which will give the same γ. Therefore, each non-occurrence which resulted as a T' or a T'' is counted at most twice by members of Q. □

Summarizing the above, we have the total number of subsequences of length k, $\binom{n}{k}$, is at least the number of τ-occurrences in both P and Q plus the number of non-occurrences described above (at least $\frac{1}{2}|Q|$):

$$|P| + |Q| + \frac{1}{2}|Q| \leq \binom{n}{k}.$$

Then

$$\frac{3}{2}|Q| \leq \binom{n}{k} - |P| \leq \binom{n}{k},$$

implying that

$$\frac{|Q|}{\binom{n}{k}} \leq \frac{2}{3}.$$

Also notice that $\frac{|P|}{\binom{n}{k}} \leq \frac{\binom{n}{k-1}}{\binom{n}{k}} \to 0$ as $n \to \infty$. Therefore,

$$\delta_c(\tau) = \lim_{n\to\infty} \frac{g(n,\tau)}{\binom{n}{k}} \leq \lim_{n\to\infty} \frac{|P|+|Q|}{\binom{n}{k}} \leq \frac{2}{3},$$

which proves the result. □

Remark 4.2. As a consequence of Proposition 4.1, noticing that all length 3 patterns are equivalent to a monotonic pattern through circular shift, we see that all length 3 patterns have circular packing density 1. This is certainly not the case for the traditional packing densities.

On the other hand, it is easy to see the following relationship between the packing density of a pattern τ and the circular packing density of τ.

Proposition 4.3. *Let τ be a pattern of any length. Then, $\delta_c(\tau) \geq \delta(S^i(\tau))$ for all i.*

Proof. Suppose that τ is of length k, and recall that

$$g_c(n,\tau) = \max\{f(\pi,\tau) + f(\pi,S(\tau)) + \cdots + f(\pi,S^{k-1}(\tau)) : \pi \text{ is a permutation of length } n\}.$$

Let i be such that $g(n,S^i(\tau)) \geq g(n,S^j(\tau))$ for any $0 \leq j \leq k-1$. Then $g_c(n,\tau) \geq g(n,S^i(\tau))$. Thus, $\delta_c(\tau) \geq \delta(S^i(\tau))$. □

Next we point out an interesting observation (similar in nature to Theorem 3.3) that, when enough revolutions are allowed, every long enough permutation is optimal for any given pattern. For the sake of our argument we first introduce another notation.

Definition 4.4. For permutations π and τ, we define $f_R(\pi,\tau)$ to be the number of R-rev occurrences of τ in π. Likewise,

$$g_R(n,\tau) = \max\{f_R(\sigma,\tau) : \sigma \text{ is a permutation of length } n\}.$$

The following is an analogue of Theorem 3.3 in terms of packing density.

Proposition 4.5. *Given $k \geq 2$ and $R \geq k-1$, every permutation π of length at least k is R-rev τ-optimal for any pattern τ of length k.*

Proof. We only consider the case $R = k-1$. Take any k-subsequence of π (of length $n \geq k$), say ϕ. Theorem 3.3 claims that ϕ is a $(k-1)$-rev k-superpattern, implying that ϕ contains τ as a pattern when $k-1$ revolutions are allowed. Hence ϕ is an occurrence of τ.

Thus, $f_R(\pi,\tau) = \binom{n}{k} = g_R(n,\tau)$, showing that π is $(k-1)$-rev τ-optimal. □

5. Concluding Remarks

In this note we generalized the study of pattern containment to circular permutations where a permutation or pattern is allowed to wrap around a circle one or more times. We provided some basic observations on the minimum length of a circular superpattern. Furthermore, we showed that any long enough permutation, when enough revolutions are allowed, is a superpattern. This also implies that the classic pattern avoidance problem should not be considered for circular permutations when too many revolutions are allowed. While there has been some study on pattern avoidance in circular permutations [6], it is not clear how it plays out when multiple (but not too many) revolutions are allowed. In terms of packing patterns in a circular permutation, all patterns equivalent to monotonic patterns still have the highest packing density of 1. By Proposition 4.3, we have that $\delta_c(\tau) \geq \delta(S^i(\tau))$ for any pattern τ and for any i. It is still unknown whether there exists a pattern τ for which this inequality is strict for all i, however, we make the following conjecture:

Conjecture 5.1. *If τ is not a layered pattern, then $\delta_c(\tau) > \delta(\tau)$.*

Here the so-called layered patterns, a concept extensively studied in pattern packing and superpatterns (see [1, 9] for instance), are patterns that can be represented as

$$n_1(n_1-1)\dots 1 \oplus n_2(n_2-1)\dots 1 \oplus \dots \oplus n_k(n_k-1)\dots 1$$

for some k and $(n_1, n_2, \dots, n_k)$. As an example, $215436 = 21 \oplus 321 \oplus 1$ is a layered pattern while 3142 is not a layered pattern.

Using brute force, we have found for the pattern $\tau = 3142$, that $g_c(n,\tau) > g(n,\tau)$ for $6 \leq n \leq 10$. We have outlined our results in the Table 1. However, for all layered patterns of length 3 or 4, we have found no difference between the circular and linear packing densities. While this is not proof, it does lend some support to Conjecture 5.1.

Table 1: Optimal permutations for 3142 and its circular shifts

Length		n=10	n=9	n=8
Circular 3142-optimal permutations		6 7 2 8 3 9 4 5 1 10	4 5 6 1 7 2 8 3 9	5 6 2 7 3 4 1 8
			5 6 7 2 8 3 4 1 9	
			6 7 2 8 3 4 5 1 9	
	$g_c(n, 3142)$	73	45	27
Linear permutations	3142-optimal	6 7 1 8 2 9 3 10 4 5	5 6 1 7 2 8 9 3 4	6 4 1 7 2 8 5 3
		6 7 1 8 2 9 3 10 5 4	5 6 1 7 2 8 9 4 3	
		7 5 1 8 2 9 3 10 6 4	6 4 1 7 2 8 9 5 3	
		7 6 1 8 2 9 3 10 4 5	6 5 1 7 2 8 9 3 4	
		7 6 1 8 2 9 3 10 5 4	6 5 1 7 2 8 9 4 3	
		8 6 4 1 2 9 10 7 5 3	6 7 4 1 2 8 9 5 3	
		8 6 4 2 1 9 10 7 5 3	6 7 4 2 1 8 9 5 3	
		8 6 4 1 9 2 10 7 5 3	7 6 4 1 2 8 9 5 3	
		8 6 4 1 2 10 9 7 5 3	7 6 4 2 1 8 9 5 3	
		8 6 4 2 1 10 9 7 5 3	6 7 1 2 8 3 9 4 5	
			6 7 2 1 8 3 9 4 5	
			6 7 1 2 8 3 9 5 4	
			6 7 2 1 8 3 9 5 4	
			6 7 4 1 8 2 9 5 3	
			7 5 1 2 8 3 9 6 4	
			7 5 2 1 8 3 9 6 4	
			7 6 1 2 8 3 9 4 5	
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			7 5 2 1 9 8 6 3 4	
			7 5 1 2 9 8 6 4 3	
			7 5 2 1 9 8 6 4 3	
	$g(n, 3142)$	41	26	17
Linear permutations	2314-optimal	3 4 5 6 7 2 8 1 9 10	3 4 5 6 2 7 1 8 9	3 4 5 6 1 2 7 8
		4 5 6 7 2 3 8 1 9 10	3 4 5 6 7 1 2 8 9	3 4 5 6 2 1 7 8
		4 5 6 7 2 8 3 1 9 10	3 4 5 6 7 2 1 8 9	3 4 5 6 1 2 8 7
		4 5 6 7 3 8 1 2 9 10	3 4 5 6 2 7 1 9 8	3 4 5 6 2 1 8 7
		4 5 6 7 3 8 2 1 9 10	3 4 5 6 7 1 2 9 8	
		4 5 6 7 8 2 3 1 9 10	3 4 5 6 7 2 1 9 8	
		3 4 5 6 7 2 8 1 10 9		
		4 5 6 7 2 3 8 1 10 9		
		4 5 6 7 2 8 3 1 10 9		
		4 5 6 7 3 8 1 2 10 9		
		4 5 6 7 3 8 2 1 10 9		
		4 5 6 7 8 2 3 1 10 9		
	$g(n, 2314)$	62	40	24
Linear permutations	4231-optimal	10 8 9 4 5 6 7 2 3 1	8 9 4 5 6 7 2 3 1	7 8 3 4 5 6 1 2
			9 7 8 3 4 5 6 1 2	7 8 3 4 5 6 2 1
			9 7 8 3 4 5 6 2 1	8 5 6 7 2 3 4 1
			9 8 4 5 6 7 2 3 1	8 7 3 4 5 6 1 2
				8 7 3 4 5 6 2 1
	$g(n, 4231)$	68	42	24
Linear permutations	1423-optimal	1 2 9 10 3 8 4 5 6 7	1 2 8 9 3 4 5 6 7	1 2 7 8 3 4 5 6
		2 1 9 10 3 8 4 5 6 7	2 1 8 9 3 4 5 6 7	2 1 7 8 3 4 5 6
		1 2 10 3 8 9 4 5 6 7	1 2 9 3 8 4 5 6 7	1 2 8 7 3 4 5 6
		2 1 10 3 8 9 4 5 6 7	2 1 9 3 8 4 5 6 7	2 1 8 7 3 4 5 6
		1 2 10 8 3 9 4 5 6 7	1 2 9 8 3 4 5 6 7	
		2 1 10 8 3 9 4 5 6 7	2 1 9 8 3 4 5 6 7	
		1 2 10 3 9 4 5 6 7 8		
		2 1 10 3 9 4 5 6 7 8		
		1 2 10 8 9 3 4 5 6 7		
		2 1 10 8 9 3 4 5 6 7		
		1 2 10 9 3 8 4 5 6 7		
		2 1 10 9 3 8 4 5 6 7		
	$g(n, 1423)$	62	40	24

Analogous to the study of superpatterns, we also showed that a sufficient number of revolutions will automatically achieve the maximum packing density. It would also be interesting to continue this study for a limited number of revolutions.

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