A CAMERON AND ERDŐS CONJECTURE ON COUNTING PRIMITIVE SETS

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Abstract

Let $f(n)$ count the number of subsets of $\{1, \ldots, n\}$ without an element dividing another. In this paper we show that $f(n)$ grows like the n-th power of some real number, in the sense that $\lim_{n\to\infty} f(n)^{1/n}$ exists. This confirms a conjecture of Cameron and Erdős, proposed in a paper where they studied a number of similar problems, including the well known "Cameron-Erdős Conjecture" on counting sum-free subsets.

1. The Result

Let $f(n)$ be the number of subsets of $[n] = \{1, \ldots, n\}$ such that no element divides another - call these sets *primitive*. One easily notices that $2^n \geq f(n) \geq 2^{n/2}$ (since subsets of the second half are all primitive), motivating Cameron and Erdős to question whether there is an exact real number characterizing the exponential growth of this function [1]. We confirm their conjecture.

Theorem. $\lim_{n\to\infty} f(n)^{1/n}$ exists.

Proof. We will study the auxiliary and more structured $f(n, k)$, which we define to be the number of subsets of $[n]$ such that no two elements have an integer ratio for which all prime factors are at most p_k (the k-th prime number). Call these sets k-core. The crux of the proof will be a little argument that shows that if, for each k, $\lim_{n\to\infty} f(n, k)^{1/n}$ exists, then $\lim_{n\to\infty} f(n)^{1/n}$ also exists. This is somewhat surprising, because if one doesn't think about this in the right way, it may seem that it is necessary to send k to infinity together with n, in order to obtain the desired limit. That said, we divide the proof into two parts:

`Part 1:` If we assume that, for each k, $\lim_{n\to\infty} f(n, k)^{1/n} = \alpha_k$ exists, then the α_k decrease to some limit α and $\lim_{n\to\infty} f(n)^{1/n}$ exists and is equal to α.

`Part 2:` For each fixed k, $\lim_{n\to\infty} f(n, k)^{1/n}$ in fact exists.

Proof of part 1. Let $\lim_{n\to\infty} f(n,k)^{1/n} = \alpha_k$. Clearly $f(n,k+1) \le f(n,k)$ (because the condition of being $k+1$-core is more restrictive than the condition of being k-core). By taking $1/n$ powers and limits we get that α_k are a decreasing sequence. Since they are non-negative, it follows that the α_k must have a limit α.

Since $f(n) \le f(n,k)$ for all k, we get $\limsup f(n)^{1/n} \le \alpha_k$ for each k, which gives $\limsup f(n)^{1/n} \le \alpha$.

Now we need an inequality for the other side. For that, we notice that for a k-core subset of $[n]$, if the elements less than $\frac{n}{k}$ are removed we get a primitive subset. That is because the ratio of any two remaining elements is less than k, so if one element divided another then their ratio would be an integer less than k. All prime factors of such an integer are less than p_k, which contradicts that the original set is k-core. Hence this operation maps k-core sets to primitive sets. Also, it is clear that this operation maps at most $2^{n/k}$ sets to the same set (because two sets mapped to the same set may disagree only on the first n/k elements). This gives the inequality

$$f(n,k) \le 2^{n/k} f(n).$$

By taking $1/n$ power and taking n to infinity this gives

$$\liminf f(n)^{1/n} \ge \alpha_k 2^{-1/k},$$

for all k. By making $k \to \infty$ we get $\liminf f(n)^{1/n} \ge \alpha$. So

$$\alpha \le \liminf f(n)^{1/n} \le \limsup f(n)^{1/n} \le \alpha,$$

which completes the proof that $\lim_{n\to\infty} f(n)^{1/n}$ exists and is equal to α.

Proof of part 2. Fix k. Let $S = \{p_1, \ldots, p_k\}$ and let $D = p_1 \cdots p_k$ be the product of the first k primes. Each integer can be written uniquely as a product aR where a only has prime factors in S and $(R,D) = 1$. Integers with distinct values of R cannot have an integer ratio with prime factors in S. So we partition the integers in $[n]$ according to their value of R, and the total number of k-core subsets of $[n]$ is just the product of the number of k-core subsets of each part. We also notice that each part consists of the naturals of the form aR, where a runs over the naturals $1 \le \frac{n}{R}$ with all prime factors in S. Hence, if we define $P_k(x)$ to be the number of k-core (or simply primitive) subsets of the set of naturals $\le x$ with all prime factors in S, we get

$$f(n,k) = \prod_{1\le R\le n, (R,D)=1} P_k\left(\left\lfloor \frac{n}{R} \right\rfloor\right).$$

Now set $\epsilon > 0$ to be chosen later. We first want to show that the first ϵn terms of this product do not contribute substantially. For these terms we use the bound

$$P_k(x) \le 2^{(1+\log x)^k}.$$

We obtain this by bounding $P_k(x)$ above by the number of subsets of the set of naturals $\le x$ with all prime factors in S, and we bound the size of this set by $(1+\log x)^k$ by noticing that each $p_1^{a_1}\cdots p_k^{a_k} \le x$ is associated to a distinct k-tuple $(a_1,\ldots,a_k)$ with $a_i \le \log x$. Hence,

$$\prod_{1\le R\le \epsilon n,(R,D)=1} P_k\left(\left\lfloor \frac{n}{R}\right\rfloor\right) \le \prod_{1\le R\le \epsilon n} 2^{(1+\log \frac{n}{R})^k} \le 2^{\epsilon n(1+\log n)^k}.$$

The product of the first ϵn terms is also ≥ 1, so we get

$$f(n,k) = 2^{O(\epsilon n(1+\log n)^k)} \prod_{\epsilon n < R\le n,(R,D)=1} P_k\left(\left\lfloor \frac{n}{R}\right\rfloor\right).$$

Now $\frac{n}{R}$ is always between 1 and $\frac{1}{\epsilon}$. For each integer l between 1 and $\frac{1}{\epsilon}$ there are $n(\frac{1}{l}-\frac{1}{l+1})+O(1)$ integers R from ϵn to n with $\lfloor \frac{n}{R}\rfloor = l$. And this is a run of consecutive numbers, so $n(\frac{1}{l}-\frac{1}{l+1})\frac{\phi(D)}{D}+O(D)$ of these numbers are prime with D (ϕ is the Euler totient function). Hence:

$$\begin{aligned} f(n,k)^{1/n} &= 2^{O(\epsilon(1+\log n)^k)} \prod_{1\le l\le \frac{1}{\epsilon}} P_k(l)^{(\frac{1}{l}-\frac{1}{l+1})\frac{\phi(D)}{D}+\frac{O(D)}{n}} \\ &= 2^{O(\epsilon(1+\log n)^k+\frac{D(1+\log 1/\epsilon)^k}{\epsilon n})} \prod_{1\le l\le \frac{1}{\epsilon}} P_k(l)^{(\frac{1}{l}-\frac{1}{l+1})\frac{\phi(D)}{D}}. \end{aligned}$$

Here we used the bound $P_k(l) \le 2^{(1+\log l)^k} \le 2^{(1+\log 1/\epsilon)^k}$ again. Finally, we choose $\epsilon = \frac{1}{\sqrt{n}}$. By making $n\to\infty$, the error terms go to zero, and the number of terms in the product goes to infinity, so in order to prove that $\lim_{n\to\infty} f(n,k)^{1/n}$ exists it is enough to show that

$$\prod_{l=1}^{\infty} P_k(l)^{(\frac{1}{l}-\frac{1}{l+1})\frac{\phi(D)}{D}}$$

is a convergent product (and the limit will be equal to this product). Indeed, by the same bound for $P_k(x)$ as before, it is enough to prove that $\sum_{l=1}^{\infty}\frac{(1+\log l)^k}{l(l+1)}$ is convergent, which is true. Hence the proof is complete. □

Unfortunately our attempts up to now have failed to find the value of α (in some reasonable sense). This solution essentially reduces the original limit to a "smoothed" version of itself, in terms of the P_k, which is guaranteed to converge - but because we don't know much else about the P_k, attempts to find the limit end up being circular. It is also amusing to notice that if one looks only at the infinite product formula we found for α_k, it is not obvious that these form a decreasing sequence. One needs the "combinatorial" argument from part 1 to establish that, and this seems to be a considerable barrier to making sense out of the limit of α_k through this formula.

Reference

[1] Cameron, P.J; Erdős, P. On the number of sets of integers with various properties, *Number Theory (Banff, AB, 1988)*, 61-79, de Gruyter, Berlin, 1990.