ON ℓ-TH ORDER GAP BALANCING NUMBERS

S. S. Rout
Institute of Mathematics and Applications, Bhubaneswar, Odisha, India
lbs.sudhansu@gmail.com; sudhansu@iomaorissa.ac.in

R. Thangadurai
Harish-Chandra Research Institute, HBNI, Allahabad, Uttar Pradesh, India
thanga@hri.res.in

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Abstract

In this article, we prove the finiteness of the positive integral solutions of

$$1^\ell + 2^\ell + \cdots + \left(x - \frac{k+1}{2}\right)^\ell = \left(x + \frac{k+1}{2}\right)^\ell + \cdots + (y-1)^\ell$$

for any given integer $\ell > 1$ and an odd integer $k \geq 1$. Moreover, when $k = 3$ and $\ell = 2, 3$ and 5, we explicitly compute the positive integral solutions using the elliptic logarithmic method.

1. Introduction

Let $k \geq 1$ be an odd positive integer and $\ell \geq 1$ be given integer. We say a positive integer x is a *k-gap balancing number of order ℓ*, if there exists an integer $y \geq x + (k+3)/2$ such that

$$1^\ell + 2^\ell + \cdots + \left(x - \frac{k+1}{2}\right)^\ell = \left(x + \frac{k+1}{2}\right)^\ell + \cdots + (y-1)^\ell. \tag{1}$$

This was introduced in the paper of Rout and Panda [10]. In the literature, the classical case $k = 1$ and $\ell = 1$ was introduced in [1], and in this case x is called a *balancing number*. Further, in [4], the case $k = 1$ and $\ell > 1$ was considered and the following conjecture was stated.

Conjecture 1. For any given integer $y \geq 2$, the integer $x = 1$ is the only positive integral solution of the equation

$$1^\ell + 2^\ell + \cdots + (x-1)^\ell + x^\ell = x^\ell + (x+1)^\ell + \cdots + (y-1)^\ell. \tag{2}$$

If $\ell = 2$, solving (2) is equivalent to solving the Thue equations $m^3 + 2n^3 = 11$ or 33. But the only integral solution of $m^3 + 2n^3 = 11$ is $(m, n) = (3, -2)$ and therefore the only positive integral solution of (2) is $(y, x) = (2, 1)$ which is trivial. Also, $m^3 + 2n^3 = 33$ has no integral solution. Thus, altogether for $\ell = 2$, Conjecture 1 is true. Conjecture 1 is also true for $\ell = 3$ [11] and $\ell = 5$ [5]. Using a result of Bilu and Tichy [2] on the Diophantine equation $f(x) = g(y)$, for a fixed integer $\ell > 1$, Ingram [5] proved that the equation (2) has at most finitely many solutions by keeping x and y as unknowns with $x \le y - 1$.

The balancing numbers are further generalized as follows. For fixed positive integers p and q, we call a positive integer x $(\le y - 2)$ *a (p, q)-balancing number* if

$$1^p + 2^p + \cdots + (x-1)^p = (x+1)^q + \cdots + (y-1)^q \tag{3}$$

holds for some natural number y. Liptai et al. [7] proved the finiteness of the number of solutions of (3).

For an arbitrary odd integer $k \ge 1$ and $\ell = 1$, (1) has some classes of solutions which were given in [10]. In this article, we study the explicit positive integral solutions of (1) with $k = 3$ and $\ell = 2, 3$ and 5. First, we shall prove the general case as follows.

Theorem 1. *For fixed $\ell > 1$ and an odd positive integer $k \ge 1$, (1) has finitely many positive integral solutions.*

We prove Theorem 1 along the same line of proof given in [7]. Moreover, using the explicit lower bound for linear forms in elliptic logarithms given in [3], we construct rational points of certain elliptic curves and prove the following results.

Theorem 2.

1. *If $k = 3$ and $\ell = 2$ then (1) has precisely one solution, namely, $(x, y) = (44, 56)$.*
2. *The equation (1) has no positive integral solution, when $k = 3$ and $\ell = 3$.*
3. *The equation (1) has no positive integral solution, when $k = 3$ and $\ell = 5$.*

Theorem 2, part 1 asserts that the analogue of Conjecture 1 is not true for (1) with $k = 3$ and $\ell = 2$.

In the proof of Theorem 2, we use the notations as in [13].

2. Proof of Theorem 1

For any integer $\ell \ge 1$, let us denote

$$S_\ell(x) = 1^\ell + 2^\ell + \cdots + (x-1)^\ell. \tag{4}$$

Then note that the polynomial $S_\ell(x) \in \mathbb{Q}[x]$ and is of degree $\ell+1$ whose leading coefficient is $1/(\ell+1)$. It is known that for an odd integer $\ell > 1$, we can express

$$S_\ell(x) = \psi_\ell\left(\left(x - \frac{1}{2}\right)^2\right)$$

for some polynomial $\psi_\ell(x) \in \mathbb{Q}[x]$ of degree $(\ell+1)/2$.

To prove Theorem 1, we need the following result of Rakaczki [8].

Theorem 3. *Let m be a positive integer. For any polynomial $g(x) \in \mathbb{Q}[x]$ of degree ≥ 2, the Diophantine equation*

$$S_m(x) = g(y)$$

has finitely many integer solutions in x and y, unless $(m, g(x))$ is a special pair where all the 7 types of special pairs are defined in [8].

Proof of Theorem 1. Let $\ell > 1$ be a fixed integer and $k \geq 1$ be an odd positive integer. Then we want to prove that

$$1^\ell + 2^\ell + \cdots + \left(x - \frac{k+1}{2}\right)^\ell = \left(x + \frac{k+1}{2}\right)^\ell + \ldots + (y-1)^\ell \tag{5}$$

has finitely many integer solutions. First we rewrite (5) as

$$\begin{aligned}\left(x + \frac{k+1}{2}\right)^\ell + \cdots + (y-1)^\ell &= 1^\ell + 2^\ell + \cdots + (y-1)^\ell - \left(1^\ell + \cdots + \left(x + \frac{k-1}{2}\right)^\ell\right) \\ &= S_\ell(y) - S_\ell\left(x + \frac{k+1}{2}\right).\end{aligned}$$

Therefore, (1) becomes

$$S_\ell\Big(x - \frac{k-1}{2}\Big) + S_\ell\Big(x + \frac{k+1}{2}\Big) = S_\ell(y). \tag{6}$$

Let

$$\begin{aligned} g(x) &:= 2S_\ell\Big(x - \frac{k-1}{2}\Big) + \Big(x - \frac{k-1}{2}\Big)^\ell + \cdots + \Big(x + \frac{k-1}{2}\Big)^\ell \\ &= S_\ell\Big(x - \frac{k-1}{2}\Big) + S_\ell\Big(x + \frac{k+1}{2}\Big). \end{aligned} \tag{7}$$

Clearly, $g(x)$ is a polynomial in $\mathbb{Q}[x]$ of degree $\ell+1$. Also, by the definition of $g(x)$, it is clear that the degrees of all terms in $g(x)$ except the first term is ℓ.

With these notations, it is now enough to prove that there are only finitely many integer solutions to the equation $S_\ell(y) = g(x)$.

On the contrary, we assume that the equation $S_\ell(y) = g(x)$ has infinitely many integer solutions. Then, by Theorem 3, we must have the pair $(\ell, g(x))$ as one of the 7 special pairs described in [8].

Type I. $g(x) = S_\ell(q(x))$ for some non-constant polynomial $q(x) \in \mathbb{Q}[x]$.

In this case, by comparing the degrees from both sides, we get $q(x) = ux+v$ where $u, v \in \mathbb{Q}$ with $u \neq 0$. Then the leading coefficient of $S_\ell(q(x))$ is $\frac{u^{\ell+1}}{\ell+1}$. However, the leading coefficient of $g(x)$ is $\frac{2}{\ell+1}$. Hence, $u^{\ell+1} = 2$, which implies that $u = 2$ and $\ell = 0$. This is a contradiction to $\ell > 1$.

Type II. ℓ is an odd integer, $g(x) = \psi_\ell(\delta(x)q(x)^2)$ for some non-zero polynomial $q(x) \in \mathbb{Q}[x]$, and $\delta(x) \in \mathbb{Q}[x]$ is a linear polynomial.

By comparing the degrees, in this case, we conclude that degree of $\delta(x)q(x)^2$ is equal to 2. But this is not possible as $\delta(x)$ is a linear polynomial.

Type III. ℓ is an odd integer and $g(x) = \psi_\ell(c\delta(x)^t)$ for some $c \in \mathbb{Q}\backslash\{0\}$, and $t \geq 3$ is an odd integer.

In this case, the degree of $\psi_\ell(c\delta(x)^t)$ is greater than $(t(\ell+1))/2$. Since $(t(\ell+1))/2 > \ell+1$, it follows that the degree of $g(x)$ is greater than $\ell+1$, which is a contradiction.

Type IV. ℓ is an odd integer and $g(x) = \psi_\ell((a\delta(x)^2+b)q(x)^2)$ with $a, b \in \mathbb{Q}\backslash\{0\}$.

Here we see, by comparing degrees, that δ is linear (say $\delta = ux+v$) and $q(x) = q$, a constant (non-zero). The leading coefficient of $\psi_\ell((a\delta(x)^2+b)q(x)^2)$ is $(au^2q^2)^{(\ell+1)/2}$. Hence by comparing the leading coefficients, we get

$$(au^2q^2)^{(\ell+1)/2} = 2 \text{ implies } \ell \leq 1,$$

a contradiction to $\ell > 1$.

Type V. ℓ is an odd integer and $g(x) = \psi_\ell(q(x)^2)$.

By comparing the degrees, we conclude that $q(x) = ux+v$ is a linear polynomial with $u \neq 0$. Note that the leading coefficient of $\psi_\ell(q(x)^2)$ is $u^{\ell+1}/(l+1)$. As the leading coefficient of $g(x)$ is $2/(\ell+1)$, we have

$$u^{\ell+1} = 2 \text{ implies } u = 2 \text{ and } \ell = 0,$$

a contradiction to $\ell > 1$.

Type VI. $\ell = 3$ and $g(x) = \delta(x)q(x)^2$.

Since the degree of $S_3(x)$ is 4 and $\delta(x)$ is a linear polynomial, we get a contradiction, by comparing the degrees of both sides.

Type VII. $\ell = 3$ and $g(x) = q(x)^2$.

In this case, $q(x)$ must be a quadratic polynomial, say, $q(x) = ux^2 + vx + w \in \mathbb{Q}[x]$. Therefore, by comparing the leading coefficients, we get $u^2 = 2$, which is a contradiction to $u \in \mathbb{Q}$.

This proves that (6) has only finitely many integer solutions. □

3. Proof of Theorem 2

`Case 1.` ($k = 3$ and $\ell = 2$).

In this case, we completely solve the following equation

$$1^2 + 2^2 + \cdots + (x-2)^2 = (x+2)^2 + \cdots + y^2, \tag{8}$$

over integers. Indeed, we shall prove the following statement: *The equation* (8) *has only one integral solution* $(x, y) = (44, 55)$. Equivalently, we have only one relation of the form

$$1^2 + 2^2 + \cdots + 42^2 = 46^2 + \cdots + 55^2. \tag{9}$$

The idea is to use the well-known formula

$$\sum_{r=1}^{a} r^2 = \frac{a(a+1)(2a+1)}{6}$$

in (8) to convert it into an elliptic curve and look for an integral point of that elliptic curve. By applying this formula in (8), after writing (8) as

$$1^2+\cdots+(x-2)^2 = (1^2+2^2+\cdots+(x+1)^2)+(x+2)^2+\ldots+y^2-(1^2+2^2+\cdots+(x+1)^2),$$

and by using the change of variable $u := 2x$ and $v := 2y + 1$, we arrive at the cubic equation

$$2u^3 + 52u = v^3 - v. \tag{10}$$

Note that the asymptote of (10) is $v = \alpha u$, where α is a real cube root of 2. Now, using *Magma*, we see that the transformation

$$X = \frac{-2u + 2704v}{52u + v}, \quad Y = -\frac{140610}{52u + v} \tag{11}$$

yields a minimal model for (10), i.e.,

$$E: \ \ Y^2 = X^3 + 312X - 281212. \tag{12}$$

One can compute the Mordell-Weil group for the elliptic curve given in (12) and the group is $E(\mathbb{Q}) \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$. By Mordell's theorem, this group is generated by the points $P_1 = (64, 30), P_2 = (76, 426)$ and $P_3 = (88, 654)$.

Since the point at infinity of this group is the limit point of points on (12) arising from the asymptote of (10), we let $Q_0 = (X_0(\alpha), Y_0(\alpha))$ as the point at infinity. Therefore,

$$\begin{aligned} X_0(\alpha) &= \lim_{(u,v)\to\infty} \frac{-2u + 2704v}{52u + v} \\ &= \frac{-2 + 2704\alpha}{52 + \alpha} \\ &= a + b\alpha + c\alpha^2 \text{ (say).} \end{aligned}$$

Then, we get $a = 0, b = 52$ and $c = -1$, and hence $X_0(\alpha) = -\alpha^2 + 52\alpha$. By similar calculation, we get $Y_0(\alpha) = 0$.

Let $P \in E$ be an arbitrary point on E with integer co-ordinates u and v on (10). Since $E(\mathbb{Q})$ is generated by the points P_1, P_2 and P_3, we write

$$P = m_1P_1 + m_2P_2 + m_3P_3 \text{ where } m_1, m_2, m_3 \in \mathbb{Z}.$$

Our aim is to get an upper bound for these m_i's. By letting $M = \max\{|m_1|, |m_2|, |m_3|\}$, we need to find the upper bound for M. One way to get an upper bound for M is to apply the elliptic logarithmic method.

From (10), we get

$$\frac{d(u(v))}{dv} = \frac{3v^2 - 1}{6u^2 + 52}.$$

For $v \geq 10, u(v)$ given by (10) can be viewed as a strictly increasing function of v. Now, using Lemma 2 of [13], it follows from (11) that

$$\frac{dv}{6u^2 + 52} = \frac{1}{2}\frac{dX}{Y}. \tag{13}$$

Therefore, we get

$$\int_v^\infty \frac{dv}{6u^2 + 52} = \frac{1}{2}\int_X^{X_0(\alpha)} \frac{dX}{Y}. \tag{14}$$

One can observe from (10) that $6u^2 + 52 > 3v^2$ for $v \geq 10$. Therefore, we have

$$\int_v^\infty \frac{dv}{6u^2 + 52} < \frac{1}{3}\int_v^\infty \frac{dv}{v^2} = \frac{1}{3v}. \tag{15}$$

We let

$$L(P) = -\int_X^{X_0(\alpha)} \frac{dX}{Y}.$$

The main idea is to get the upper and lower bound for $|L(P)|$ involving M. Indeed,

$$-\int_X^{X_0(\alpha)} \frac{dX}{Y} = \int_{X_0(\alpha)}^X \frac{dX}{Y} = \int_{X_0(\alpha)}^\infty \frac{dX}{Y} - \int_X^\infty \frac{dX}{Y} = \omega(\phi(Q_0) - \phi(P))$$

where $\omega = 0.302283\ldots$ is the real part of the fundamental period of $E(\mathbb{C})$ and ϕ is the elliptic logarithm. Thus, we get

$$L(P) = \omega(\phi(Q_0) - \phi(P)).$$

Also,

$$\phi(P) = \phi(m_1P_1 + m_2P_2 + m_3P_3) = m_1\phi(P_1) + m_2\phi(P_2) + m_3\phi(P_3) + m_0$$

with $m_0 \in \mathbb{Z}$. Since $\phi(P_i) \in [0, 1)$, we see that

$$|m_0| \leq |\phi(P) - (m_1\phi(P_1) + m_2\phi(P_2) + m_3(P_3))| \leq 1 + 3M.$$

Let $u_0 = \omega\phi(Q_0), u_1 = \omega\phi(P_1), u_2 = \omega\phi(P_2)$ and $u_3 = \omega\phi(P_3)$. Then, by using Zagier's algorithm [14], we arrive at $u_1 = 0.297514\ldots, u_2 = 0.242130\ldots, u_3 = 0.219679\ldots$ and $u_0 = \omega\phi(Q_0) = 0.151141\ldots$. Since

$$\begin{aligned} L(P) = \int_{X_0}^{X} \frac{dX}{Y} &= \omega(\phi(Q_0) - \phi(P)) \\ &= u_0 - m_1u_1 - m_2u_2 - m_3u_3 - m_0\omega, \end{aligned}$$

by (14) and (15), we have

$$|L(P)| = 2\int_{v}^{\infty} \frac{dv}{6u^2 + 52} < \frac{2}{3v}. \tag{16}$$

Since our aim to get the upper bound as a function of M, we need to replace v by a function of M. Since the Néron-Tate height has the lower bound involving M as

$$\hat{h}(P) \geq c_1M^2 \tag{17}$$

where $c_1 = 0.3776939\ldots$ is the least eigenvalue of the Néron-Tate height pairing matrix, we use this height calculation as follows. First note that since $v = \alpha u$ is the asymptote, we see that $\alpha u - v < 0.002$ for all $v \geq 10$ and hence $u < (0.002 + v)/\alpha$. Now we get an upper bound for the Weil height $h(P) := h(X(P))$ as follows:

$$\begin{aligned} h(X(P)) &= \log\max\{|-2u + 2704v|, |52u + v|\} \leq \log(|2u + 2704v|) \\ &\leq \log\left(2\left(\frac{0.002 + |v|}{|\alpha|}\right) + 2704|v|\right) \\ &\leq \log\left(\left(\frac{2}{\alpha} + 2704\right)|v| + \frac{0.004}{\alpha}\right) \\ &\leq 7.906249 + \log|v|. \end{aligned} \tag{18}$$

Silverman [9] proved that

$$2\hat{h}(P) - h(P) < 8.7926994.$$

Using (17) and this difference, we get

$$h(P) > 2c_1M^2 - 8.7926994. \tag{19}$$

From (17), (18) and (19), we conclude that

$$-\log|v| < 16.698948 - 0.7553879M^2. \tag{20}$$

Therefore, by (16), we get

$$|L(P)| < \exp(16.293483 - 0.7553879M^2). \tag{21}$$

David [3] proved the general lower bound of $|L(P)|$ as follows:

$$L(P) > \exp(-c_4(\log M' + c_5))(\log\log M' + c_6)^6) \tag{22}$$

where $M' := 3M + 1, c_4 := 9 \times 10^{158}, c_5 := 1.69315$ and $c_6 := 21.81715$. Now, by comparing (21) and (22), we arrive at an upper bound $M_0 = 6.9 \times 10^{82}$ for M. To find all solutions below this large bound, we use the reduction procedure based on the LLL-algorithm [12] and we get the reduced new bound for M which is $M = 11$. A direct computer search finds that the only points $P = m_1P_1 + m_2P_2 + m_3P_3$ with $|m_i| \leq 11$ and $(u(P), v(P))$ on (10) are listed in the following table:

m_1	m_2	m_3	$X(P)$	$Y(P)$	$u(P)$	$v(P)$
-1	0	0	64	-30	88	111
0	0	-1	88	-654	4	7
0	0	1	88	654	-4	-7
1	0	0	64	30	-88	-111

Note that when $u = 88$ and $v = 111$, we get $x = 44$ and $y = 55$. In this case, we have a solution for (8). When $u = 4$ and $v = 7$, the corresponding $x = 2$ and $y = 3$ violate the condition $y \geq x + 2$. In rest of the cases x or y may not be positive integers. Hence, $(x, y) = (44, 55)$ is the only positive integral solution for (8).

Case 2. ($k = 3$ and $\ell = 3$).

In this case, we study the Diophantine equation

$$1^3 + 2^3 + \cdots + (x-2)^3 = (x+2)^3 + \cdots + y^3 \tag{23}$$

for $y \geq x + 2$ and we show that this equation has no solution. Let x be a 3-gap balancing number of order 3. Since the sum of the cubes of the first n positive integers is given by

$$\sum_{m=1}^{n} m^3 = \Big[\frac{n(n+1)}{2}\Big]^2,$$

from (23), we have

$$\left[\frac{(x-2)(x-1)}{2}\right]^2 + \left[\frac{(x+1)(x+2)}{2}\right]^2 = \left[\frac{y(y+1)}{2}\right]^2. \tag{24}$$

Simplifying (24) and putting $u = 2x$ and $v = 2y(y+1)$, we get

$$2v^2 = u^4 + 52u^2 + 64. \tag{25}$$

Multiplying by 2 and setting $Y = 2v$ and $X = u$, we get the quartic elliptic curve as follows:

$$E: \quad Y^2 = 2X^4 + 104X^2 + 128. \tag{26}$$

Using *Magma*, we find the following set of integral points on E:

$$(X, Y) = (\pm 2, \pm 24), (\pm 4, \pm 48), (\pm 14, \pm 312), (\pm 68, \pm 6576).$$

Now, note that the point $(X, Y) = (2, 24)$ (respectively $(4, 48)$) gives the integral solution $(x, y) = (1, 2)$ (respectively $(2, 3)$). However, this is contradicting to the condition that $y \geq x+2$. If $(X, Y) = (14, 312)$ (respectively $(68, 6576)$), then $y \notin \mathbb{Z}$. Thus, we conclude that there is no positive integral solution of (23).

`Case 3.` ($k = 3$ and $\ell = 5$).

In this case, we study the Diophantine equation

$$1^5 + 2^5 + \cdots + (x-2)^5 = (x+2)^5 + \cdots + y^5. \tag{27}$$

In particular, we show that equation (27) has no solution in positive integers. Let x be a 3-gap balancing number of order 5. By the well-known formula

$$\sum_{i=1}^{n} i^5 = \frac{n^2(n+1)^2(2n^2+2n-1)}{12},$$

and setting $u := 4x^2$ and $v := (2y+1)^2$, we get

$$2u^3 + 140u^2 + 944u = v^3 - 5v^2 + 7v - 387. \tag{28}$$

We shift the v-coordinate using (9) to get a better model. As this transformation preserves the integrality of the points, it will not affect our result. Hence writing $v := v + 9$, we get

$$2u^3 + 140u^2 + 944u = v^3 + 22v^2 + 160v. \tag{29}$$

Using *Magma*, we find that the birational transformations

$$\begin{aligned} X &= \frac{14363u + 11354v + 250328}{59u - 10v}, \\ Y &= -4\left(\frac{52187488u + 6179535u^2 + 24345398v + 1650669v^2}{(59u-10v)^2}\right) \end{aligned} \tag{30}$$

and

$$\begin{aligned} u &= \frac{-3481X^3 + 3481X^2 + 3481Y^2 + 101423033X - 167971860617}{31291X^2 - 6855974X + 813516Y + 55912827099}, \\ v &= -8\left(\frac{26093744X + 1846169Y + 6757133452}{31291X^2 - 6855974X + 813516Y + 55912827099}\right) \end{aligned} \tag{31}$$

relate (29) and the minimal model

$$E: \; Y^2 = X^3 - X^2 - 1161X + 32535321. \tag{32}$$

Using *Magma*, we see that the torsion subgroup of the Mordell-Weil group of $E(\mathbb{Q})$ is trivial and its rank is 3 with generators, namely, $P_1 = (-\frac{968}{9}, \frac{151307}{27}), P_2 = (-67, 5684)$ and $P_3 = (7523, 652484)$.

We apply the same method as discussed in Case 1. The asymptote of (29) is $v = \alpha u + \beta$ where α is a real cube root of 2 and $\beta = \frac{11\alpha^2 - 70}{80}$. Let $Q_0 = (X_0, Y_0)$ be the point at infinity of E. Then, we can compute $X_0 = 40\alpha^2 + 236\alpha + 257$ and $Y_0 = -2596\alpha^2 - 2800\alpha - 8700$. Also, by (29), we view u as a strictly increasing function of v when $u > 4$ and $v > -6.6$, or $u < -8$ and $v < -44$. For each such point (u, v), there is a unique point $(X, Y) \in \mathbb{R}^2$ with $Y \geq 0$. We assume that $v \geq 10^2$ for the rest of the calculation. In this case, we see that $6u^2 + 280u + 944 > 3v^2$. Thus,

$$\int_v^\infty \frac{dv}{6u^2 + 280u + 944} < \frac{1}{3} \int_v^\infty \frac{dv}{v^2} = \frac{1}{3v}. \tag{33}$$

Let $P = m_1P_1 + m_2P_2 + m_3P_3$ be an arbitrary point on E with integral coordinates u, v on (29). Thus, the linear form

$$\begin{aligned} L(P) = \int_{X_0}^X \frac{dX}{Y} &= \omega(\phi(Q_0) - \phi(P)) \\ &= u_0 - m_0\omega - m_1u_1 - m_2u_2 - m_3u_3, \end{aligned}$$

where $u_0 = 0.235708\ldots, u_1 = 0.176128\ldots, u_2 = 0.168950\ldots, u_3 = 0.023059\ldots$ and $\omega = 0.4714160\ldots$ is the real part of the fundamental period of $E(\mathbb{C})$. In order to get the upper bound for $|L(P)|$, we proceed with height calculations. For $v \geq 10^2$, it is easy to verify that

$$h(X(P)) \leq 10.1399 + \log|v|. \tag{34}$$

Moreover, we have

$$\hat{h}(P) \geq c_1M^2 \tag{35}$$

where $c_1 = 1.538118\ldots$ is the least eigenvalue of the Néron-Tate height pairing matrix and $M = \max_{1 \leq i \leq 3} |m_i|$. Also, the Silverman's bound for the heights on elliptic curve gives that

$$2\hat{h}(P) - h(p) < 12.0302566. \tag{36}$$

Now, using Lemma 2 of [13] and (29), we get

$$\frac{dv}{6u^2 + 280u + 944} = \frac{1}{4}\frac{dX}{Y}. \tag{37}$$

By (37), for $v \geq 10^2$, we have

$$\int_v^\infty \frac{dv}{6u^2 + 280u + 944} = \frac{1}{4} \int_X^{X_0} \frac{dX}{Y}. \tag{38}$$

Thus, using (33),(34),(35) and (36), we get the upper bound as

$$L(P) < 22.457837 - 3.076238M^2. \tag{39}$$

As $|m_0| \leq 3M + 1$, we obtain the lower bound ([3])

$$L(P) > \exp(-c_4(\log M' + c_5))(\log\log M' + c_6)^6), \tag{40}$$

where $M' := 3M + 1, c_4 := 6 \times 10^{159}, c_5 := 1.69315$ and $c_6 := 33.3467$. Together with the upper bound (39), this lower bound yields an absolute upper bound $M_0 = 3.575 \times 10^{85}$ for M. To find all solutions below this large bound, we use the reduction procedure based on the LLL-algorithm [12]. Using this procedure, we have further reduced our bound to $M_0 = 7$.

The direct computation reveals that for all integral points $P = m_1P_1 + m_2P_2 + m_3P_3$ for $|m_i| \leq 7$ with $(X(P), Y(P))$ on (32) gives

$$P = \left(\frac{335962176073}{979126681}, -\frac{260722045187502036}{30637852975171}\right)$$

for $m_1 = m_2 = m_3 = 1$. The corresponding integral point on (29) is $(u, v) = (0, 0)$ which is a trivial point and hence we conclude that there is no integer solution to (27). This completes the proof of the theorem. □

4. Concluding Remarks

Considering the equation $1^2 + 2^2 + \cdots + (x-2)^2 = (x+2) + (x+3) + \cdots + y$, and using the elliptic logarithmic method (the method used in the proof of Theorem 2), one can prove that

$$(x, y) = (7, 13), (8, 16), (54, 315), (182, 1988), (253, 3266), (1338, 39916)$$

are the only positive integral solutions.

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