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RUTH-AARON PAIRS CONTAINING RIESEL OR SIERPIŃSKI NUMBERS

Elliot R. Emadian
Department of Dance, University of Illinois, Urbana, Illinois

Carrie E. Finch-Smith
Department of Mathematics, Washington and Lee University, Lexington, Virginia
finchc@wlu.edu

Margaret G. Kallus
Department of Mathematics, Washington and Lee University, Lexington, Virginia

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Abstract

In this paper, we construct a Ruth-Aaron pair that contains a Riesel number. We also construct a Ruth-Aaron pair that contains a Sierpiński number. Finally, we construct a Ruth-Aaron pair with the smaller number in the pair a Riesel number and the larger number in the pair a Sierpiński number.

1. Introduction

On April 8, 1974, Hank Aaron hit his 715th career home run in Atlanta, Georgia, breaking Babe Ruth's longstanding record of 714 career home runs. Soon after, mathematicians at the University of Georgia noticed that 714 and 715 were an interesting pair of numbers. We have the factorizations

$$2 \cdot 3 \cdot 7 \cdot 17 = 714 \quad \text{and} \quad 5 \cdot 11 \cdot 13 = 715.$$

We see that the sum of the factors is the same for these numbers:

$$2 + 3 + 7 + 17 = 29 \quad \text{and} \quad 5 + 11 + 13 = 29.$$

From this, Ruth-Aaron pairs are defined. For a positive integer n with prime factorization $n = p_1^{e_1} \cdot p_2^{e_2} \cdots p_i^{e_i}$, we define $S(n) = e_1p_1 + e_2p_2 + \cdots + e_ip_i$. If $S(n) = S(n+1)$, then n and $n+1$ form a Ruth-Aaron pair.

Nelson, Penney, and Pomerance [1] showed that Ruth-Aaron pairs can be constructed using particular values of the polynomials $f(k)$ and $f(k)+1$ shown below:

$$\begin{aligned} f(k) &= 384k^3+432k^2+112k-5 = (8k+5)(48k^2+24k-1) \\ f(k)+1 &= 384k^3+432k^2+112k-4 = 4(2k+1)(48k^2+30k-1). \end{aligned}$$

If all of the factors

$$\begin{aligned} p(k) &= 8k+5, & q(k) &= 48k^2+24k-1, \\ r(k) &= 2k+1, \quad \text{and} & s(k) &= 48k^2+30k-1 \end{aligned}$$

are prime for a particular value of k, then we have

$$S(f(k)) = p(k)+q(k) = 4+r(k)+s(k) = S(f(k)+1).$$

That is, for such a value of k, $f(k)$ and $f(k)+1$ form a Ruth-Aaron pair.

In 1956, H. Riesel [2] showed that there are no prime numbers in the set $\{k\cdot 2^n-1 : n \in \mathbf{N}\}$ for any $k \equiv 509203 \pmod{1184810}$. These investigations give rise to the following definition: an odd positive integer k is called a *Riesel number* if $k\cdot 2^n-1$ is composite for all natural numbers n. Riesel's original example 509203 is the smallest known Riesel number. Currently, there are 49 remaining candidates smaller than 509203 that have not been ruled out as Riesel numbers. See `primegrid.com` for the most up-to-date information.

In a similar manner, W. Sierpiński [3] showed in 1960 that there are infinitely many odd positive integers k with the property that $k \cdot 2^n + 1$ is composite for all positive integers n. Such an integer k is called a *Sierpiński number* in honor of Sierpiński's work. Two years later, J. Selfridge (unpublished) showed that 78557 is a Sierpiński number. To this day, this is the smallest known Sierpiński number, and it is conjectured to be the smallest Sierpiński number. As of this writing, there are five candidates smaller than 78557 to consider: 21181, 22699, 24737, 55459, 67607. The most current progress on this conjecture can be found at `seventeenorbust.com`.

A natural extension of questions related to Riesel and Sierpiński numbers is to search for numbers with both properties. There are infinitely many odd positive integers k with the property that *both* $k \cdot 2^n + 1$ and $k \cdot 2^n - 1$ are composite for all natural numbers n. The smallest known example of a number that is both Riesel and Sierpiński has 22-digits: 3316923598096294713661. For details, see `oeis.org/A076335/a076335.txt`.

In the remainder of this paper, we use the polynomials constructed by Nelson and his coauthors to form Ruth-Aaron pairs that contain a Riesel number, a Sierpiński number, or both. We suppress the requirement that k be odd when constructing Sierpiński and Riesel numbers where necessary since one of the members of each Ruth-Aaron pair must be even.

2. Ruth-Aaron Pairs with Riesel or Sierpiński Numbers

2.1. $f(k)$ Riesel

In this section, we construct a Ruth-Aaron pair in which the smaller integer is also a Riesel number. Let $f(k) = 384k^3 + 432k^2 + 112k - 5$ denote the first number in a Ruth-Aaron pair. To make this number a Riesel number, we want to ensure that $f(k) \cdot 2^n - 1$ is composite for all positive integers n. Consider the following implications:

$$\begin{array}{rlclcrl}
n \equiv 0 & \pmod 2 & \& & k \equiv 0 \pmod 3 & \Longrightarrow & 3 \mid f(k)\cdot 2^n - 1 \\
n \equiv 1 & \pmod 3 & \& & k \equiv 4 \pmod 7 & \Longrightarrow & 7 \mid f(k)\cdot 2^n - 1 \\
n \equiv 1 & \pmod 4 & \& & k \equiv 1 \pmod 5 & \Longrightarrow & 5 \mid f(k)\cdot 2^n - 1 \\
n \equiv 7 & \pmod 8 & \& & k \equiv 6 \pmod{17} & \Longrightarrow & 17 \mid f(k)\cdot 2^n - 1 \\
n \equiv 3 & \pmod{12} & \& & k \equiv 9 \pmod{13} & \Longrightarrow & 13 \mid f(k)\cdot 2^n - 1 \\
n \equiv 11 & \pmod{24} & \& & k \equiv 75 \pmod{241} & \Longrightarrow & 241 \mid f(k)\cdot 2^n - 1.
\end{array}$$

The congruences for n in the above list form a covering. If a Ruth-Aaron pair $f(k)$ and $f(k)+1$ is constructed using a k-value that satisfies all of the congruences for k, then $f(k)\cdot 2^n - 1$ will be divisible by one of the primes in the set $\{3, 5, 7, 13, 17, 241\}$. We find such k using the Chinese Remainder Theorem, and the value of k we obtain is $4263606 \pmod{3\cdot 5\cdot 7\cdot 13\cdot 17\cdot 241}$. Finally, we find a value of k in this arithmetic progression that makes all four of the factors $p(k)$, $q(k)$, $r(k)$, and $s(k)$ prime, and the smallest such k is

$$222367401 = 4263606 + 39(3\cdot 5\cdot 7\cdot 13\cdot 17\cdot 241).$$

Thus, when this value is substituted for k in $f(k)$, the resulting value is a Riesel number that is the first member of a Ruth-Aaron pair. That is, the 28-digit integers

$$4222256247122218429505668123 \text{ and } 4222256247122218429505668124$$

form a Ruth-Aaron pair in which the smaller number is Riesel.

2.2. $f(k) + 1$ Sierpiński

Next, we construct a Ruth-Aaron pair in which the larger of the two numbers is Sierpiński. We again use the polynomials $f(k)$ and $f(k)+1$ constructed by Nelson, Penney, and Pomerance [1]. Thus, we want to ensure that $(f(k)+1)\cdot 2^n + 1$ is composite for all positive integers n. Consider the following implications:

$$\begin{array}{llllllll}
n \equiv & 0 & \pmod 2 & \& & k \equiv & 0 & \pmod 3 & \Longrightarrow & 3 \mid (f(k)+1)\cdot 2^n+1 \\
n \equiv & 1 & \pmod 3 & \& & k \equiv & 5 & \pmod 7 & \Longrightarrow & 7 \mid (f(k)+1)\cdot 2^n+1 \\
n \equiv & 1 & \pmod 4 & \& & k \equiv & 4 & \pmod 5 & \Longrightarrow & 5 \mid (f(k)+1)\cdot 2^n+1 \\
n \equiv & 3 & \pmod 8 & \& & k \equiv & 3 & \pmod{17} & \Longrightarrow & 17 \mid (f(k)+1)\cdot 2^n+1 \\
n \equiv & 11 & \pmod{12} & \& & k \equiv & 5 & \pmod{13} & \Longrightarrow & 13 \mid (f(k)+1)\cdot 2^n+1 \\
n \equiv & 15 & \pmod{24} & \& & k \equiv & 93 & \pmod{241} & \Longrightarrow & 241 \mid (f(k)+1)\cdot 2^n+1.
\end{array}$$

Since the congruences for n shown above form a covering of the integers, we can see that $f(k)+1$ is a Sierpiński number for all k satisfying all of the congruences for k. The values of such k are integers in the arithmetic progression 4184094 $(\mathrm{mod}\ 3\cdot 5\cdot 7\cdot 13\cdot 17\cdot 241)$, and the smallest such k which makes the factors of $f(k)$ and $f(k)+1$ prime is

$$300581559 = 4184094 + 53(3\cdot 5\cdot 7\cdot 13\cdot 17\cdot 241).$$

From this value of k, we obtain the 29-digit Ruth-Aaron numbers

$$10428413037356521507568175931 \text{ and } 10428413037356521507568175932,$$

which is the smallest known Ruth-Aaron pair in which the larger of the numbers is Sierpiński.

2.3. $f(k)$ Riesel and $f(k)+1$ Sierpiński

In this section, we construct a Ruth-Aaron pair given by $f(k)$ and $f(k)+1$ with the additional property that $f(k)$ is Riesel and $f(k)+1$ is Sierpiński. First, we see from the following implications that any integer k satisfying all of the conditions for k will ensure that $f(k)+1$ is Sierpiński:

$$\begin{array}{llllllll}
n \equiv & 0 & \pmod 2 & \& & k \equiv & 0 & \pmod 3 & \Longrightarrow & 3 \mid (f(k)+1)\cdot 2^n+1 \\
n \equiv & 1 & \pmod 3 & \& & k \equiv & 0,5 & \pmod 7 & \Longrightarrow & 7 \mid (f(k)+1)\cdot 2^n+1 \\
n \equiv & 3 & \pmod 4 & \& & k \equiv & 3 & \pmod 5 & \Longrightarrow & 5 \mid (f(k)+1)\cdot 2^n+1 \\
n \equiv & 5 & \pmod 8 & \& & k \equiv & 5,15 & \pmod{17} & \Longrightarrow & 17 \mid (f(k)+1)\cdot 2^n+1 \\
n \equiv & 9 & \pmod{12} & \& & k \equiv & 10 & \pmod{13} & \Longrightarrow & 13 \mid (f(k)+1)\cdot 2^n+1 \\
n \equiv & 17 & \pmod{24} & \& & k \equiv & 6 & \pmod{241} & \Longrightarrow & 241 \mid (f(k)+1)\cdot 2^n+1.
\end{array}$$

The congruences for n shown above form a covering of the integers, so we have constructed k so that $(f(k)+1)\cdot 2^n+1$ is composite for all natural numbers n. Next, we continue the construction by finding congruences to ensure that $f(k)$ is Riesel. To this end, we first note that the following congruences for k are identical to those shown above, so we obtain a partial covering of the integers as follows:

$$\begin{array}{llllllll} n \equiv 0 & \pmod{2} & \& & k \equiv & 0 & \pmod{3} & \Longrightarrow & 3 \mid f(k)\cdot 2^n - 1 \\ n \equiv 2 & \pmod{3} & \& & k \equiv & 0,5 & \pmod{7} & \Longrightarrow & 7 \mid f(k)\cdot 2^n - 1 \\ n \equiv 3 & \pmod{4} & \& & k \equiv & 3 & \pmod{5} & \Longrightarrow & 5 \mid f(k)\cdot 2^n - 1 \\ n \equiv 5 & \pmod{8} & \& & k \equiv & 5,15 & \pmod{17} & \Longrightarrow & 17 \mid f(k)\cdot 2^n - 1. \end{array}$$

The congruences for n do not form a covering of the integers. In particular, they do not cover the residue classes $n \equiv 1 \pmod{24}$ and $n \equiv 9 \pmod{24}$. Filling in these residue classes using the primes 13 and 241 is not possible since this would lead to incompatible congruences for k. So we expand the covering by instead including the primes 19, 37, 73, 109, and 38737. These primes are not particularly large, and they increase the least common multiple of the moduli in the covering for n to 72. The following congruences complete the covering for n to make $f(k)$ Riesel:

$$\begin{array}{llllllll} n \equiv 15 & \pmod{18} & \& & k \equiv & 13 & \pmod{19} & \Longrightarrow & 19 \mid f(k)\cdot 2^n - 1 \\ n \equiv 1 & \pmod{36} & \& & k \equiv & 28 & \pmod{37} & \Longrightarrow & 37 \mid f(k)\cdot 2^n - 1 \\ n \equiv 0 & \pmod{9} & \& & k \equiv & 41 & \pmod{73} & \Longrightarrow & 73 \mid f(k)\cdot 2^n - 1 \\ n \equiv 25 & \pmod{36} & \& & k \equiv & 105 & \pmod{109} & \Longrightarrow & 109 \mid f(k)\cdot 2^n - 1 \\ n \equiv 57 & \pmod{72} & \& & k \equiv & 82 & \pmod{433} & \Longrightarrow & 433 \mid f(k)\cdot 2^n - 1 \\ n \equiv 49 & \pmod{72} & \& & k \equiv & 11721 & \pmod{38737} & \Longrightarrow & 38737 \mid f(k)\cdot 2^n - 1. \end{array}$$

The smallest value of k that satisfies all of the congruences in this section for k so that both $f(k)$ is Riesel and $f(k)+1$ is Sierpiński and also ensures that $f(k)$ and $f(k)+1$ form a Ruth-Aaron pair is

$$k = 126646628050461389206593.$$

This yields the smallest known Ruth-Aaron pair with the first number a Riesel number and the second number a Sierpiński number. The smaller number in this pair is the 72-digit integer

780031458617255229218576736314327792661742649117243886505103839537747867.

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