NEW CONGRUENCES AND FINITE DIFFERENCE EQUATIONS FOR GENERALIZED FACTORIAL FUNCTIONS

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Abstract

The generalized factorial product sequences we study in this article are defined symbolically by $p_n(\alpha, R) = R(R+\alpha)\cdots(R+(n-1)\alpha)$ for any indeterminate R. This definition of the generalized factorial functions $p_n(\alpha, R)$ includes variants of the classical single and double factorial functions, multiple α-factorial functions, $n!_{(\alpha)}$, rising and falling factorial polynomials such as the Pochhammer symbol, and other well-known sequences as special cases. We use the rationality of the generalized h^{th} convergent functions, $\mathrm{Conv}_h(\alpha, R; z)$, to the infinite J-fraction expansions enumerating these generalized factorial product sequences first proved by the author in 2017 to construct new congruences and h-order finite difference equations for generalized factorial functions. Applications of the new results we prove in this article include new finite sums and congruences for the α-factorial functions, restatements of classical necessary and sufficient conditions of the primality of special integer subsequences and tuples, and new finite sums for the single and double factorial functions modulo integers $h \geq 2$.

1. Notation and Other Conventions in the Article

1.1. Notation and Special Sequences

Most of the special conventions in the article are consistent with the notation employed within the *Concrete Mathematics* reference, and the conventions defined in the introduction to the first articles [11, 12]. These conventions include particular notational variants for usage of *Iverson's convention*, $[i=j]_\delta \equiv \delta_{i,j}$, bracket notation for the Stirling numbers, $\left[{n \atop k}\right] = (-1)^{n-k}s(n,k)$ and $\left\{{n \atop k}\right\} = S(n,k)$, and notation for the generalized *r-order harmonic numbers*, $H_n^{(r)} = \sum_{k=1}^{n} k^{-r}$ where $H_n = H_n^{(1)}$ denotes the sequence of *first-order harmonic numbers*. We will use the convention of denoting the falling factorial function by $x^{\underline{n}} = x!/(x-n)!$, the rising factorial function as $x^{\overline{n}} = \Gamma(x+n)/\Gamma(x)$, or equivalently by the Pochhammer symbol, $(x)_n = x(x+1)(x+2)\cdots(x+n-1)$, when x is a non-negative natural number. Within the article the notation $g_1(n) \equiv g_2(n) \pmod{N_1, N_2, \ldots, N_k}$ is understood to mean that the congruence, $g_1(n) \equiv g_2(n) \pmod{N_j}$, holds modulo any of the bases, N_j, for $1 \leq j \leq k$. Finally, we adopt the convention that the natural numbers are indexed starting from zero: $\mathbb{N} := \{0, 1, 2, 3, \ldots\}$.

1.2. Mathematica Summary Notebook Document and Computational Reference Information

The article is prepared with a more extensive set of computational data and software routines released as open source software to accompany the examples and numerous other applications suggested as topics for future research and investigation within the article. It is highly encouraged, and expected, that the interested reader obtain a copy of the summary notebook reference and computational documentation prepared in this format to assist with computations in a multitude of special case examples cited as particular applications of the new results.

The prepared summary notebook file `https://goo.gl/KK5rNnmultifact-cfracs-summary.nb`, attached to this manuscript contains the working *Mathematica* code to verify the formulas, propositions, and other identities cited within the article [13]. Given the length of this and the first article, the *Mathematica* summary notebook included with this submission is intended to help the reader with verifying and modifying the examples presented as applications of the new results cited below. The summary notebook also contains numerical data corresponding to computations of multiple examples and congruences specifically referenced in several places by the applications given in the next sections of the article.

2. Introduction

2.1. Motivation

In this article, we extend the results from [11] providing infinite J-fraction expansions for the typically divergent *ordinary generating functions* (OGFs) of generalized factorial product sequences of the form

$$p_n\left(\alpha, R\right) := R(R+\alpha)(R+2\alpha) \times \cdots \times (R+(n-1)\alpha)\left[n \geq 1\right]_{\delta} + \left[n = 0\right]_{\delta}, \tag{1}$$

when R depends linearly on n. Notable special cases of (1) that we are particularly interested in enumerating through the convergents to these J-fraction expansions include the multiple, or α-*factorial functions*, $n!_{(\alpha)}$, defined for $\alpha \in \mathbb{Z}^+$ as

$$n!_{(\alpha)} = \begin{cases} n \cdot (n-\alpha)!_{(\alpha)}, & \text{if } n > 0; \\ 1, & \text{if } -\alpha < n \leq 0; \\ 0, & \text{otherwise,} \end{cases} \tag{2}$$

and the generalized factorial functions of the form $p_n(\alpha, \beta n + \gamma)$ for $\alpha, \beta, \gamma \in \mathbb{Z}$, $\alpha \neq 0$, and β, γ not both zero. The second class of special case products are related to the *Gould polynomials*, $G_n(x; a, b) = \frac{x}{x-an} \cdot \left(\frac{x-an}{b}\right)^{\underline{n}}$, through the following identity ([12, §3.4.2],[10, §4.1.4]):

$$p_n\left(\alpha, \beta n + \gamma\right) = \frac{(-\alpha)^{n+1}}{\gamma - \alpha - \beta} \times G_{n+1}\left(\gamma - \alpha - \beta; -\beta, -\alpha\right). \tag{3}$$

The α-factorial functions, $(\alpha n - d)!_{(\alpha)}$ for $\alpha \in \mathbb{Z}^+$ and some $0 \leq d < \alpha$, form special cases of (3) where, equivalently, $(\alpha, \beta, \gamma) \equiv (-\alpha, \alpha, -d)$ and $(\alpha, \beta, \gamma) \equiv (\alpha, 0, \alpha - d)$ [11, §6]. The α-factorial

functions are expanded by the triangles of *Stirling numbers of the first kind*, $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]$, and the α-*factorial coefficients*, $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_{\alpha}$, respectively, in the following forms [5, 12]:

$$\begin{aligned} n!_{(\alpha)} &= \sum_{m=0}^{n} \left[\begin{matrix} \lceil n/\alpha \rceil \\ m \end{matrix}\right] (-\alpha)^{\lceil \frac{n}{\alpha} \rceil - m} n^m, \ \forall n \geq 1, \alpha \in \mathbb{Z}^+ \\ &= \sum_{m=0}^{n} \left[\begin{matrix} \lfloor \frac{n-1+\alpha}{\alpha} \rfloor + 1 \\ m+1 \end{matrix}\right]_{\alpha} (-1)^{\lfloor \frac{n-1+\alpha}{\alpha} \rfloor - m} (n+1)^m, \ \forall n \geq 1, \alpha \in \mathbb{Z}^+ \\ (\alpha n - d)!_{(\alpha)} &= (\alpha - d) \times \sum_{m=1}^{n} \left[\begin{matrix} n \\ m \end{matrix}\right]_{\alpha} (-1)^{n-m} (\alpha n + 1 - d)^{m-1} \\ &= \sum_{m=0}^{n} \left[\begin{matrix} n+1 \\ m+1 \end{matrix}\right]_{\alpha} (-1)^{n-m} (\alpha n + 1 - d)^m, \ \forall n \geq 1, \alpha \in \mathbb{Z}^+, 0 \leq d < \alpha. \end{aligned} \tag{4}$$

A careful treatment of the polynomial expansions of these generalized α-factorial functions through the coefficient triangles in (4) is given in [12].

2.2. Summary of the J-fraction Results

For all $h \geq 2$, we can generate the generalized factorial product sequences, $p_n(\alpha, R)$, through the strictly rational generating functions provided by the h^{th} convergent functions, denoted by $\mathrm{Conv}_h(\alpha, R; z)$, to the infinite continued fraction series established by [11]. In particular, we have series expansions of these convergent functions proved in [11] given by

$$\begin{aligned} \mathrm{Conv}_h(\alpha, R; z) &:= \cfrac{1}{1 - R \cdot z - \cfrac{\alpha R \cdot z^2}{1 - (R + 2\alpha) \cdot z - \cfrac{2\alpha(R+\alpha) \cdot z^2}{1 - (R + 4\alpha) \cdot z - \cfrac{3\alpha(R+2\alpha) \cdot z^2}{\cdots \\ \hline 1 - (R + 2(h-1)\alpha) \cdot z}}}} \\ &= \frac{\mathrm{FP}_h(\alpha, R; z)}{\mathrm{FQ}_h(\alpha, R; z)} \\ &= \sum_{n=0}^{h} p_n(\alpha, R) z^n + \sum_{n>h}^{\infty} \left[p_n(\alpha, R) \pmod h \right] z^n, \end{aligned} \tag{5}$$

where the convergent function numerator and denominator polynomial subsequences which provide the characteristic expansions of (5) are given in closed-form by (6) and (7) below. For example, when $h := 2$ we have a convergent function approximation given by

$$\begin{aligned} \mathrm{Conv}_2(\alpha, R; z) &= \frac{1 - (2\alpha + R)z}{1 - 2(\alpha + R)z + R(\alpha + R)z^2} \\ &= 1 + Rz + R(\alpha + R)z^2 + R(\alpha + R)(\alpha + 2R)z^3 \\ &\quad + R(\alpha + R)^2(4\alpha + R)z^4 + R(\alpha + R)^2(R^2 + 8\alpha R + 8\alpha^2)z^5 + \cdots . \end{aligned}$$

More generally, we have that

$$\begin{aligned}\mathrm{FQ}_h(\alpha,R;z) &= \sum_{k=0}^{h}\binom{h}{k}(-1)^k\left(\prod_{j=0}^{k-1}(R+(h-1-j)\alpha)\right)z^k \qquad (6)\\ &= \sum_{k=0}^{h}\binom{h}{k}\left(\frac{R}{\alpha}+h-k\right)_k(-\alpha z)^k\\ &= (-\alpha z)^h\cdot h!\times L_h^{(R/\alpha-1)}\left((\alpha z)^{-1}\right),\end{aligned}$$

when $L_n^{(\beta)}(x)$ denotes an *associated Laguerre polynomial*, and where

$$\begin{aligned}\mathrm{FP}_h(\alpha,R;z) &= \sum_{n=0}^{h-1}C_{h,n}(\alpha,R)z^n \qquad (7a)\\ &= \sum_{n=0}^{h-1}\left(\sum_{i=0}^{n}\binom{h}{i}(-1)^i p_i\left(-\alpha,R+(h-1)\alpha\right)p_{n-i}\left(\alpha,R\right)\right)z^n \qquad (7b)\\ &= \sum_{n=0}^{h-1}\left(\sum_{i=0}^{n}\binom{h}{i}\left(1-h-R/\alpha\right)_i\left(R/\alpha\right)_{n-i}\right)(\alpha z)^n. \qquad (7c)\end{aligned}$$

The coefficients of the polynomial powers of z in the previous several expansions of (7), denoted by $C_{h,n}(\alpha,R) := [z^n]\,\mathrm{FP}_h(\alpha,R;z)$ for $0\le n<h$, also have the following multiple, alternating sum expansions involving the Stirling number triangles [11, §5.2]:

$$\begin{aligned}C_{h,n}(\alpha,R) &= \sum_{\substack{0\le m\le k\le n\\ 0\le s\le n}}\left(\binom{h}{k}\binom{m}{s}\begin{bmatrix}k\\m\end{bmatrix}(-1)^m\alpha^n\left(\frac{R}{\alpha}\right)_{n-k}\left(\frac{R}{\alpha}-1\right)^{m-s}\right)\times h^s \qquad (8a)\\ &= \sum_{\substack{0\le m\le k\le n\\ 0\le t\le s\le n}}\left(\binom{h}{k}\binom{m}{t}\begin{bmatrix}k\\m\end{bmatrix}\begin{bmatrix}n-k\\s-t\end{bmatrix}(-1)^m\alpha^{n-s}(h-1)^{m-t}\right)\times R^s \qquad (8b)\\ &= \sum_{\substack{0\le m\le k\le n\\ 0\le i\le s\le n}}\binom{h}{k}\binom{h}{i}\binom{m}{s}\begin{bmatrix}k\\m\end{bmatrix}\begin{Bmatrix}s\\i\end{Bmatrix}(-1)^m\alpha^n\left(\frac{R}{\alpha}\right)_{n-k}\left(\frac{R}{\alpha}-1\right)^{m-s}\times i! \qquad (8c)\\ &= \sum_{\substack{0\le m\le k\le n\\ 0\le v\le i\le s\le n}}\binom{h}{k}\binom{m}{s}\binom{i}{v}\binom{h+v}{v}\begin{bmatrix}k\\m\end{bmatrix}\begin{Bmatrix}s\\i\end{Bmatrix}(-1)^{m+i-v}\alpha^n\times \qquad (8d)\\ &\qquad\qquad\times\left(\frac{R}{\alpha}\right)_{n-k}\left(\frac{R}{\alpha}-1\right)^{m-s}\times i!\\ &= \sum_{i=0}^{n}\underbrace{\left(\sum_{\substack{0\le m\le k\le n\\ 0\le t\le s\le n}}\binom{h}{k}\binom{m}{t}\begin{bmatrix}k\\m\end{bmatrix}\begin{bmatrix}n-k\\s-t\end{bmatrix}\begin{Bmatrix}s\\i\end{Bmatrix}(-1)^{m+s-i}(h-1)^{m-t}\right)}_{\text{polynomial function of } h \text{ only}}\times\alpha^n\left(\frac{R}{\alpha}\right)_i. \qquad (8e)\end{aligned}$$

Given that the non-zero h^{th} convergent functions, $\text{Conv}_h(\alpha, R; z)$, are rational in each of z, α, R for all $h \geq 1$, and that the convergent denominator sequences, $\text{FQ}_h(\alpha, R; z)$, have characteristic expansions by the Laguerre polynomials and the *confluent hypergeometric functions*, we may expand both exact finite sums and congruences modulo $h\alpha^t$ for the generalized factorial functions, $p_n(\alpha, \beta n + \gamma)$, by the distinct special zeros of these functions according to the following identities:

$$\begin{aligned}
p_n(\alpha, R) &= \sum_{j=1}^{n} c_{n,j}(\alpha, R) \times \ell_{n,j}(\alpha, R)^n && (9)\\
p_n(\alpha, R) &\equiv \sum_{j=1}^{h} c_{h,j}(\alpha, R) \times \ell_{h,j}(\alpha, R)^n && \pmod{h}\\
n!_{(\alpha)} &= \sum_{j=1}^{n} c_{n,j}(-\alpha, n) \times \ell_{n,j}(-\alpha, n)^{\lfloor \frac{n-1}{\alpha} \rfloor}\\
n!_{(\alpha)} &\equiv \sum_{j=1}^{h} c_{h,j}(-\alpha, n) \times \ell_{h,j}(-\alpha, n)^{\lfloor \frac{n-1}{\alpha} \rfloor} && \pmod{h, h\alpha, \cdots, h\alpha^h}.
\end{aligned}$$

These exact formulas and congruence properties for the generalized factorial functions $p_n(\alpha, R)$ and $n!_{(\alpha)}$ follow from expansions of the rational h^{th} convergent functions in partial fractions with respect to z as $\text{Conv}_h(\alpha, R; z) \equiv \sum_{1 \leq j \leq h} c_{h,j}(\alpha, R)/(1 - \ell_{h,j}(\alpha, R) \cdot z)$ [11, §6.2].

2.3. Key New Results Proved in the Article

In [11] we prove new exact formulas and congruence properties using the algebraic properties of the rational convergents, $\text{Conv}_h(\alpha, R; z)$. In contrast with this previous work, in this article we choose an alternate route to derive our new results. Namely, we use the rationality of the h^{th} convergent functions to establish new h-order finite difference equations for the coefficients of $\text{Conv}_h(\alpha, \beta n + \gamma; z)$, both in the exact forms of $p_n(\alpha, \beta n + \gamma)$ with respect to n, as well as for these special factorial functions expansions modulo $h\alpha^t$ for any integers $h \geq 2$ and $0 \leq t \leq h$. We state our next key proposition which we will prove in Section 3.1.

Proposition 1 (Finite Difference Equations for Generalized Factorial Functions). *For fixed $\beta, \gamma \in \mathbb{Z}$ with $\alpha = \pm 1, \pm 2$ and β, γ not both zero, h odd or prime, an integer $0 \leq t \leq h$, and any integers $n, r \geq 0$, we have the following exact expansions and congruences for the generalized product sequences, $p_n(\alpha, R)$ and $p_n(\alpha, \beta n + \gamma)$:*

$$\begin{aligned}
p_n(\alpha, R) &= \sum_{k=0}^{n-1} \tbinom{n+r}{k+1}(-1)^k p_{k+1}(-\alpha, R + (n-1+r)\alpha) p_{n-1-k}(\alpha, R) + C_{n+r,n}(\alpha, R) && (10a)\\
&= \sum_{k=0}^{n-1} \tbinom{n}{k+1}(-1)^k p_{k+1}(-\alpha, R + (n-1)\alpha) p_{n-1-k}(\alpha, R) + [n=0]_\delta\\
p_n(\alpha, \beta n + \gamma) &\equiv \sum_{k=0}^{n} \binom{h}{k}(-\alpha)^k p_k(-\alpha, \beta n + \gamma + (h-1)\alpha)\, p_{n-k}(\alpha, \beta n + \gamma) \pmod{h} && (10b)
\end{aligned}$$

$$= \sum_{k=0}^{n} \binom{h}{k} \alpha^{n+k} \left(1 - h - \tfrac{\beta n+\gamma}{\alpha}\right)_k \left(\tfrac{\beta n+\gamma}{\alpha}\right)_{n-k}$$

$$p_n\left(\alpha, \beta n+\gamma\right) \equiv \sum_{k=0}^{n} \binom{h}{k} \alpha^{n+(t+1)k} \left(1 - h - \tfrac{\beta n+\gamma}{\alpha}\right)_k \left(\tfrac{\beta n+\gamma}{\alpha}\right)_{n-k} \pmod{h\alpha^t}. \tag{10c}$$

The numerator coefficients, denoted by $C_{h,n}(\alpha, R)$ in the previous equations, are defined as

$$C_{h,k}(\alpha, R) := [z^k] \operatorname{FP}_h\left(\alpha, R; z\right).$$

Remark 1 (Stronger Statements of the Key Congruence Properties). Based on numerical evidence computed in the summary notebook [13], we conjecture, but we do not offer a conclusive proof here, that the results stated in (10b) and (10c) of the key proposition in fact hold for *all integers* $h \geq 2$, $\alpha \neq 0$, and $0 \leq t \leq h$ – not just in the odd and even prime special cases of the moduli specified in the previous proposition. This observation substantially widens the utility of the application of these results in the examples cited in Section 3 below.

More precisely, we offer the following conjecture, significantly extending the range of cases of the parameters for which we may apply Proposition 1:

Conjecture 1. Under the same assumptions on the other key parameters, the conclusions of Proposition 1 are true when we let the parameter α assume any integer value.

2.4. Examples

2.4.1. Applications in Wilson's Theorem and Clement's Theorem Concerning the Twin Primes

The first examples given in this section provide restatements of the necessary and sufficient integer-congruence-based conditions imposed in both statements of Wilson's theorem and Clement's theorem through the exact expansions of the factorial functions defined above. For odd integers $p \geq 3$, the congruences implicit to each of Wilson's theorem and Clement's theorem are enumerated as follows [9, §4.3] [6, §6.6] [2]:

$$\begin{aligned}
p \text{ prime} \quad & \text{if and only if } (p-1)! + 1 && \equiv 0 \pmod{p} \\
& \text{if and only if } [z^{p-1}] \operatorname{Conv}_p\left(-1, p-1; z\right) + 1 && \equiv 0 \pmod{p} \\
& \text{if and only if } [z^{p-1}] \operatorname{Conv}_p\left(1, 1; z\right) + 1 && \equiv 0 \pmod{p}, \\
p, p+2 \text{ prime} \quad & \text{if and only if } 4\left((p-1)! + 1\right) + p && \equiv 0 \pmod{p(p+2)} \\
& \text{if and only if } 4[z^{p-1}] \operatorname{Conv}_{p(p+2)}\left(-1, p-1; z\right) + p + 4 && \equiv 0 \pmod{p(p+2)} \\
& \text{if and only if } 4[z^{p-1}] \operatorname{Conv}_{p(p+2)}\left(1, 1; z\right) + p + 4 && \equiv 0 \pmod{p(p+2)}.
\end{aligned}$$

The rationality in z of the convergent functions, $\operatorname{Conv}_h\left(\alpha, R; z\right)$, at each h leads to further alternate formulations of other well-known congruence statements concerning the divisibility of factorial functions. For example, we may characterize the primality of the odd integers, $p > 3$, of the form $p = 4k+1$ (*i.e.,* the so-termed subset of "*Pythagorean primes*") according to the next condition [6,

§7] (http://oeis.org/A002144):

$$\left(\tfrac{p-1}{2}\right)!^2 \equiv -1 \pmod{p} \text{ if and only if } p \text{ is a prime of the form } 4k+1. \tag{11}$$

For an odd integer $p > 3$ to be both prime and satisfy $p \equiv 1 \pmod 4$, the congruence statement in (11) requires that the diagonals of the following rational two-variable convergent generating functions satisfy the following equivalent conditions where p_i is chosen so that $p \mid 2^p p_i$ for each $i = 1, 2$:

$$\begin{aligned}
\left(\tfrac{p-1}{2}\right)!^2 &= [z^{(p-1)/2}][x^0]\left(\text{Conv}_{p_1}\left(-1, \frac{p-1}{2}; x\right)\text{Conv}_{p_2}\left(-1, \frac{p-1}{2}; \frac{z}{x}\right)\right) &&\equiv -1 \pmod{p}\\
\left(\tfrac{p-1}{2}\right)!^2 &= [z^{(p-1)/2}][x^0]\left(\text{Conv}_{p_1}\left(-2, p-1; x\right)\text{Conv}_{p_2}\left(-2, p-1; \frac{z}{4x}\right)\right) &&\equiv -1 \pmod{p}\\
\left(\tfrac{p-1}{2}\right)!^2 &= [z^{(p-1)/2}][x^0]\left(\text{Conv}_{p_1}\left(-1, \frac{p-1}{2}; x\right)\text{Conv}_{p_2}\left(-2, p-1; \frac{z}{2x}\right)\right) &&\equiv -1 \pmod{p}.
\end{aligned}$$

The supplementary computational reference provides additional remarks on the harmonic-number-related fractional power series expansions of the convergent-based generating functions,

$$\text{Conv}_n\left(1, 1; z/x\right) \times \text{Conv}_n\left(1, 1; x\right),$$

and

$$\text{Conv}_n\left(2, 1; z/x\right) \times \text{Conv}_n\left(1, 1; x\right),$$

related to the single factorial function squares enumerated by the identities in the previous equations [13].

These particular congruences involving the expansions of the single factorial function are considered in [12, §6.1.6] as an example of the first product-based symbolic factorial function expansions implicit to both Wilson's theorem and Clement's theorem. Related formulations of conditions concerning the primality of prime pairs, $(p, p+d)$, and then of other prime k-tuples, are similarly straightforward to obtain by elementary methods starting from the statement of Wilson's theorem [8]. For example, the new results proved in Section 3 are combined with the known congruences established in [8, §3, §5] to obtain the cases of the next particular forms of alternate necessary and sufficient conditions for the twin primality of the odd positive integers $p_1 := 2n+1$ and $p_2 := 2n+3$ when $n \geq 1$ (http://oeis.org/A001359, http://oeis.org/A001097):

$$\begin{aligned}
&2n+1, 2n+3 \text{ odd primes} \qquad\qquad (12)\\
&\quad \text{if and only if } 2\left(\sum_{i=0}^{n}\tbinom{(2n+1)}{i}^{(2n+3)}{}^2(-1)^i i!(n-i)!\right)^2 + (-1)^n(10n+7) \equiv 0\\
&\qquad\qquad\qquad\qquad\qquad\qquad \pmod{(2n+1)(2n+3)}\\
&\quad \text{if and only if } 4\left(\sum_{i=0}^{2n}\tbinom{(2n+1)}{i}^{(2n+3)}{}^2(-1)^i i!(2n-i)!\right) + 2n + 5 \equiv 0\\
&\qquad\qquad\qquad\qquad\qquad\qquad \pmod{(2n+1)(2n+3)}.
\end{aligned}$$

Section 3.4.2 of this article considers the particular cases of these two classically-phrased congruence statements as applications of the new polynomial expansions for the generalized product sequences, $p_n(\alpha, \beta n + \gamma)$, derived from the expansions of the convergent function sequences by finite difference equations.

2.4.2. Congruences for the Wilson Primes and the Single Factorial Function Modulo n^2

The sequence of *Wilson primes*, or the subsequence of odd integers $p \geq 5$ satisfying $n^2 \mid (n-1)!+1$, is characterized through each of the following additional divisibility requirements placed on the expansions of the single factorial function implicit to Wilson's theorem considered in Section 3.4 (`http://oeis.org/A007540`):

$$\underbrace{\sum_{i=0}^{n-1} \binom{n^2}{i}^2 (-1)^i i!(n-1-i)!}_{\equiv\ (n-1)! \pmod{n^2}} \equiv -1 \pmod{n^2}$$

$$\underbrace{\sum_{i=0}^{n-1} \binom{n^2}{i} (n^2-n)^{\underline{i}} \times (-1)^{n-1-i}(n-1)^{\underline{n-1-i}}}_{\equiv (n-1)! \pmod{n^2}} \equiv -1 \pmod{n^2}$$

$$\sum_{s=0}^{n-1} \sum_{i=0}^{s} \sum_{v=0}^{i} \left(\sum_{k=0}^{n-1} \sum_{m=0}^{k} \binom{n^2}{k} \binom{m}{s} \binom{i}{v} \binom{n^2+v}{v} \left[{k \atop m} \right] \left\{ {s \atop i} \right\} (-1)^{i-v} (n-1)^{\underline{n-1-k}} (-n)^{m-s} i! \right) \equiv -1 \pmod{n^2}.$$

The results providing the new congruence properties for the α-factorial functions modulo the integers p, and $p\alpha^i$ for some $0 \leq i \leq p$, expanded in Section 3.2 also lead to alternate phrasings of the necessary and sufficient conditions on the primality of several notable subsequences of the odd positive integers $n \geq 3$ [11, *cf.* §6.4].

2.5. Expansions of Congruences for the α-factorial Functions and Related Sequences

2.5.1. Expansions of Other New Congruences for the Double and Triple Factorial Functions

A few representative examples of the new congruences for the double and triple factorial functions obtained from the statements of Corollary 1 given in Section 3 also include the following particular expansions for integers $p_1, p_2 \geq 2$, and where $0 \leq s \leq p_1$ and $0 \leq t \leq p_2$ assume some prescribed values over the non-negative integers (see the computations contained in [13]):

$$\begin{aligned}
(2n-1)!! &\equiv \sum_{i=0}^{n} \binom{p_1}{i} 2^n (-2)^{(s+1)i} \left(\tfrac{1}{2} - p_1\right)_i \left(\tfrac{1}{2}\right)_{n-i} && \pmod{p_1 2^s} \\
&\equiv \sum_{i=0}^{n} \binom{p_1}{i} (-2)^n 2^{(s+1)i} \left(\tfrac{1}{2} + n - p_1\right)_i \left(\tfrac{1}{2} - n + i\right)_{n-i} && \pmod{p_1 2^s} \\
(3n-1)!!! &\equiv \sum_{i=0}^{n} \binom{p_2}{i} 3^n (-3)^{(t+1)i} \left(\tfrac{1}{3} - p_2\right)_i \left(\tfrac{2}{3}\right)_{n-i} && \pmod{p_2 3^t}
\end{aligned}$$

$$\equiv \sum_{i=0}^{n} \binom{p_2}{i} (-3)^n 3^{(t+1)i} \left(\tfrac{1}{3} + n - p_2\right)_i \left(\tfrac{1}{3} - n + i\right)_{n-i} \pmod{p_2 3^t}$$

$$(3n-2)!!! \equiv \sum_{i=0}^{n} \binom{p_2}{i} 3^n (-3)^{(t+1)i} \left(\tfrac{2}{3} - p_2\right)_i \left(\tfrac{1}{3}\right)_{n-i} \pmod{p_2 3^t}$$

$$\equiv \sum_{i=0}^{n} \binom{p_2}{i} (-3)^n 3^{(t+1)i} \left(\tfrac{2}{3} + n - p_2\right)_i \left(\tfrac{2}{3} - n + i\right)_{n-i} \pmod{p_2 3^t}.$$

2.5.2. Semi-polynomial Congruences for Double Factorial Functions and the Central Binomial Coefficients

The integer congruences satisfied by the double factorial function, $(2n-1)!!$, and the Pochhammer symbol cases, $2^n \times \left(\frac{1}{2}\right)_n$, expanded in Section 3.5 provide the next variants of the polynomial congruences for the *central binomial coefficients*, $\binom{2n}{n} = 2^n \times (2n-1)!!/n!$, reduced modulo the respective integer multiples of $2n+1$ and the polynomial powers, n^p, for fixed integers $p \geq 2$ in the following equations (`http://oeis.org/A000984`) (see the computations in [13])[1]:

$$\binom{2n}{n} \equiv \left\{ \sum_{i=0}^{n} \frac{\binom{2x+1}{i} (-2)^i \left(\frac{1}{2} + 2x\right)^{\underline{i}} \left(\frac{1}{2}\right)_{n-i} \times \frac{2^{2n}}{n!}}{\bmod 2x+1 \quad \curvearrowright \quad x :\mapsto n} \right\} \pmod{2n+1}$$

$$\binom{2n}{n} \equiv \left\{ \frac{\sum_{i=0}^{n} \binom{x^p}{i} \binom{2n-2i}{n-i} \left(\frac{1}{2} - x^p\right)_i \times \frac{8^i \cdot (n-i)!}{n!}}{\bmod x^p \quad \curvearrowright \quad x :\mapsto n} \right\} \pmod{n^p}.$$

3. Finite Difference Equations for Generalized Factorial Functions and Applications

The rationality of the convergent functions, $\mathrm{Conv}_h(\alpha, R; z)$, in z for all h suggests new forms of h-order finite difference equations with respect to h, α, and R satisfied by the product sequences, $p_n(\alpha, R)$, when α and R correspond to fixed parameters independent of the sequence indices n. In particular, the rationality of the h^{th} convergent functions immediately implies the first two results stated in Proposition 1 above, which also provides both forms of the congruence properties stated in (10b) modulo odd and even prime integers $h \geq 2$ [7, §2.3] [5, §7.2].

When the initially indeterminate parameter, R, assumes an implicit dependence on the sequence index, n, the results phrased by the previous equations, somewhat counter-intuitively, do not immediately imply difference equations satisfied between only the generalized product sequences, either exactly, or modulo the prescribed choices of $h \geq 2$ (*cf.* (13) and Example 2 on page 11). The new formulas connecting the generalized product sequences, $p_n(\alpha, \beta n + \gamma)$, resulting from (10b) and (10c) in these cases are, however, reminiscent of the relations satisfied between the generalized Stirling polynomial and convolution polynomial sequences expanded in [12] and [5, *cf.* §6.2].

[1]The unconventional notation for computing the underlined polynomial mod operations in the next equations is defined by the footnote on page 28.

3.1. Proof of the Key Proposition

Proof of (10a). Since the h^{th} convergent functions, $\mathrm{Conv}_h(\alpha, R; z)$, are rational for all $h \geq 2$, we obtain the next h-order finite difference equation exactly generating the coefficients, $[z^n]\,\mathrm{Conv}_h(\alpha, R; z)$, from (6) and (7) above when $n \geq 0$ given by [7, §2.3]

$$p_n(\alpha, R) \equiv \sum_{k=1}^{n} \binom{h}{k} (-1)^{k+1} p_k(-\alpha, R + (h-1)\alpha) p_{n-k}(\alpha, R) \pmod{h}$$
$$+ C_{h,n}(\alpha, R)\,[h > n \geq 1]_\delta + [n = 0]_\delta\,,$$

where $\mathrm{Conv}_h(\alpha, R; z)$ exactly enumerates the sequence of $p_n(\alpha, R)$ for $0 \leq n \leq h$ and where $C_{n,n}(\alpha, R) = 0$ for all n. Thus we see that the previous equation implies both formulas in (10a). □

Proof of (10b) *and* (10c). We will prove the first statement in the two special cases of $\alpha := \pm 1, \pm 2$ which we explicitly employ in our applications given in the subsections below. The method we use easily generalizes to further cases of $|\alpha| \geq 3$, but we only conjecture that the formulas hold for these subsequent special values of α. We begin by noticing that since h is odd or $h = 2$, we have that $h | \binom{h}{k}$ for all $1 \leq k < h$ where $\binom{h}{0} = \binom{h}{h} = 1$. So it suffices to evaluate the sum only for the indices k corresponding to these two corner cases. If $n < h$, then the sum is trivially exactly equal to the product function, $p_n(\alpha, R)$. Next, we let $R := \beta n + \gamma$ and suppose that $n \geq h$ in order to evaluate the terms in the sum at both indices $k := 0, h$ where the binomial coefficient $\binom{h}{k} \neq 0 \pmod{h}$ and as follows:

$$\begin{aligned}
\mathrm{RHS}(n) &= p_n(\alpha, R) + (-\alpha)^h p_h(-\alpha, R + (h-1)\alpha) \times p_{n-h}(\alpha, R) \\
&= p_n(\alpha, R) - \alpha^h \times \prod_{j=0}^{h-1} (R + (h-1)\alpha - (h-1-j)\alpha) \times \prod_{j=0}^{n-h-1} (R + (n-h-1-j)\alpha) \\
&\equiv p_n(\alpha, R) - \alpha^h \times \prod_{j=0}^{h-1} (R + j\alpha) \times \prod_{j=0}^{n-h-1} (R + (n-1-j)\alpha) \pmod{h} \\
&= p_n(\alpha, R) - \alpha^h \times (R + (n-1)\alpha) \cdots (R + h\alpha) \times (R + (h-1)\alpha) \cdots R \\
&= (1 - \alpha^h) p_n(\alpha, R).
\end{aligned}$$

In both cases of $\alpha = \pm 1, 2$, we easily see that when $n \geq h$ the respective α-factorial function has a factor of h as $h | n!, (2n)!!, (2n-1)!!$, which implies that both of $p_n(\alpha, R), (1 - \alpha^h) p_n(\alpha, R) \equiv 0 \pmod{h}$. The second formula in (10b) is a rearrangement of the inner terms of the first formula. The third formula in (10c) also follows easily from the first formula modulo $h\alpha^t$. □

3.2. Combinatorial Identities for the Double Factorial Function and Finite Sums Involving the α-factorial Functions

The double factorial function, $(2n-1)!!$, satisfies a number of known expansions through the finite sum identities summarized in [4, 1]. For example, when $n \geq 1$, the double factorial function is

generated by the expansion of finite sums of the form [1, §4.1],

$$(2n-1)!! = \sum_{k=0}^{n-1} \binom{n}{k+1} (2k-1)!!(2n-2k-3)!!. \tag{13}$$

The particular combinatorial identity for the double factorial function expanded in the form of equation (13) above is remarkably similar to the statement of the second sum in (10a) satisfied by the more general product function cases, $p_n(\alpha_0, R_0)$, generating the α-factorial functions, $(\alpha n-1)!_{(\alpha)}$, when $(n, \alpha_0, R_0) :\mapsto (n, \alpha, \alpha-1), (n, -\alpha, \alpha n-1)$.

Example 2 (Exact Finite Sums Involving the α-Factorial Functions). More generally, if we assume that $\alpha \geq 2$ is integer-valued, and proceed to expand these cases of the α-factorial functions according to the expansions from (10a) above, we readily see that [5, *cf.* §5.5]

$$\begin{aligned}
(\alpha n-1)!_{(\alpha)} &= \sum_{k=0}^{n-1} \binom{n-1}{k+1} (-1)^k \times \left(\tfrac{1}{\alpha}\right)_{-(k+1)} \left(\tfrac{1}{\alpha} - n\right)_{k+1} \\
&\qquad \times (\alpha(k+1)-1)!_{(\alpha)} (\alpha(n-k-1)-1)!_{(\alpha)} \\
(\alpha n-1)!_{(\alpha)} &= \sum_{k=0}^{n-1} \binom{n-1}{k+1} (-1)^k \times \binom{\frac{1}{\alpha}+k-n}{k+1} \binom{\frac{1}{\alpha}-1}{k+1}^{-1} \\
&\qquad \times (\alpha(k+1)-1)!_{(\alpha)} (\alpha(n-k-1)-1)!_{(\alpha)}.
\end{aligned}$$

We note the simplification $\left(\frac{1}{\alpha}\right)_{-(k+1)} = \frac{(-\alpha)^{k+1}}{(\alpha(k+1)-1)!_{(\alpha)}}$ where the expansions of the α-factorial functions, $(\alpha n-1)!_{(\alpha)}$, by the Pochhammer symbol correspond to the results given in [16]. The Pochhammer symbol identities cited in [16] provide other related simplifications of the terms in these sums.

The first sum above combined with the expansions of the Pochhammer symbols, $(\pm x)_n$, given in [11, Lemma 12], and the form of Vandermonde-convolution-like identities restated in (7) also lead to the following pair of double sum identities for the α-factorial functions when $\alpha, n \geq 2$ are integer-valued:

$$\begin{aligned}
(\alpha n-1)!_{(\alpha)} &= \sum_{k=0}^{n-1} \sum_{i=0}^{k+1} \tbinom{n-1}{k+1} \tbinom{k+1}{i} (-1)^k \alpha^{k+1-i} (\alpha i-1)!_{(\alpha)} (\alpha(n-1-k)-1)!_{(\alpha)} \times \\
&\qquad \times (n-1-k)_{k+1-i} \\
&= \sum_{k=0}^{n-1} \sum_{i=0}^{k+1} \tbinom{n-1}{k+1} \tbinom{k+1}{i} \tbinom{n-1-i}{k+1-i} (-1)^k \alpha^{k+1-i} (\alpha i-1)!_{(\alpha)} (\alpha(n-1-k)-1)!_{(\alpha)} \times \\
&\qquad \times (k+1-i)!.
\end{aligned}$$

The construction of further analogues for generalized variants of the finite summations and more well-known combinatorial identities satisfied by the double factorial function cases when $\alpha := 2$ from the references is suggested as a topic for future investigation in Section 4.2.2.

Corollary 1 (Congruences for the α-Factorial Functions). *If we let $\alpha = \pm 1, \pm 2$, $0 \leq d < \alpha$, and suppose that h is odd or prime with $0 \leq t \leq h$, we can generalize the congruence results for the double and triple factorial functions from the introduction according to the next equations:*

$$(\alpha n - d)!_{(\alpha)} \equiv \sum_{i=0}^{n} \binom{h}{i} \alpha^n (-\alpha)^{(t+1)i} \left(\tfrac{d}{\alpha} - h\right)_i \left(\tfrac{\alpha - d}{\alpha}\right)_{n-i} \pmod{h\alpha^t}$$

$$(\alpha n - d)!_{(\alpha)} \equiv \sum_{i=0}^{n} \binom{h}{i} (-\alpha)^n \alpha^{(t+1)i} \left(\tfrac{d}{\alpha} + n + 1 - h\right)_i \left(\tfrac{d}{\alpha} - n\right)_{n-i} \pmod{h\alpha^t}.$$

Proof. By Lemma 10 stated in [11, §6.1], we know that for $\alpha \in \mathbb{Z}^+$ and any integers $0 \leq t < \alpha$, we have that $(\alpha n - d)!_{(\alpha)} = p_n(\alpha, \alpha - d)$ and that $(\alpha n - d)!_{(\alpha)} = p_n(-\alpha, \alpha n - d)$. Then if we let $\beta n + \gamma :\mapsto R$ in (10c), we can use the two results from the lemma in combination with the result that

$$p_n(\alpha, R) \equiv \sum_{k=0}^{n} \binom{h}{k} \alpha^{n+(t+1)k} \left(1 - h - \tfrac{R}{\alpha}\right)_k \left(\tfrac{R}{\alpha}\right)_{n-k} \pmod{h\alpha^t},$$

to easily prove our two formulas. □

3.3. Multiple Summation Identities and Finite-degree Polynomial Expansions of the Generalized Product Sequences in n

We are primarily concerned with cases of generalized factorial-related sequences formed by the products, $p_n(\alpha, R_n)$, when the parameter $R_n := \beta n + \gamma$ depends linearly on n for some $\beta, \gamma \in \mathbb{Q}$ (not both zero). Strictly speaking, once we evaluate the indeterminate, R, as a function of n in these cases of the generalized product sequences, $p_n(\alpha, R)$, the corresponding generating functions over the coefficients enumerated by the approximate convergent function series no longer correspond to predictably rational functions of z. We may, however, still prefer to work with these sequences formulated as finite-degree polynomials in n through a few useful forms of the next multiple sums expanded below, which are similar to the forms of the identities given in (8) of the introduction for the coefficients of the numerator convergent functions.

3.3.1. Generalized Polynomial Expansions and Multiple Sum Identities With Applications to Prime Congruences

Proposition 2. *For integers $n, s, n-s \geq 1$ and fixed $\alpha, \beta, \gamma \in \mathbb{Q}$, the following exact finite, multiple sum identities provide particular polynomial expansions in n satisfied by the generalized factorial function cases:*

$$p_{n-s}(\alpha, \beta n + \gamma) = \sum_{\substack{0 \leq m \leq k < n-s \\ 0 \leq r \leq p \leq n-s}} \sum_{t=0}^{n-s-k} \binom{m}{r} \binom{n-s}{k} \binom{t}{p-r} \begin{bmatrix} k \\ m \end{bmatrix} \begin{bmatrix} n-s-k \\ t \end{bmatrix} \times \tag{14}$$
$$\times (-1)^{p-r-1} \alpha^{n-s-m-t} \beta^r \gamma^{m-r} (\alpha + \beta)^{p-r} \times$$
$$\times (\alpha(s+1) - \gamma)^{t-(p-r)} \times n^p$$

$$p_{n-s}(\alpha,\beta n+\gamma) = \sum_{\substack{0\leq r\leq p\leq u\leq 3n\\ 0\leq m,i\leq k<n-s}} \sum_{t=0}^{n-s-k} \binom{m}{r}\binom{i}{u-p}\binom{t}{p-r}\begin{bmatrix}k\\ m\end{bmatrix}\begin{bmatrix}k\\ i\end{bmatrix}\begin{bmatrix}n-s-k\\ t\end{bmatrix}\times$$
$$+[0\leq n\leq s]_\delta$$
$$\times \frac{(-1)^{u-r+k+1}}{k!}\alpha^{n-s-m-t}\beta^r\gamma^{m-r}(\alpha+\beta)^{p-r}\times$$
$$\times (\alpha(s+1)-\gamma)^{t-(p-r)}\times s^{p-u+i}n^u$$
$$+[0\leq n\leq s]_\delta\,.$$

Proof Sketch. The forms of these expansions for the generalized factorial function sequence variants stated in (14) are provided by this proposition without citing the complete details to a somewhat tedious, and unnecessary, proof derived from the well-known polynomial expansions of the products, $p_n(\alpha,R)=\alpha^n\,(R/\alpha)_n$ by the Stirling number triangles. More concretely, for $n,k\geq 0$ and fixed $\alpha,\beta,\gamma,\rho,n_0\in\mathbb{Q}$, the following particular expansions suffice to show enough of the detail needed to more carefully prove each of the multiple sum identities cited in (14) starting from the first statements provided in (10a):

$$\begin{aligned} p_k(\alpha,\beta n+\gamma+\rho) &= \alpha^k\cdot\left(\frac{\beta n+\gamma+\rho}{\alpha}\right)_k\\ &= \sum_{m=0}^{k}\begin{bmatrix}m\\ k\end{bmatrix}\alpha^{k-m}\,(\beta n+\gamma+\rho)^m\\ &= \sum_{p=0}^{k}\left(\sum_{m=p}^{k}\begin{bmatrix}k\\ m\end{bmatrix}\binom{m}{p}\alpha^{k-m}\beta^p\,(\gamma+\rho+\beta n_0)^{m-p}\right)\times(n-n_0)^p. \end{aligned}$$ □

One immediate consequence of Proposition 2 phrases the form of the next multiple sums that exactly generate the single factorial functions, $(n-s)!$, modulo any prescribed integers $h\geq 2$. The simplified triple sum expansions of interest in Example 4 below correspond to a straightforward simplification of the more general multiple finite quintuple 5-sums and 6-sum identities that exactly enumerate the functions, $p_{n-s}(\alpha,\beta n+\gamma)$, when $(s,\alpha,\beta,\gamma):=(1,-1,1,0)$. In particular, these results lead to the following finite, triple sum expansions of the single factorial function cases implicit to the statements of both Wilson's theorem and Clement's theorem from the introduction and which are considered as examples in the next subsection [6, *cf.* §7]:

$$\begin{aligned} (n-1)! &= \sum_{p=0}^{n}\left(\sum_{0\leq t\leq k<n}\binom{n}{n-1-k}\begin{bmatrix}n-1-k\\ p\end{bmatrix}\begin{bmatrix}k\\ k-t\end{bmatrix}(-1)^{n-1-p}\right)\times(n-1)^p \qquad (15)\\ &= \sum_{p=0}^{n}\left(\sum_{\substack{0\leq k<n\\ 0\leq t\leq n-1-k}}\binom{n}{k}\begin{bmatrix}k\\ p\end{bmatrix}\begin{bmatrix}n-1-k\\ n-1-k-t\end{bmatrix}(-1)^{n-1-p}\right)\times(n-1)^p\\ &= \sum_{p=0}^{n}\left(\sum_{0\leq t\leq k<n}\binom{n}{n-1-k}\begin{bmatrix}n-1-k\\ n-p\end{bmatrix}\begin{bmatrix}k\\ k-t\end{bmatrix}(-1)^{p+1}\right)\times(n-1)^{n-p}. \end{aligned}$$

The third exact triple sum identity given in (15) is further expanded through the formula of Riordan cited in [3, p. 173] and [5, *cf.* Ex. 5.65, p. 534] in the following form:

$$n^n = \sum_{0\leq k<n} \binom{n-1}{k}(k+1)! \times n^{n-1-k} = \sum_{k=0}^{n-1} \binom{n-1}{k}(n-k)! \times n^k. \qquad \text{(A Formula of Riordan)}$$

A couple of the characteristic examples of these polynomial expansions in n by the Stirling numbers of the first kind in (15) are considered by Example 4 in the next section to illustrate the notable special cases of Wilson theorem and Clement's theorem modulo some as yet unspecified odd prime, $n \geq 3$.

For comparison, the next several equations provide related forms of finite, double and triple sum identities for the double factorial function, $(2n-1)!!$:

$$\begin{aligned}
(2n-1)!! &= \sum_{1\leq j\leq k\leq n} \begin{bmatrix} k-1 \\ j-1 \end{bmatrix} 2^{n-j}(-1)^{n-k}\,(1-n)_{n-k} \qquad \text{(Double Factorial Triple Sums)} \\
&= \sum_{\substack{1\leq j\leq k\leq n \\ 0\leq m\leq n-k}} \begin{bmatrix} k-1 \\ j-1 \end{bmatrix} \begin{bmatrix} n-k+1 \\ m+1 \end{bmatrix} 2^{n-j}(-1)^{n-k-m} n^m \\
(2n-1)!! &= \sum_{1\leq j\leq k\leq n} \binom{2n-k-1}{k-1} \begin{bmatrix} k \\ j \end{bmatrix} (2n-2k-1)!! \\
&= \sum_{\substack{1\leq j\leq k\leq n \\ 0\leq m\leq n-k}} \binom{2n-k-1}{k-1} \begin{bmatrix} k \\ j \end{bmatrix} \begin{bmatrix} n-k \\ m \end{bmatrix} 2^{n-k-m} \\
&= \sum_{\substack{1\leq j\leq k\leq n \\ 0\leq m\leq n-k}} \binom{2n-k-1}{k-1} \begin{bmatrix} k \\ j \end{bmatrix} \begin{bmatrix} n-k+1 \\ m+1 \end{bmatrix}_2 (-1)^{n-k-m}\,(2n-2k)^m\,.
\end{aligned}$$

The expansions of the double factorial function in the previous equations are obtained from the lemma in (14) applied to the known double sum identities involving the Stirling numbers of the first kind given in [1, §6].

3.3.2. Expansions of Parameterized Congruences Involving the Single Factorial Function

Definition 3. We define the next parameterized congruence variants, denoted by $F_{\omega,n}(x_{\mathrm{P}}, x_{\mathrm{T}}, x_{\mathrm{K}})$, corresponding to the first triple sum identity expanded in (15) for some application-dependent, prescribed functions, $N_{\omega,p}(n)$ and $M_{\omega}(n)$, and where the formal variables $\{x_{\mathrm{P}}, x_{\mathrm{T}}, x_{\mathrm{K}}\}$, index the terms in each individual sum over the respective variables, p, t, and k.

$$\begin{aligned}
F_{\omega,n}(x_{\mathrm{P}}, x_{\mathrm{T}}, x_{\mathrm{K}}) := \sum_{\substack{0\leq t\leq k<n \\ 0\leq p\leq n}} \binom{n}{n-1-k} \begin{bmatrix} n-1-k \\ p \end{bmatrix} \begin{bmatrix} k \\ k-t \end{bmatrix} \times \qquad (16) \\
\times (-1)^{n-1-p} \times N_{\omega,p}(n) \times \{x_{\mathrm{P}}^p x_{\mathrm{T}}^t x_{\mathrm{K}}^k\} \qquad \pmod{M_{\omega}(n)}
\end{aligned}$$

Notice that when $N_{\omega,p}(n) := (n-1)^p$, the function $F_{\omega,n}(1,1,1)$ exactly generates the single factorial function, $(n-1)!$, modulo any specific choice of the function, $M_\omega(n)$, depending on n.

Example 4 (Wilson's Theorem and Clement's Theorem on Twin Primes). The next specialized forms of the parameters implicit to the congruence in (16) of the previous definition are chosen as follows to form another restatement of Wilson's theorem, given immediately below:

$$(\omega, N_{\omega,p}(n), M_\omega(n)) :\mapsto (\text{WT}, (-1)^p, n)\,. \qquad \text{(Wilson Parameter Definitions)}$$

Then we see that

$$n \geq 2 \text{ prime} \text{ if and only if } F_{\text{WT},n}(1,1,1) \equiv -1 \pmod{M_{\text{WT}}(n)}. \qquad \text{(Wilson's Theorem)}$$

The special case of these parameterized expansions of the congruence variants defined by (16) corresponding to the classical congruence-based characterization of the *twin primes* (`http://oeis.org/A001359`, `http://oeis.org/A001097`) formulated in the statement of Clement's theorem is of particular interest in continuing the discussion from Section 2.4. When $k := 2$ in the first congruence result given by (17) of Lemma 1 below, the parameters in (16) are formed as the particular expansions

$$(\omega, N_{\omega,p}(n), M_\omega(n)) :\mapsto \left(\text{CT}, \tfrac{(-1)^p}{2}\left(2 + (1-3^p)\cdot n\right), n(n+2)\right). \qquad \text{(Clement Parameters)}$$

The corresponding expansion of this alternate formulation of Clement's theorem initially stated as in Section 2.4.1 of the introduction, then results in the restatement of this result given in following form [9, §4.3]:

$$n, n+2 \text{ prime} \text{ if and only if } 4 \cdot F_{\text{CT},n}(1,1,1) + 4 + n \equiv 0 \pmod{M_{\text{CT}}(n)}.$$
(Clement's Theorem)

3.3.3. Conjectures From the Formal Polynomial Computations in the Summary Notebook

Numerical computations with *Mathematica*'s `PolynomialMod` function suggest several noteworthy properties satisfied by the trivariate polynomial sequences, $F_{\text{WT},n}(x_{\text{P}}, x_{\text{T}}, x_{\text{K}})$, defined by (16) when n is prime, particularly as formed in the cases taken over the following polynomial configurations of the three formal variables, x_{P}, x_{T}, and x_{K}:

$$(x_{\text{P}}, x_{\text{T}}, x_{\text{K}}) \in \{(x,1,1), (1,x,1), (1,1,x)\}\,.$$

In particular, these computations suggest the following properties satisfied by these sums for integers $n \geq 2$ where the coefficients of the functions, $F_{\text{WT},n}(x_{\text{P}}, x_{\text{T}}, x_{\text{K}})$ and $F_{\text{CT},n}(x_{\text{P}}, x_{\text{T}}, x_{\text{K}})$, are computed termwise with respect to the formal variables, $\{x_{\text{P}}, x_{\text{T}}, x_{\text{K}}\}$, modulo each $M_{\text{WT}}(n) :\mapsto n$ and $M_{\text{CT}}(n) :\mapsto n(n+2)$:

(1) $F_{\text{WT},n}(x_{\text{P}}, 1, 1) \equiv n-1 \pmod{n}$ when n is prime where $\deg_{x_{\text{P}}}\{F_{\text{WT},n}(x_{\text{P}}, 1, 1) \pmod{n}\} > 0$ when n is composite;

(2) $F_{\mathrm{WT},n}(1,1,x_{\mathrm{K}}) \equiv (n-1)\cdot x_{\mathrm{K}}^{n-1}\,[n \text{ prime}]_{\delta} \pmod{n}$; and

(3) $F_{\mathrm{WT},n}(1,x_{\mathrm{T}},1) \equiv \sum_{i=0}^{n-2} x_{\mathrm{T}}^{i} \pmod{n}$ when n is prime, and where $\deg_{x_{\mathrm{T}}}\{F_{\mathrm{WT},n}(1,1,x_{\mathrm{T}}) \pmod{n}\} < n-2$ when n is composite.

(4) For fixed $0 \leq p < n$, the outer sums in the definition of (16), each implicitly indexed by powers of the formal variable x_{P} in the parameterized congruence expansions defined above, yield the Stirling number terms given by the coefficients

$$[x_{\mathrm{P}}^{p}]F_{\omega,n}(x_{\mathrm{P}},1,1) = N_{\omega,p}(n) \times (-1)^{n-1}(p+1)\begin{bmatrix} n \\ p+1 \end{bmatrix}.$$

Moreover, for any fixed lower index, $p+1 \geq 1$, the Stirling number terms resulting from these sums are related to factorial multiples of the r-order harmonic number sequences expanded by the properties stated in Section 3.3.4 below when $r \in \mathbb{Z}^{+}$ [12, *cf.* §4.3].

(5) One other noteworthy property computationally verified for the sums, $F_{\mathrm{CT},n}(x_{\mathrm{P}},x_{\mathrm{T}},x_{\mathrm{K}})$, modulo each prescribed $M_{\mathrm{CT}}(n) := n(n+2)$ for the first several cases of the integers $n \geq 3$, suggests that whenever n is prime and $n+2$ is composite we have that

$$F_{\mathrm{CT},n}(1,1,x_{\mathrm{K}}) \equiv n+4+(n^2-4)x_{\mathrm{K}}^{n-1} \pmod{n(n+2)},$$

where $\deg_{x_{\mathrm{K}}}\{F_{\mathrm{CT},n}(1,1,x_{\mathrm{K}}) \pmod{n(n+2)}\} > 0$ when n is prime.

The computations in the attached computational summary notebook provide several specific examples of the properties suggested by these configurations of the special congruence polynomials for these cases [13]. See the summary notebook file in [13] for more detailed calculations of these, and other, formal polynomial congruence properties.

There are numerous additional examples of prime-related congruences that are also easily adapted by extending the procedure for the classical cases given above. A couple of related approaches to congruence-based primality conditions for prime pairs formulated through the triple sum expansions phrased in Example 4 above are provided by the applications given in the next examples of the prime-related tuples highlighted in Section 3.4.

Lemma 1 (Congruences for Powers Modulo Double and Triple Integer Products). *For integers $p \geq 0$, $n \geq 1$, and any fixed $j > k \geq 1$, the following congruence properties hold:*

$$(n-1)^{p} \equiv \frac{(-1)^{p}}{k}\left(k + (1-(k+1)^{p})\cdot n\right) \pmod{n(n+k)} \tag{17a}$$

$$(n-1)^{p} \equiv (-1)^{p}\left(\tfrac{(n+k)(n+j)}{jk} + \tfrac{n(n+j)(k+1)^{p}}{k(k-j)} + \tfrac{n(n+k)(j+1)^{p}}{j(j-k)}\right) \pmod{n(n+k)(n+j)}. \tag{17b}$$

Proof. First, notice that a naïve expansion by repeated appeals to the binomial theorem yields the next exact expansions of the fixed powers of $(n-1)^{p}$:

$$(n-1)^{p} = (-1)^{p} + \sum_{s=1}^{p} \tbinom{p}{s}\tbinom{s-1}{0} n \cdot (-1)^{p-s} \cdot (-k)^{s-1}$$

$$+\underbrace{\sum_{s=1}^{p}\binom{p}{s}\binom{s-1}{1}n\cdot(n+k)\times(-1)^{p-s}(-k)^{s-2}}_{\equiv 0 \pmod{n(n+k)}}$$

$$+\underbrace{\sum_{s=1}^{p}\sum_{r=2}^{s-1}\binom{p}{s}\binom{s-1}{r}\binom{r-1}{0}n\cdot(n+k)\times(-1)^{p-s}(-k)^{s-1-r}(k-j)^{r-1}}_{\equiv 0 \pmod{n(n+k)}}$$

$$+\underbrace{\sum_{s=1}^{p}\sum_{r=2}^{s-1}\sum_{t=1}^{r-1}\binom{p}{s}\binom{s-1}{r}\binom{r-1}{t}n\cdot(n+k)\cdot(n+j)\times(-1)^{p-s}(-k)^{s-1-r}(k-j)^{r-1-t}(n+j)^{t-1}.}_{\equiv 0 \pmod{n(n+k),n(n+k)(n+j)}}$$

Each of the stated congruences are then easily obtained by summing the non-trivial remainder terms modulo the cases of the integer double products, $n(n+k)$, and the triple products, $n(n+k)(n+j)$, respectively. □

Example 5 (Prime Triples and Sexy Prime Triplets). A special case of the generalized congruences results for prime k-tuples, obtained by induction from Wilson's theorem in the supplementary results computed in [13], implies the next statement characterizing odd integer triplets, or 3-tuples, of the form $(n, n+d_2, n+d_3)$, for some $n \geq 3$ and some prescribed, application-specific choices of the even integer-valued parameters, $d_3 > d_2 \geq 2$:

$(n, n+d_2, n+d_3) \in \mathbb{P}^3$ if and only if (Wilson's Theorem for Prime Triples)

$$(1+(n-1)!)(1+(n+d_2-1)!)(1+(n+d_3-1)!) \equiv 0 \pmod{n(n+d_2)(n+d_3)}.$$

A partial characterization of the *sexy prime triplets*, or prime-valued odd integer triples of the form, $(n, n+6, n+12)$, defined by convention so that $n+18$ is composite, then occurs whenever (`http://oeis.org/A046118`, `http://oeis.org/A046124`)

$$P_{\text{SPT},1}(n)(n-1)! + P_{\text{SPT},2}(n)(n-1)!^2 + P_{\text{SPT},3}(n)(n-1)!^3 \equiv -1 \pmod{n(n+6)(n+12)},$$

where the three polynomials, $P_{\text{SPT},i}(n)$ for $i := 1, 2, 3$, in the previous equation are expanded by the definitions given in the following equations:

$$\begin{aligned}
P_{\text{SPT},1}(n) &:= 1 + (n)_6 + (n)_{12} \\
P_{\text{SPT},2}(n) &:= (n)_6 + (n)_{12} + (n)_6 \times (n)_{12} \\
P_{\text{SPT},3}(n) &:= (n)_6 \times (n)_{12}\,.
\end{aligned}$$

Let the congruence parameters in (16) corresponding to the sexy prime triplet congruence expansions of the single factorial function powers from the previous equations be defined as follows:

$(\omega, N_{\omega,p}(n), M_\omega(n))$ (Sexy Prime Triplet Congruence Parameters)

$$:\mapsto \left(\text{SPT}, \tfrac{(-1)^p}{72}\left((n+6)(n+12) - 2n(n+12)\cdot 7^p + n(n+6)\cdot 13^p\right), n(n+6)(n+12)\right).$$

We similarly see that the elements of an odd integer triple of the form $(n, n+6, n+12)$, are all prime whenever $n \geq 3$ satisfies the next divisibility requirement modulo the integer triple products, $n(n+6)(n+12)$.

$$\sum_{1 \leq i \leq 3} P_{\text{SPT},i}(n) \times F_{\text{SPT},n}(1,1,1)^i \equiv -1 \pmod{M_{\text{SPT}}(n)} \qquad \text{(Sexy Prime Triplets)}$$

The other notable special case triples of interest in the print references, and in the additional polynomial congruence cases computed in the supplementary reference data [13], include applications to the prime 3-tuples of the forms $(p+d_1, p+d_2, p+d_3)$ for $(d_1, d_2, d_3) \in \{(0,2,6), (0,4,6)\}$ [6, *cf.* §1.4], [9, §4.4] (`http://oeis.org/A022004`, `http://oeis.org/A022005`).

3.3.4. Remarks on Expansions of the Stirling Number Triangles by the r-order Harmonic Numbers

The divisibility of the Stirling numbers of the first kind in (14) and in (15) is tied to well-known expansions of the triangle involving the generalized *r-order harmonic numbers*, $H_n^{(r)} := \sum_{k=1}^{n} k^{-r}$, for integer-order $r \geq 1$ [5, §6] [3, *cf.* §5.7] [6, *cf.* §7-8]. The applications cited in [12, §4.3] and [5, §6.3] provide statements of the following established special case identities for these coefficients (`http://oeis.org/A001008`, `http://oeis.org/A002805`, `http://oeis.org/A007406`, `http://oeis.org/A007407`, `http://oeis.org/A007408`, `http://oeis.org/A007409`):

$$\begin{aligned}
\begin{bmatrix} n+1 \\ 2 \end{bmatrix} &= n! \cdot H_n \qquad \text{(Harmonic Number Expansions of the Stirling Numbers)} \\
\begin{bmatrix} n+1 \\ 3 \end{bmatrix} &= \frac{n!}{2} \left(H_n^2 - H_n^{(2)} \right) \\
\begin{bmatrix} n+1 \\ 4 \end{bmatrix} &= \frac{n!}{6} \left(H_n^3 - 3H_n H_n^{(2)} + 2H_n^{(3)} \right) \\
\begin{bmatrix} n+1 \\ 5 \end{bmatrix} &= \frac{n!}{24} \left(H_n^4 - 6H_n^2 H_n^{(2)} + 3\left(H_n^{(2)} \right)^2 + 8H_n H_n^{(3)} - 6H_n^{(4)} \right). \qquad (18)
\end{aligned}$$

In [5, p. 554, Ex. 6.51] a related precise statement of the necessary condition on the primality of odd integers $p > 3$ implied by Wolstenholme's theorem is given in the following form [6, *cf.* §7.8]:

$$p > 3 \text{ prime} \implies \qquad \text{(Stirling Number Variant of Wolstenholme's Theorem)}$$
$$p^2 \mid \begin{bmatrix} p \\ 2 \end{bmatrix},\ p^2 \mid p\begin{bmatrix} p \\ 3 \end{bmatrix} - p^2 \begin{bmatrix} p \\ 4 \end{bmatrix} + \cdots + p^{p-2} \begin{bmatrix} p \\ p \end{bmatrix}.$$

The expansions given in the next remarks of Section 3.3.5 suggest similar expansions of congruences involving the α-factorial functions through more general cases r-order harmonic number sequences, such as the sequence variants, $H_{n,\alpha}^{(r)}$, defined in the next subsection of the article.

3.3.5. More General Expansions of the New Congruence Results by Multiple Factorial Functions and Generalized Harmonic Number Sequences

The noted relations of the divisibility of the Stirling numbers of the first kind to the r-order harmonic number sequences expanded by the special cases from (18) are generalized to the α-factorial function

coefficient cases through the following forms of the exponential generating functions given in [11] [12, *cf.* §3.3]:

$$\sum_{n\geq 0} \begin{bmatrix} n+1 \\ m+1 \end{bmatrix}_{\alpha} \frac{z^n}{n!} = \frac{(1-\alpha z)^{-1/\alpha}}{m! \cdot \alpha^m} \times \operatorname{Log}\left(\frac{1}{1-\alpha z}\right)^m. \tag{19}$$

The special cases of these coefficients generated by the previous equation when $m := 1, 2$ are then expanded by the sums involving the r-order harmonic number sequences in the following equations:

$$\begin{bmatrix} n+1 \\ 2 \end{bmatrix}_{\alpha} \frac{1}{n!} = \alpha^{n-1} \times \sum_{k=0}^{n} \binom{1-\frac{1}{\alpha}}{k} (-1)^k H_{n-k} \tag{20}$$
$$\begin{bmatrix} n+1 \\ 3 \end{bmatrix}_{\alpha} \frac{1}{n!} = \frac{\alpha^{n-2}}{2} \times \sum_{k=0}^{n} \binom{1-\frac{1}{\alpha}}{k} (-1)^k \left(H_{n-k}^2 - H_{n-k}^{(2)}\right).$$

Identities providing expansions of the generalized α-factorial triangles from [12] at other specific cases of the lower indices $m \geq 3$ that involve the slightly generalized cases of the ordinary r-order harmonic number sequences, $H_{\alpha,n}^{(r)}$, defined by the next equation are expanded by related constructions:

$$H_{\alpha,n}^{(r)} := \sum_{k=1}^{n} \frac{1}{(\alpha k + 1 - \alpha)^r}, \; n \geq 1, \alpha, r > 0. \qquad \text{(Generalized Harmonic Number Definitions)}$$

For comparison with the Stirling number identities noted in (18) above, the first few cases of the coefficient identities in (20) are expanded explicitly by these more general integer-order harmonic number sequence cases in the following equations:

$$\begin{bmatrix} n+1 \\ 2 \end{bmatrix}_{\alpha} \frac{1}{n!} = \alpha^n \binom{n+\frac{1-\alpha}{\alpha}}{n} \times H_{n,\alpha}^{(1)}$$
$$\begin{bmatrix} n+1 \\ 3 \end{bmatrix}_{\alpha} \frac{1}{n!} = \frac{\alpha^n}{2} \binom{n+\frac{1-\alpha}{\alpha}}{n} \times \left(\left(H_{n,\alpha}^{(1)}\right)^2 - H_{n,\alpha}^{(2)}\right)$$
$$\begin{bmatrix} n+1 \\ 4 \end{bmatrix}_{\alpha} \frac{1}{n!} = \frac{\alpha^n}{6} \binom{n+\frac{1-\alpha}{\alpha}}{n} \times \left(\left(H_{n,\alpha}^{(1)}\right)^3 - 3H_{n,\alpha}^{(1)} H_{n,\alpha}^{(2)} + 2H_{n,\alpha}^{(3)}\right).$$

When $\alpha := 2$, we have a relation between the sequences, $H_{n,\alpha}^{(r)}$, and the r-order harmonic numbers of the form $H_{2,n}^{(r)} = H_{2n}^{(r)} - 2^{-r} H_n^{(r)}$, which yields particular coefficient expansions for the double factorial functions involved in stating several of the congruence results from the examples given below. The expansions of the prime-related congruences involving the double factorial function cited in Section 3.5 above also suggest additional applications to finding integer congruence properties and necessary conditions involving these harmonic number sequences related to other more general forms of these expansions for prime pairs and prime-related subsequences (see also Section 3.6).

3.4. Expansions of Several New Forms of Prime-related Congruences and Other Prime Subsequence Identities

3.4.1. Statements of Several New Results Providing Finite Sum Expansions of the Single Factorial Function Modulo Fixed Integers

The second cases of the generalized factorial function congruences in (10b) are of particular assistance in expanding several of the non-trivial results given in Section 3.3.2 below when $h-(n-s) \geq 1$. The results related to the double factorial functions and the central binomial coefficients expanded through the congruences in Section 3.5 employ the second cases of (10b) and (10c) stated in Proposition 1. The next few results stated in (21a) and (21b) are provided as lemmas needed to state many of the congruence results for the prime-related sequence cases given as examples in this section and in Section 3.6.

Lemma 2 (Congruences for the Single Factorial Function). *For natural numbers, $n, n-s \geq 0$, the single factorial function, $(n-s)!$, satisfies the following congruences whenever $h \geq 2$ is fixed (or when h corresponds to some fixed function with an implicit dependence on the sequence index n),*

$$\begin{aligned}
(n-s)! &\equiv C_{h,n-s}(1,1) \pmod{h} \qquad (21a)\\
&= \sum_{i=0}^{n-s} \binom{h}{i} (-h)_i \, (n-s-i)! \\
&= \sum_{i=0}^{n-s} \binom{h}{i}^2 (-1)^i i! (n-s-i)! \\
&= \sum_{i=0}^{n-s} \binom{h}{i} \binom{i-h-1}{i} i! (n-s-i)! \\
(n-s)! &\equiv C_{h,n-s}(-1,n-s) \pmod{h} \qquad (21b)\\
&= \sum_{i=0}^{n-s} \binom{h}{i} (n+1-s-h)_i \times (-1)^{n-s-i} \, (-(n-s))_{n-s-i} \\
&= \sum_{i=0}^{n-s} \binom{h}{i} \binom{n-s}{i} \binom{h-n+s-1}{i} (-1)^i i! \times (n-s-i)!.
\end{aligned}$$

The right-hand-side terms, $C_{h,n-s}(\alpha, R)$, in the previous two equations correspond to the auxiliary convergent function sequences implicitly defined by (7), and the corresponding multiple sum expansions stated in (8), whose expansions are highlighted by the listings given in the tables from [11, §9].

Proof. Since $n! = p_n(-1,n)$ and $n! = p_n(1,1)$ for all $n \geq 1$, the identities in (10b) of Proposition 1 imply the pair of congruences stated in each of (21a) and (21b) modulo any fixed, prescribed setting of the integer-valued $h \geq 2$. The expansions of the remaining sums follow first from (7), and then from the results stated in Lemma 12 from [11] applied to each of the expansions of these first two sums. □

Corollary 2 (Special Cases). *If $n, n-s, d, an+r \in \mathbb{Z}^+$ are selected so that $n+d, an+r > n-s$, the coefficient identities for the sequences, $p_{n-s}(1,1) = [z^{n-s}]\,\mathrm{FP}_{n+d}(1,1;z) \pmod{n+d, an+r}$, stated in* (7c) *and* (8) *provide that*

$$(n-s)! \equiv \sum_{i=0}^{n-s} \binom{n+d}{i}^2 (-1)^i i! \times (n-s-i)! \pmod{n+d} \tag{22}$$

$$(n-s)! \equiv \sum_{i=0}^{n-s} \binom{an+r}{i} (-(an+r))_i\, (n-s-i)! \pmod{an+r}.$$

Proof. The expansions of the congruences for the single factorial function provided by the lemmas stated in (21a) follow as immediate consequences of the results in Proposition 1. The previous equations then correspond to the particular cases of these results when $h :\mapsto n+d$ and $h :\mapsto an+r$ respectively in the equations stated above. □

The results expanded through the symbolic computations with these sums obtained from Carsten Schneider's `http://www.risc.jku.at/research/combinat/software/Sigma/index.phpSigma` package for Mathematica outlined in Section 3.4.3 provide additional non-trivial forms of the prime-related congruences involving the single factorial function cases defined by the previous two results.

3.4.2. Examples: Consequences of Wilson's Theorem

Example 6 (Expansions of Variants of Wilson's theorem). The previous identities lead to additional examples phrasing congruences equivalent to the primality condition in Wilson's theorem involving products of the single factorial functions, $n!$ and $(n+1)!$, modulo some odd integer $p := 2n+1$ of unspecified primality to be determined by an application of these results. For example, we can prove that for $n \geq 1$, an odd integer $p := 2n+1$ is prime if and only if [6, *cf.* §8.9][2],

$$2^{1-n} \cdot n! \cdot (n+1)! \equiv (-1)^{\binom{n+2}{2}} \pmod{2n+1} \tag{23}$$

$$(n!)^2 \equiv (-1)^{n+1} \pmod{2n+1}.$$

The first congruence in (23) yields the following additional forms of necessary and sufficient conditions on the primality of the odd integers, $p := 2n+1$, resulting from Wilson's theorem:

$$\frac{1}{2^{n-1}} \times \left(\prod_{s \in \{0,1\}} \sum_{i=0}^{n+s} \binom{2n+1}{i}^2 (-1)^i i! (n+s-i)! \right) \equiv (-1)^{(n+1)(n+2)/2} \pmod{2n+1} \tag{24}$$

$$\left(\sum_{i=0}^{n} \binom{2n+1}{i}^2 (-1)^i i! (n-i)! \right)^2 \equiv (-1)^{n+1} \pmod{2n+1}.$$

Example 7 (Congruences for Primes of the Form n^2+1). If we further seek to determine new properties of the odd primes of the form $p := n^2+1 \geq 5$, obtained from adaptations of

[2]The first equation restates a result proved by Szántó in 2005 given on the `http://mathworld.wolfram.com/WilsonsTheorem.html` *MathWorld* website.

the new forms given by these sums, the second consequence of Wilson's theorem provided in (23) above leads to an analogous requirement expanded in the form of the next equations [9, §3.4(D)] (http://oeis.org/A002496):

$$n^2+1 \text{ prime} \text{ if and only if } \left(\sum_{i=0}^{n^2/2} \binom{n^2+1}{i} \left(-(n^2+1)\right)_i \left(\tfrac{1}{2}(n^2-2i)\right)!\right)^2 \equiv (-1)^{n^2/2+1} \pmod{n^2+1}$$
$$\text{if and only if } \left(\sum_{i=0}^{n^2/2} \binom{n^2+1}{i} \binom{i-n^2-2}{i} i! \left(\tfrac{1}{2}(n^2-2i)\right)!\right)^2 \equiv (-1)^{n^2/2+1} \pmod{n^2+1}.$$

For comparison with the previous two congruences, the first classical statement of Wilson's theorem stated as in the introduction is paired with the next expansions of the fourth and fifth multiple sums stated in (8) to show that an odd integer $p \geq 5$ of the form $p := n^2+1$ is prime for some even $n \geq 2$ whenever

$$\sum_{\substack{0 \leq m \leq k \leq n^2 \\ 0 \leq v \leq i \leq s \leq n^2}} \frac{\binom{n^2+1}{k}\binom{m}{s}\binom{i}{v}\binom{n^2+1+v}{v}\begin{bmatrix} k \\ m \end{bmatrix}\left\{ {s \atop i} \right\}(-1)^{m+i-v} i! \left(-n^2\right)_{n^2-k} \left(n^2+1\right)^{m-s}}{C_{h,k}(\alpha,R) \text{ where } h{:}\mapsto n^2+1,\ k{:}\mapsto n^2,\ \alpha{:}\mapsto -1,\ R{:}\mapsto n^2 \text{ in (8d)}} \equiv -1 (\text{mod } n^2+1)$$

$$\sum_{\substack{0 \leq i \leq n^2 \\ 0 \leq m \leq k \leq n^2 \\ 0 \leq t \leq s \leq n^2}} \frac{\binom{n^2+1}{k}\binom{m}{t}\begin{bmatrix} k \\ m \end{bmatrix}\begin{bmatrix} n^2-k \\ s-t \end{bmatrix}\left\{ {s \atop i} \right\}(-1)^{m+s-i} n^{2m-2t} \times i!}{C_{h,n}(\alpha,R) \text{ where } h{:}\mapsto n^2+1,\ n{:}\mapsto n^2,\ \alpha{:}\mapsto 1,\ R{:}\mapsto 1 \text{ in (8e)}} \equiv -1 \pmod{n^2+1};$$

These congruences are straightforward to adapt to form related results characterizing subsequences of primes of the form $p := an^2+bn+c$ for some fixed constants $a, b, c \in \mathbb{Z}$ satisfying the constraints given in [6, §2.8] at natural numbers $n \geq 1$.

Example 8 (Congruences for the Wilson Primes). The sequence of *Wilson primes* denotes the subsequence odd primes n such that the *Wilson quotient*, $W_{\text{quot}}(n) := \frac{((n-1)!+1)}{n}$, is divisible by n, or equivalently the sequence of odd integers $n \geq 3$ with the divisibility property of the single factorial function, $(n-1)!$, modulo n^2 defined in the next equation [9, §5.4] [6, §6.6] (http://oeis.org/A007619, http://oeis.org/A007540).

$$\mathbb{P}_{\text{Wilson}} := \left\{n \geq 3 : n^2 | (n-1)!+1\right\} \xrightarrow{\text{A007540}} (5, 13, 567, \ldots). \qquad \text{(Wilson Primes)}$$

A few additional expansions of congruences characterizing the Wilson primes correspond to the imposing the following additional equivalent requirements on the divisibility of the single factorial function (modulo n) in Wilson's theorem:

$$\underbrace{\sum_{i=0}^{n-1} \binom{n^2}{i}^2 (-1)^i i!(n-1-i)!}_{C_{n^2,n-1}(1,1) \ \equiv \ (n-1)! \pmod{n^2}} \equiv -1 \pmod{n^2} \quad \text{(Wilson Prime Congruences)}$$

$$\underbrace{\sum_{i=0}^{n-1} \binom{n^2}{i}\binom{i-n^2-1}{i} i!(n-1-i)!}_{(n-1)! \pmod{n^2}} \equiv -1 \pmod{n^2}$$

$$\underbrace{\sum_{i=0}^{n-1}\binom{n^2}{i}(n^2-n)^{\underline{i}}\times(-1)^{n-1-i}(n-1)^{\underline{n-1-i}}}_{C_{n^2,n-1}(-1,n-1)\ \equiv\ (n-1)!\pmod{n^2}}\equiv -1 \pmod{n^2}.$$

The congruences in the previous equation are verified numerically in [13] to hold for the first few hundred primes, p_n, only when $p_n \in \{5, 13, 563\}$. The third and fourth multiple sum expansions of the coefficients, $C_{n^2,n-1}(-1, n-1)$, given in (8) similarly provide that an odd integer $n > 3$ is a Wilson prime if and only if either of the following congruences holds modulo the integer squares n^2,

$$\underbrace{\sum_{s=0}^{n-1}\sum_{i=0}^{s}\left(\sum_{k=0}^{n-1}\sum_{m=0}^{k}\tbinom{n^2}{k}\tbinom{n^2}{i}\tbinom{m}{s}\begin{bmatrix}k\\m\end{bmatrix}\begin{Bmatrix}s\\i\end{Bmatrix}(-1)^{n-1-k}(1-n)_{n-1-k}(-n)^{m-s}i!\right)}_{C_{n^2,n-1}(-1,n-1)\text{ in (8c)}}\equiv -1 \pmod{n^2}$$

$$\underbrace{\sum_{s=0}^{n-1}\sum_{i=0}^{s}\sum_{v=0}^{i}\left(\sum_{k=0}^{n-1}\sum_{m=0}^{k}\tbinom{n^2}{k}\tbinom{m}{s}\tbinom{i}{v}\tbinom{n^2+v}{v}\begin{bmatrix}k\\m\end{bmatrix}\begin{Bmatrix}s\\i\end{Bmatrix}(-1)^{i-v}(n-1)^{\underline{n-1-k}}(-n)^{m-s}i!\right)}_{C_{n^2,n-1}(-1,n-1)\text{ in (8d)}}\equiv -1 \pmod{n^2}.$$

Example 9 (Congruences for Special Prime Pair Sequences). The constructions of the new results expanded above are combined with the known congruences established in [8, §3, §5] to obtain the alternate necessary and sufficient conditions for the twin prime pairs stated in (12) of the introduction (`http://oeis.org/A001359`, `http://oeis.org/A001097`). The results in the previous reference also provide analogous expansions of the congruence statements corresponding to characterizations of the *cousin prime* and *sexy prime* pairs expanded in the following equations (`http://oeis.org/A023200`, `http://oeis.org/A023201`):

$2n+1, 2n+5$ odd primes (Cousin Prime Pairs)

$$\begin{aligned}
&\text{if and only if } 36\left(\sum_{i=0}^{n}\left(\tbinom{(2n+1)(2n+5)}{i}\right)^2(-1)^i i!(n-i)!\right)^2 \\
&\qquad +(-1)^n(29-14n) &&\equiv 0 \pmod{(2n+1)(2n+5)},\\
&\text{if and only if } 96\left(\sum_{i=0}^{2n}\left(\tbinom{(2n+1)(2n+5)}{i}\right)^2(-1)^i i!(2n-i)!\right) \\
&\qquad +46n+119 &&\equiv 0 \pmod{(2n+1)(2n+5)},
\end{aligned}$$

$2n+1, 2n+7$ odd primes

$$\begin{aligned}
&\text{if and only if } 1350\left(\sum_{i=0}^{n}\left(\tbinom{(2n+1)(2n+7)}{i}\right)^2(-1)^i i!(n-i)!\right)^2 \qquad \text{(Sexy Prime Pairs)}\\
&\qquad +(-1)^n(578n+1639) &&\equiv 0 \pmod{(2n+1)(2n+7)},\\
&\text{if and only if } 4320\left(\sum_{i=0}^{2n}\left(\tbinom{(2n+1)(2n+7)}{i}\right)^2(-1)^i i!(2n-i)!\right) \\
&\qquad +1438n+5039 &&\equiv 0 \pmod{(2n+1)(2n+7)}.
\end{aligned} \tag{25}$$

The multiple sum expansions of the single factorial functions in the congruences given in the previous two examples also yield similar restatements of the pair of congruences in (12) from Section 2.4.1 providing that for some $n \geq 1$, the odd integers, $(p_1, p_2) := (2n+1, 2n+3)$, are both prime whenever either of the following divisibility conditions hold:

$$2 \times \underbrace{\left(\sum_{\substack{0 \leq i \leq s \leq n \\ 0 \leq m \leq k \leq n}} \binom{(2n+1)(2n+3)}{i} \binom{(2n+1)(2n+3)}{k} \binom{m}{s} \begin{bmatrix} k \\ m \end{bmatrix} \begin{Bmatrix} s \\ i \end{Bmatrix} (-1)^{s+k} i! \times n^{\underline{n-k}} (n+1)^{m-s} \right)^2}_{C_{(2n+1)(2n+3),n}(-1,n) \text{ in (8c)}}$$
$$+ (-1)^n (10n+7) \equiv 0 \pmod{(2n+1)(2n+3)}$$

$$4 \underbrace{\left(\sum_{\substack{0 \leq v \leq i \leq s \leq 2n \\ 0 \leq m \leq k \leq 2n}} \binom{(2n+1)(2n+3)}{k} \binom{(2n+1)(2n+3)+v}{v} \binom{i}{v} \binom{m}{s} \begin{bmatrix} k \\ m \end{bmatrix} \begin{Bmatrix} s \\ i \end{Bmatrix} (-1)^{s-i+v+k} i! \times (2n)^{\underline{2n-k}} (2n+1)^{m-s} \right)}_{C_{(2n+1)(2n+3),2n}(-1,2n) \text{ in (8d)}}$$
$$+ (2n+5) \equiv 0 \pmod{(2n+1)(2n+3)}.$$

The treatment of the integer congruence identities involved in these few notable example cases from Example 4, in Example 5, and in the last several examples from the remarks above, is by no means exhaustive, but serves to demonstrate the utility of this approach in formulating several new forms of non-trivial prime number results with many notable applications.

3.4.3. Computations of symbolic sums with Schneider's Sigma package for Mathematica

Example 10 (Expansions of the First Sum from Lemma 2). The working summary notebook attached to the article [13] includes computations with the `http://www.ris.jku.at/research/combinat/software/Sigma/index.phpSigma` package in Mathematica that yield additional forms of the identities expanded in (22), (23), and (24), for the single factorial function, $(n-s)!$, when $s := 0$. For example, alternate variants of the identity for the first sum in (22) are expanded as,

$$n! \equiv \sum_{i=0}^{n} \binom{n+d}{i} (-(n+d))_i \, (n-i)! \pmod{n+d},$$

$$n! \equiv \sum_{i=0}^{n} \binom{n+d}{i} \binom{i-n-d-1}{i} i!(n-i)! \pmod{n+d},$$

$$n! \equiv \sum_{i=0}^{n} \binom{n+d}{i}^2 (-1)^i i!(n-i)! \pmod{n+d},$$

$$= (-1)^n (2d)_n \times \left(1 + d^2 \times \sum_{i=1}^{n} \binom{i+d}{i} \frac{(-1)^i (-(i+d))_i}{(i+d)^2 (2d)_i} \right)$$

$$= \underbrace{(-1)^n d^2 \times \sum_{i=0}^{n} \binom{i+d}{d}^2 \times \frac{i! \cdot (2d+i)_{n-i}}{(i+d)^2}}_{:= S_{1,d}(n)}. \tag{26}$$

The first special cases of the sums, $S_{1,d}(n)$, defined in the last equation are expanded in terms of the first-order harmonic numbers for integer-valued cases of $d \geq 1$ as follows:

$$\begin{aligned}
S_{1,1}(n) &= (-1)^n (2)_n \times H_{n+1} \\
&= (-1)^n (n+1)! \times H_{n+1} \\
S_{1,2}(n) &= (-1)^n (n+2)! \times ((n+3)H_{n+2} - 2(n+2)) \\
&= \frac{3}{2} \times (-1)^n (4)_n \times \left(H_n - \frac{(2n^3+8n^2+7n-1)}{(n+1)(n+2)(n+3)} \right) \\
S_{1,3}(n) &= \frac{1}{4} \times (-1)^n (n+4)! \times ((n+5)H_{n+3} - 3(n+3)) \\
&= \frac{10}{3} \times (-1)^n (6)_n \times \left(H_n - \frac{(3n^4+24n^3+60n^2+46n-1)}{(n+1)(n+2)(n+3)(n+5)} \right) \\
S_{1,4}(n) &= \frac{1}{108} \times (-1)^n (n+4)! \times \left(3(n+5)(n+6)(n+7)H_{n+4} - (n+4)(11n^2+118n+327) \right) \\
&= \frac{35}{12} \times (-1)^n (8)_n \times \left(3H_n - \frac{(11n^7+260n^6+2498n^5+12404n^4+33329n^3+45548n^2+24426n-108)}{(n+1)(n+2)(n+3)(n+4)(n+5)(n+6)(n+7)} \right).
\end{aligned}$$

More generally, we can provide a somewhat intricate proof (omitted here) of another expansion of the last sum in (26) in terms of the first-order harmonic numbers given by (`http://oeis.org/A001008`, `http://oeis.org/A002805`)

$$n! \equiv (-1)^n (2d)_n \left(1 + \frac{d}{2} \binom{2d}{d} \sum_{1 \leq i \leq d} \frac{(-1)^{d-i}(d+i-2)!}{(i-1)!^2 (d-i)!} \left[H_{n-1+d+i} - H_{d+i-1} \right] \right) (\text{mod } n+d).$$

When the parameter $d :\mapsto d_n$ in the previous expansions of the sums, $S_{1,d}(n)$, depends linearly, or quadratically on n, the harmonic-number-based identities expanding these congruence forms yield the forms of the next examples considered in this subsection.

Example 11 (Expansions of Sums with a Linear Dependence of h on n). Further computation with the `http://www.risc.jku.at/research/combinat/software/Sigma/index.phpSigma` package for *Mathematica* similarly yields the following alternate form of the second sums in (22) implicit to the congruence identities stated in (23) and (24) above:

$$n! \equiv \sum_{i=0}^{n} \tbinom{2n+1}{i}^2 (-1)^i i! (n-i)! \qquad (\text{mod } 2n+1), \tag{27a}$$

$$= \tfrac{(-1)^n (3n+1)!}{8(n!)^2} \times \left(8 - \sum_{i=1}^{n} \tbinom{2i+1}{i}^2 \tfrac{(i!)^3}{2 \cdot (3i+1)!} \left(11 + \tfrac{20}{i} - \tfrac{8}{(2i+1)} + \tfrac{1}{(2i+1)^2} \right) \right), \tag{27b}$$

$$= \tfrac{(-1)^n (3n+1)!}{(n!)^2} - \tfrac{(-1)^n}{16} \times \sum_{i=1}^{n} \tbinom{2i+1}{i}^2 \tfrac{i! \cdot (3i+2)_{3n-3i}}{(i+1)_{n-i}^2} \left(11 + \tfrac{20}{i} - \tfrac{8}{(2i+1)} + \tfrac{1}{(2i+1)^2} \right),$$

$$= \frac{(-1)^n(3n+1)!}{8(n!)^2} \times \left(8 - \sum_{i=1}^{n} \binom{2i+1}{i}^2 \frac{(i!)^3}{(3i)!} \left(\frac{10}{i} + \frac{5}{(2i+1)} - \frac{1}{(2i+1)^2} - \frac{32}{(3i+1)}\right)\right), \tag{27c}$$

$$= \frac{(-1)^n(3n+1)!}{(n!)^2} - \frac{(3n+1)(-1)^n}{8} \times \sum_{i=1}^{n} \binom{2i+1}{i}^2 \frac{i!\cdot(3i+1)_{3n-3i}}{(i+1)_{n-i}^2} \left(\frac{10}{i} + \frac{5}{(2i+1)} - \frac{1}{(2i+1)^2} - \frac{32}{(3i+1)}\right);$$

The documentation for the `http://www.ris.jku.at/research/combinat/software/Sigma/index.phpSigma` package in [14, Ex. 3.3] contains several identities related to the partial harmonic-number-related expansions of the single factorial function sums given in the next examples, and for the computations contained in [13].

Example 12 (Expansions of Sums Involving a Quadratic Dependence of h on n). The second non-square variants of the sums implicit to the congruences providing characterizations of the twin prime pairs given in (12) of the introduction, and of the cousin and sexy primes expanded in (25) of the previous subsection, are easily generalized to form related results for other prime pairs (`http://oeis.org/A023202`, `http://oeis.org/A023203`, `http://oeis.org/A046133`). In particular, for positive integers $d \geq 1$, the special cases of these expansions for the prime pair sequences considered above lead to more general congruence-based characterizations of the odd prime pairs, $(2n+1, 2n+1+2d)$, in the form of the following equation for some $a_d, b_d, c_d \in \mathbb{Z}$ and where the parameter $h_d := (2n+1)(2n+1+2d) > 2n$ implicit to these sums depends quadratically on n [8, *cf.* §3, §5].

$$\begin{aligned} &2n+1, 2n+1+2d \text{ prime} \\ &\quad \text{if and only if} \quad a_d \times \underbrace{\sum_{i=0}^{2n} \binom{h_d}{i}^2 (-1)^i i!(2n-i)!}_{:= S_n(h_d) \equiv (2n)! \pmod{h_d}} + b_d n + c_d \equiv 0 \pmod{h_d} \end{aligned} \tag{28}$$

For natural numbers $h, i, n \geq 0$, let the shorthand for the functions, $T_{h,n}$ and $H_{h,i}$, be defined as in the next equations.

$$\begin{aligned} S_n(h) &:= \sum_{i=0}^{2n} \binom{h}{i}^2 (-1)^i i!(2n-i)! \\ T_{h,n} &:= \prod_{j=1}^{n} \left(\frac{(h-2j)^2(h+1-2j)^2}{2\cdot(2h+1-2j)(h-j)}\right) \\ &= 4^n \times \frac{\left(\frac{1-h}{2}\right)_n^2 \left(1-\frac{h}{2}\right)_n^2}{\left(\frac{1}{2}-h\right)_n (1-h)_n} = \frac{(1-h)_{2n}^2}{(1-2h)_{2n}} \\ H_{h,i} &:= \frac{h(h+1)(2h-1)}{(h-1)(h-i)} + \frac{2(h+1)^2(2h+1)}{h(2h+1-2i)} + \frac{2(h+1)}{h(h-1)(h+1-2i)} \\ &= \frac{(h+1)(2h+1-4i)(h-2i)}{(2h+1-2i)(h+1-2i)(h-i)} \end{aligned} \tag{29}$$

Computations with the `http://www.ris.jku.at/research/combinat/software/Sigma/index.phpSigma` package yield the next alternate expansion of the first sum defined in (29) given

by

$$S_n(h) = \sum_{i=0}^{n} \binom{h}{2i}^2 (2i)! \times \frac{T_{h,n}}{T_{h,i}} \times H_{h,i},$$

where the ratios of the product functions in the previous equation are simplified by the identities cited in [16] as follows:

$$\frac{T_{h,n}}{T_{h,i}} = \frac{(1-h)_{2n}^2}{(1-2h)_{2n}} \times \frac{(1-2h)_{2i}}{(1-h)_{2i}^2} = \frac{(1-h+2i)_{2n-2i}^2}{(1-2h+2i)_{2n-2i}},\ n \geq i. \tag{30}$$

The forms of the generalized sums, $S_n(h)$, obtained from the special case identity above using the `http://www.risc.jku.at/research/combinat/software/Sigma/index.phpSigma` software package routines are then expanded by the harmonic-number-related sums over the originally fixed indeterminate parameter h in the following forms:

$$\begin{aligned} S_n(h) &= \sum_{i=0}^{n} \tbinom{h}{2i}^2 \tfrac{(2i)! \times (1-h+2i)_{2n-2i}^2}{(1-2h+2i)_{2n-2i}} \times \left(\tfrac{h(h+1)(2h-1)}{(h-1)(h-i)} + \tfrac{2(h+1)^2(2h+1)}{h(2h+1-2i)} + \tfrac{2(h+1)}{h(h-1)(h+1-2i)} \right) \\ &= \sum_{i=0}^{n} \tbinom{h}{2n-2i}^2 (2n-2i)! \times \tfrac{(1-h+2n-2i)_{2i}^2}{(1-2h+2n-2i)_{2i}} \times \left(\tfrac{(h+1)(2h+1-4(n-i))(h-2(n-i))}{(2h+1-2(n-i))(h+1-2(n-i))(h-n+i)} \right) \end{aligned}$$

The first sum on the right-hand-side of (29) denotes the special prime pair congruence expansions for the twin, cousin, and sexy prime pairs already defined by the examples cited in the last sections corresponding to the respective forms of (28) where

$$\{(d, a_d, b_d, c_d)\}_{d=1}^{3} := \{(1,4,2,5), (2,96,48,119), (3,4320,1438,5039)\},$$

and where $h :\mapsto (2n+1)(2n+1+2d)$, as computed for reference in [13].

Remark 13. Note that the binomial coefficient identity, $\binom{\binom{k}{2}}{2} = 3\binom{k+1}{4}$, given in the exercises section of [5, p. 535, Ex. 5.67], suggests simplifications, or "*reductions*" in order of the sums, $S_n(h)$, when h denotes some fixed, implicit application-dependent quadratic function of n obtained by first expanding the inner terms, $\binom{h}{i}$, as a (finite) linear combination of binomial coefficient terms whose upper index corresponds to a linear function of n [13]. The `http://www.ris.jku.at/research/combinat/software/Sigma/index.phpSigma` package is able to obtain alternate forms of these pre-processed finite sums defining the functions, $S_n(\beta n + \gamma)$, for scalar-valued β, γ that generalize the last two expansions provided above in (27). The summary notebook document prepared with this manuscript contains additional remarks and examples related to the results in the article. For example, several specific expansions of the binomial coefficients, $\binom{(2n+1)(2n+2d+1)}{i}$, at upper index inputs varying quadratically on n suggested by the first upper index reduction identity above are computed as a starting point for simplifying the terms in these sums in [13]. Additional notes providing documentation and more detailed computational examples will be added to updated versions of the computational summary document.

3.5. Expansions of Congruences Involving the Double Factorial Function

3.5.1. Statements of Congruences for the Double Factorial Function

Proposition 3 (Congruences for the Double Factorial Function). *Let $h \geq 2$ be odd or prime and suppose that s is an integer satisfying $0 \leq s \leq h$. We have the following congruences for the double factorial function, $(2n-1)!!$:*

$$\begin{aligned}(2n-1)!! &\equiv \sum_{i=0}^{n} \binom{h}{i} 2^{n+(s+1)i} \left(\tfrac{1}{2}-h\right)_i \left(\tfrac{1}{2}\right)_{n-i} && \pmod{2^s h}\\ &\equiv \sum_{i=0}^{n} \binom{h}{i} \binom{2n-2i}{n-i} \frac{2^{n+(s+1)i}}{4^{n-i}} \times \left(\tfrac{1}{2}-h\right)_i (n-i)! && \pmod{2^s h}\\ &\equiv \sum_{i=0}^{n} \binom{h}{i} (-2)^{n+(s+1)i} \left(\tfrac{1}{2}+n-h\right)_i \left(\tfrac{1}{2}-n\right)_{n-i} && \pmod{2^s h}.\end{aligned}$$

Proof. The coefficient expansion given by the last identity in (10c) provides the alternate forms of congruences for the double factorial functions, $(2n-1)!! = p_n(-2, 2n-1)$ and $2^n (1/2)_n = p_n(2,1)$, modulo $2^s \cdot h$ stated in the first and third of the previous equations for fixed integers $h \geq 2$ and any integer $0 \leq s \leq h$. For natural numbers $n \geq 0$, the central binomial coefficients satisfy an expansion by the following identity given in [5, §5.3]:

$$\left(\tfrac{1}{2}\right)_n = \binom{-\frac{1}{2}}{n} \times (-1)^n n! = \binom{2n}{n} \times \frac{n!}{4^n}. \qquad \text{(Binomial Coefficient Half-Index Identities)}$$

The first identity in the proposition together with the previous half-index identity for the binomial coefficients imply the second congruence stated in the proposition. □

3.5.2. Semi-polynomial Congruences Expanding the Central Binomial Coefficients

The next polynomial congruences satisfied by the central binomial coefficients modulo integer multiples of the individual polynomial powers, n^p, of n for some fixed $p \geq 1$ also provide additional examples of some of the double-factorial-related phrasings of the expansions of (10b) and (10c) following from the noted identity given in (7c) (*cf.* Lemma 2 and the computations contained in [13])[3].

$$\begin{aligned}\binom{2n}{n} &= \frac{2^n}{n!} \times (2n-1)!! && (31)\\ &\equiv \left\{ \frac{\sum_{i=0}^{n} \binom{x^p}{i} 2^i \left(\frac{1}{2}-x^p\right)_i \left(\frac{1}{2}\right)_{n-i} \times \frac{2^{2n}}{n!}}{\bmod x^p \quad \curvearrowright \quad x :\mapsto n} \right\} && \pmod{n^p}\end{aligned}$$

[3] In this context, the notation $\bmod f(x) \curvearrowright x :\mapsto n$ denotes that the underlined expression should first be reduced as a polynomial in x modulo $f(x)$, and then after this operation is performed that x should be set to the explicit value of n. Surprisingly, this procedure inplemented in *Mathematica* using the function `PolynomialMod` in [13] produces correct, integer-valued congruence expressions even when the double-factorial-related numerator of the first equation in (31) is divided through by the reciprocal of $n!$, though we would typically expect this not to necessarily be the case if $2^n(2n-1)!!$ were first reduced modulo the right-hand-side functions of n.

$$\equiv \left\{ \frac{\sum_{i=0}^{n} \binom{x^p}{i} \binom{2x-2i}{x-i} \left(\frac{1}{2} - x^p\right)_i \times \frac{8^i \cdot (n-i)!}{n!}}{\bmod x^p \quad \circlearrowleft \quad x :\mapsto n} \right\} \qquad (\text{mod } n^p)$$

The special cases of the congruences in (31) corresponding to $p := 3$ and $p := 4$, respectively, are related to the necessary condition for the primality of odd integers $n > 3$ in Wolstenholme's theorem and to the sequence of *Wolstenholme primes* defined as [9, §2.2] [6, *cf.* §7] (`http://oeis.org/A088164`)

$$\begin{aligned} \mathbb{P}_{\text{Wolstenholme}} &:= \left\{ n \geq 5 : \ n \text{ prime and } \binom{2n}{n} \equiv 2 \pmod{n^4} \right\} \qquad \text{(Wolstenholme Primes)} \\ &= (16843, 2124679, \ldots) \, . \end{aligned}$$

3.5.3. An Identity for the Single Factorial Function Involving Expansions of Double Factorial Functions

As another example of the applications of these new integer congruence applications expanded through the double factorial function, notice that the following identity gives the form of another exact, finite double sum expansion of the single factorial function over convolved products of the double factorials:

$$\begin{aligned} (n-1)! = (2n-3)!! &+ \sum_{k=1}^{n-2} \sum_{j=k}^{n-1} (-1)^{j+1} \, (-j)_k \, (-(2n-k-j-2))_{j-k} \, (2n-2j-3)!! \\ &+ \sum_{k=1}^{n-2} \sum_{j=k+1}^{n-1} (-1)^{j} \, (-j)_{k+1} \, (-(2n-k-j-3))_{j-k-1} \, (2n-2j-3)!! \end{aligned}$$

This identity is simple to prove starting from the first non-round sum given in §5.1 of [1] combined with second identity for the component summation terms given in §6.3 of the same article. A modified approach involving the congruence techniques outlined in either the first cases cited in Example 4, or as suggested in the previous few example cases from the last subsections of the article, then suggests even further applications adapting the results for new variants of the established, or otherwise well-known, special case congruence-based identities expanded for the notable prime number subsequences cited above in terms of the double factorial function (see Example 14 in the next subsection).

3.6. Applications of Wilson's theorem in other famous and notable special case prime subsequences

The integer congruences obtained from Wilson's theorem for the particular special sequence cases noted in Section 3.4.2 are easily generalized to give constructions over the forms other prime subsequences including the following special cases:

- The "*Pierpont primes*" of the form $p := 2^u 3^v + 1$ for some $u, v \in \mathbb{N}$ (`http://oeis.org/A005109`);
- The subsequences of primes of the form $p := n2^n \pm 1$ (`http://oeis.org/A002234`, `http://oeis.org/A080075`);
- The "*Wagstaff primes*" corresponding to prime pairs of the form $\left(p, \frac{1}{3}(2^p + 1)\right) \in \mathbb{P}^2$ (`http://oeis.org/A000978`, `http://oeis.org/A123176`); and

- The generalized cases of the multifactorial prime sequences tabulated as in [9, Table 6, §2.2] consisting of prime elements of the form $p := n!_{(\alpha)} \pm 1$ for a fixed integer-valued $\alpha \geq 2$ and some $n \geq 1$.

Example 14 (The Factorial Primes and the Fermat Prime Subsequences). The sequences of *factorial primes* of the form $p := n! \pm 1$ for some $n \geq 1$ satisfy congruences of the following form modulo $n! \pm 1$ given by the expansions of (7c) and (8) ([9, *cf.* §2.2], `http://oeis.org/A002981`, `http://oeis.org/A002982`):

$n! + 1$ prime if and only if (Factorial Prime Congruences)

$$\underbrace{\sum_{i=0}^{n!} \binom{n!+1}{i}^2 (-1)^i i!(n!-i)!}_{(n!)! \equiv C_{n!+1,n!}(1,1) \pmod{n!+1}} \equiv -1 \pmod{n!+1}$$

$n! - 1$ prime if and only if

$$\underbrace{\sum_{i=0}^{n!-2} \binom{n!-1}{i}^2 (-1)^i i!(n!-2-i)!}_{(n!-2)! \equiv C_{n!-1,n!-2}(1,1) \pmod{n!-1}} \equiv -1 \pmod{n!-1}.$$

The *Fermat numbers*, F_n, generating the subsequence of *Fermat primes* of the form $p := 2^m + 1$ where $m = 2^n$ for some $n \geq 0$ similarly satisfy the next congruences expanded through the identities for the single and double factorial functions given above [9, §2.6] [6, §2.5] (`http://oeis.org/A000215`, `http://oeis.org/A019434`):

$F_n := 2^{2^n} + 1$ prime (Fermat Prime Congruences)

$$\begin{aligned}
&\text{if and only if } 2^{2^n}+1 \mid \left(2^{2^n}\right)! + 1\\
&\text{if and only if } 2^{2^n}+1 \mid 2^{2^{2^n}-1} \sum_{i=0}^{2^{2^n}} \binom{2^{2^n}+1}{i}^2 (-1)^i i! \left(2^{2^n} - i\right)! + 1\\
&\text{if and only if } 2^{2^n}+1 \mid 2^{2^{2^n}-1} \left(2^{2^n-1}\right)! \left(2^{2^n}-1\right)!! + 1\\
&\text{if and only if } 2^{2^n}+1 \mid 2^{\frac{3}{4}\cdot 2^{2^n}} \left(2^{2^n-2}\right)! \left(2^{2^n-1}-1\right)!! \left(2^{2^n}-1\right)!! + 1\\
&\text{if and only if } 2^{2^n}+1 \mid 2^{\frac{7}{8}\cdot 2^{2^n}} \left(2^{2^n-3}\right)! \left(2^{2^n-2}-1\right)!! \left(2^{2^n-1}-1\right)!! \left(2^{2^n}-1\right)!! + 1.
\end{aligned}$$

For integers $h \geq 2$ and $r \geq 1$ such that $2^r \mid h$, the expansions of the congruences in the previous several equations correspond to forming the products of the single and double factorial functions modulo $h + 1$ from the previous examples to require that

$$2^{\left(1-2^{-r}\right)\cdot h} \times \left(\frac{h}{2^r}\right)! \left(\frac{h}{2^{r-1}} - 1\right)!! \times \cdots \times \left(\frac{h}{2} - 1\right)!! \, (h-1)!! \equiv -1 \pmod{h+1},$$

though more general expansions by products of the α-factorial functions, $\alpha!_{(n)}$, for $\alpha > 2$ are apparent [11, *cf.* §6.4]. The *generalized Fermat numbers*, $F_n(\alpha) := \alpha^{2^n} + 1$, and the corresponding *generalized Fermat prime* subsequences when $\alpha := 2, 4, 6$ suggest generalizations of the approach

to the results in the previous equations through the procedure to the multiple, α-factorial function expansions suggested in Section 3.3.5 that generalizes the procedure to expanding the congruences above for the Fermat primes when $\alpha := 2$.

The next concluding examples provide an approach to generalized congruences providing necessary and sufficient conditions on the primality of integers in special prime subsequences involving mixed expansions of the single and double factorial functions.

Example 15 (Mersenne Primes). The *Mersenne primes* correspond to prime pairs of the form (p, M_p) for p prime and where $M_n := 2^n - 1$ is a *Mersenne number* for some (prime) integer $n \geq 2$ [9, §2.7], [6, §2.5; §6.15], [5, *cf.* §4.3, §4.8] (`http://oeis.org/A001348`, `http://oeis.org/A000668`, `http://oeis.org/A000043`). The requirements in Wilson's theorem for the primality of both p and M_p provide elementary proofs of the following equivalent necessary and sufficient conditions for the primality of the prime pairs of these forms:

$$(p, 2^p - 1) \in \mathbb{P}^2 \qquad \text{(Mersenne Prime Congruences)}$$

$$\text{if and only if}$$

$$p(2^p - 1) \mid (p-1)!(2^p - 2)! + (p-1)! + (2^p - 2)! + 1$$

$$\text{if and only if } p(2^p - 1) \mid \big(C_{p(2^p-1),p-1}(-1, p-1)C_{p(2^p-1),2^p-2}(-1, 2^p - 2) + C_{p(2^p-1),p-1}(-1, p-1) + C_{p(2^p-1),2^p-2}(-1, 2^p - 2) + 1\big)$$

$$\text{if and only if } p(2^p - 1) \mid \big(C_{p(2^p-1),p-1}(1, 1)C_{p(2^p-1),2^p-2}(1, 1) + C_{p(2^p-1),p-1}(1, 1) + C_{p(2^p-1),2^p-2}(1, 1) + 1\big)$$

$$\text{if and only if } p(2^p - 1) \mid 2^{2^{p-1}-1}(p-1)!(2^{p-1} - 1)!(2^p - 3)!! + (p-1)! + (2^p - 2)! + 1;$$

The congruences on the right-hand-sides of the previous equations are then expanded by the results in (7c) and (8), and through the second cases of the more general product function congruences stated in (10b).

Example 16 (Sophie Germain Primes). Wilson's theorem similarly implies the next related congruence-based characterizations of the *Sophie Germain primes* corresponding to the prime pairs of the form $(p, 2p + 1)$ where $p, p - 1, 2p < p(2p + 1)$ [9, §5.2] (`http://oeis.org/A005384`).

$$(p, 2p + 1) \in \mathbb{P}^2 \qquad \text{(Sophie Germain Prime Congruences)}$$

$$\text{if and only if } (p-1)!(2p)! + (p-1)! + (2p)! \equiv -1 \pmod{p(2p+1)}$$

$$\text{if and only if } 2^p p!(p-1)!(2p-1)!! + (p-1)! + (2p)! \equiv -1 \pmod{p(2p+1)}$$

The expansions of the generalized forms of the Sophie Germain primes noted in [9, §5.2] also provide applications of the multiple, α-factorial function identities outlined in Example 14 and suggested in Section 3.3.5 of the article which result from expansions of the arithmetic progressions of the single factorial functions given in the examples from [11, §6.4].

Remark 17 (Congruences for Integer Powers and Sequences of Binomials). Notice that most of the factorial function expansions involved in the results formulated by the previous few

examples do not immediately imply corresponding congruences obtained from (10) satisfied by the *Wieferich prime* sequence defined by [9, §5.3] (`http://oeis.org/A001220`)

$$\mathbb{P}_{\text{Wieferich}} := \left\{n \geq 2 : \ n \text{ prime and } 2^{n-1} \equiv 1 \pmod{n^2}\right\} \qquad \text{(Wieferich Primes)}$$
$$\xrightarrow{\texttt{A001220}} \quad (1093, 3511, \ldots),$$

nor results for the variations of the sequences of binomials enumerated by the rational convergent-function-based generating function identities over the binomial coefficient sums constructed in [11, §6.7] modulo prime powers p^m for $m \geq 2$. However, indirect expansions of the sequences of binomials, 2^{n-1} and $2^{n-1}-1$, by the Stirling numbers of the second kind through the lemma provided in [11, Lemma 12], yield the following divisibility requirements characterizing the sequence of Wieferich primes where $(2)_n = (n+1)!$:

$$\begin{aligned}
2^{n-1} &= \sum_{k=0}^{n-1} \left\{ n-1 \atop k \right\} (-1)^{n-1-k} (2)_k \\
&\equiv \sum_{k=0}^{n-1} \sum_{i=0}^{k+1} \left\{ n-1 \atop k \right\} \binom{n^2}{i}^2 (-1)^{n-1-k-i} i!(k+1-i)! \qquad \pmod{n^2} \\
2^{n-1}-1 &= 2^{n-2} + 2^{n-3} + \cdots + 2 + 1 \\
&= \sum_{0 \leq j \leq i \leq n-2} \left\{ i \atop j \right\} (-1)^{i-j} (2)_j \\
&\equiv \sum_{0 \leq m \leq j \leq i \leq n-2} \left\{ i \atop j \right\} \binom{n^2}{m} (-1)^{i-j+m} (n^2+1)^{\underline{m}} (2)_{j-m} \qquad \pmod{n^2}.
\end{aligned}$$

The constructions of the corresponding congruences for the sequences of binomials, $a^{n-1}-1 \pmod{n^2}$, are obtained by a similar procedure [9, *cf.* §5.3; Table 45]. Additional expansions of related congruences for the terms $3^t + 1 \pmod{2t+1}$ for prime $2t+1 \in \mathbb{P}$ in the particular forms of

$$\begin{aligned}
3^t + 1 &\equiv \sum_{j=0}^{t} \sum_{i=0}^{j} \left\{ t \atop j \right\} \binom{2t+1}{m} (-1)^{t-j+m} (2t+3)^{\underline{m}} (3)_{j-m} + 1 \qquad \pmod{2t+1} \\
3^t + 1 &\equiv 4 \times \sum_{m=0}^{t-1} \sum_{j=0}^{m} \sum_{i=0}^{j} \left\{ m \atop j \right\} \binom{2t+1}{i} (-1)^{t-1-j+i} (2t+3)^{\underline{i}} (3)_{j-i} \qquad \pmod{2t+1},
\end{aligned}$$

where $(3)_j = \frac{1}{2}(i+2)!$, lead to double and triple sums providing the necessary and sufficient condition for the primality of the Fermat primes, F_k, from [6, §6.14] when $t :\mapsto 2^{2^k-1}$ for some integer $k \geq 1$.

4. Conclusions

4.1. Summary

In Section 3.1, we proved the key results stated in Proposition 1 for the special cases where $\alpha := \pm 1, 2$, which includes the congruences involving the single and double factorial functions cited in the other applications from Section 3. We note that our numerical evidence provided in the summary notebook [13] suggests that these congruences do in fact also hold more generally for all integers $\alpha \neq 0$ and all $h \geq 2$ (not just the odd and prime cases of these integer-valued moduli). The applications of the key proposition given in the article provide a number of propositions which follow as corollaries of the first set of results. The specific examples of the new results we prove within the article include new finite sum expansions and congruences for the α-factorial functions, as well as applications of our results to formulating new statements of necessary and sufficient conditions on the primality of integer subsequences, pairs, and triples.

In many respects, this article is a follow-up to the first article published in 2017 [11]. For comparision with the results given in that article, we note that the results in Proposition 1 and Corollary 1 do not provide a simple or otherwise apparent mechanism for formulating new congruences and recurrence relations satisfied by the triangles of α-factorial coefficients, $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_{\alpha}$. However, such congruences and recurrence relations for the corresponding coefficients, $[R^k]p_n(\alpha, R)$, of the generalized product sequences defined in (1) are in contrast easy to obtain from the results in the key proposition. We conclude the article by posing several remaining open questions related to the expansions of the generalized factorial functions considered in this article and in [11, 12].

4.2. Open Questions and Topics for Future Research

4.2.1. Open Questions

We pose the following open questions as topics for future research on the generalized factorial functions studied in this article and in [11, 12]:

- Can we determine bounds on the zeros of the convergent denominator functions, $\mathrm{FQ}_h(\alpha, \alpha - d; z)$ and $\mathrm{FQ}_h(-\alpha, \alpha n - d; z)$, in order to determine more accurate Stirling-like approximations for the generalized multiple factorial function cases of $(\alpha n - d)!_{(\alpha)}$?
- What is the best way to evaluate the α-factorial functions, $n!_{(\alpha)}$, for fractional α or non-negative rational n? For example, are the finite sum representations in Proposition 1 a good start at generalizing these functions to non-integer arguments and the strictly rational-valued parameters that occur in applications intentionally not discussed in these articles?
- What is the best way to translate the new congruence results for the single, double, and α-factorial functions to form integer congruences for the binomial coefficients, $\binom{x}{k} = \frac{x^{\underline{k}}}{k!}$?

4.2.2. Applications to Generalizations of Known Finite Sum Identities Involving the Double Factorial Function

The construction of further analogues for generalized variants of the finite summations and more well-known combinatorial identities satisfied by the double factorial function cases when $\alpha := 2$

from the references is suggested as a topic for future investigation. The identities for the more general α-factorial function cases stated in Example 2 of Section 3.2 suggest one possible approach to generalizing the known identities summarized in [4, 1] for the next few particularly interesting special cases corresponding to the triple and quadruple factorial function cases, $n!!!$ and $n!!!!$, respectively.

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