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INTEGERS WHICH CANNOT BE PARTITIONED INTO AN EVEN NUMBER OF CONSECUTIVE PARTS

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Abstract

Consider the algebraic torus (as a variety) $X := \mathbb{C}^\times \times \mathbb{C}^\times$ and its Hilbert scheme of n points, denoted $X^{[n]}$. Let $E\left(\widetilde{X}^{[n]}; q\right)$ be the E-polynomial of the GIT[1] quotient $\widetilde{X}^{[n]} := X^{[n]}//G$, where $G := \mathbb{C}^\times \times \mathbb{C}^\times$ is the algebraic torus (as a Lie group) and the action of G on $X^{[n]}$ is induced by the obvious action of G on X. We show a relationship between the coefficients of $E\left(\widetilde{X}^{[n]}; q\right)$ and the existence of a partition

$$n = m + (m+1) + (m+2) + \cdots + (m+k-1),$$

with $m, k \in \mathbb{Z}_{\geq 1}$, such that k even.

1. Introduction

Let $X := \mathbb{C}^\times \times \mathbb{C}^\times$ be the algebraic torus as a variety. Denote $X^{[n]}$ the Hilbert scheme of n points on X. The Lie group $G := \mathbb{C}^\times \times \mathbb{C}^\times$ acts on X in the obvious way. This action can be extended to $X^{[n]}$. So, we can define the GIT quotient $\widetilde{X}^{[n]} := X^{[n]}//G$.

The E-polynomial of $\widetilde{X}^{[n]}$, denoted $E\left(\widetilde{X}^{[n]}; q\right)$, was studied by T. Hausel, E. Letellier and F. Rodriguez-Villegas [4] and independently[2] by C. Kassel and C. Reutenauer [6, 7, 8]. For more arithmetical results about these polynomials, see [2, 3].

[1]GIT is the abbreviation for Geometric Invariant Theory.

[2]C. Kassel and C. Reutenauer defined $E\left(\widetilde{X}^{[n]}; q\right)$, in a rather combinatorial way, as the unique polynomial $P_n(q) = E\left(\widetilde{X}^{[n]}; q\right)$ satisfying $(q-1)^2 P_n(q) = C_n(q)$, where $C_n(q)$ is the number of n-codimensional ideals of the algebra $\mathbb{F}_q[x, y, x^{-1}, y^{-1}]$.

The degree of $E\left(\widetilde{X}^{[n]};q\right)$ is $2n-2$, it is self-reciprocal and all its coefficients are non-negative integers [7], i.e.

$$E\left(\widetilde{X}^{[n]};q\right) = a_{n,0}q^{n-1} + \sum_{i=1}^{n-1} a_{n,i}\left(q^{n-1+i}+q^{n-1-i}\right), \qquad (1)$$

for some nonnegative integers $a_{n,0}, a_{n,1}, a_{n,2}, \ldots, a_{n,n-1}$. The aim of this paper is to prove the following result.

Theorem 1. *For each integer $n \geq 1$, the following statements are equivalent:*

(i) all odd divisors of n are smaller than $\sqrt{2n}$;

(ii) $a_{n,0} \geq a_{n,1} \geq a_{n,2} \geq \cdots \geq a_{n,n-1}$;

(iii) for all integers $m \geq 1$ and $k \geq 1$, the equality[3]

$$n = m + (m+1) + (m+2) + \cdots + (m+k-1)$$

implies that k is odd.

It is worth mentioning that a finite sequence $s_1, s_2, \ldots, s_n$ is *unimodal* if and only if there is some $1 \leq t \leq n$ such that

$$s_1 \leq s_2 \leq s_3 \leq \cdots \leq s_t \geq s_{t+1} \geq s_{t+2} \geq s_{t+3} \geq \cdots \geq s_n.$$

A polynomial having non-negative coefficients is said to be *unimodal* if its sequence of coefficients is unimodal. So, if $E\left(\widetilde{X}^{[n]};q\right)$ satisfies condition (ii) in Theorem 1, then it is unimodal.

2. Proof of the Main Result

We will use the generating function

$$\prod_{m=1}^{\infty} \frac{(1-t^m)^2}{(1-qt^m)(1-q^{-1}t^m)} = 1 + \left(q+q^{-1}-2\right)\sum_{n=1}^{\infty} \frac{E\left(\widetilde{X}^{[n]};q\right)}{q^{n-1}} t^n, \qquad (2)$$

due to T. Hausel, E. Letellier and F. Rodriguez-Villegas [4] and independently to C. Kassel and C. Reutenauer [6].

[3]The expression of a number as a sum of consecutive numbers is named *polite representation*.

Lemma 1. *The number of solutions* $(m,k) \in (\mathbb{Z}_{\geq 1})^2$ *of the equation*

$$n = m + (m+1) + (m+2) + \cdots + (m+k-1),$$

with k *even, coincides with the number of odd divisors* d *of* n *satisfying the inequality* $d > \sqrt{2n}$.

Proof. This result is due to M. D. Hirschhorn and P. M. Hirschhorn [5]. □

Lemma 2. *Let* $n \geq 1$ *be an integer. For any divisor* d *of* $2n$*, if* $d > \sqrt{2n}$ *then*

$$n + \frac{1}{2}\left(d - \frac{2n}{d} - 1\right) \geq n > n - 1 \geq n + \frac{1}{2}\left(\frac{2n}{d} - d - 1\right) \geq 0.$$

Proof. Consider an integer $n \geq 1$. Let d be a divisor of $2n$. Suppose that $d > \sqrt{2n}$. The inequality $d > \sqrt{2n}$ implies that $d - \frac{2n}{d} > 0$. Using the fact that $d - \frac{2n}{d}$ is an integer, it follows that $d - \frac{2n}{d} \geq 1$. So, $n + \frac{1}{2}\left(d - \frac{2n}{d} - 1\right) \geq n$.

The inequality $d - \frac{2n}{d} > 0$ implies that $\frac{2n}{d} - d < 0$. Using the fact that $\frac{2n}{d} - d$ is an integer, it follows that $\frac{2n}{d} - d \leq -1$. So, $n + \frac{1}{2}\left(\frac{2n}{d} - d - 1\right) \leq n - 1$.

For $x \geq 1$ and $y \geq 1$, we have the trivial inequality

$$(x-1)(y-1) \geq 2(1-y).$$

From the above inequality, it follows that $xy \geq x - y + 1$. Substituting $x = d$ and $y = \frac{2n}{d}$, we obtain

$$2n \geq d - \frac{2n}{d} + 1,$$

which is equivalent to

$$n + \frac{1}{2}\left(\frac{2n}{d} - d - 1\right) \geq 0.$$

□

Lemma 3. *Let* $n \geq 1$ *be an integer. For any divisor* d *of* $2n$*, if* $d < \sqrt{2n}$*, then*

$$n + \frac{1}{2}\left(\frac{2n}{d} - d - 1\right) \geq n > n - 1 \geq n + \frac{1}{2}\left(d - \frac{2n}{d} - 1\right) \geq 0.$$

Proof. It is enough to apply Lemma 2 with $d = \frac{2n}{d}$. □

We proceed to prove our main result.

Proof. (Theorem 1) The equality

$$E\left(\widetilde{X}^{[n]}; q\right) = \sum_{\substack{d \mid n \\ d \equiv 1 \pmod 2}} \frac{q^{n+(2n/d-d-1)/2} - q^{n+(d-2n/d-1)/2}}{q-1}$$

follows from the combination of (2) with the classical identity [4, p. 113]

$$\frac{1}{\theta(w)} - \frac{1}{1-w} = \sum_{\substack{n,\, m \geq 1 \\ n \not\equiv m \pmod 2}} (-1)^n q^{nm/2} w^{(m-n-1)/2},$$

attributed to L. Kronecker and C. Jordan, where $\theta(w)$ is the formal product

$$\theta(w) := (1-w) \prod_{n \geq 1} \frac{(1-q^n w)\left(1-q^n w^{-1}\right)}{(1-q^n)^2}.$$

We can express $E\left(\widetilde{X}^{[n]}; q\right)$ as the difference $E\left(\widetilde{X}^{[n]}; q\right) = R_n(q) - S_n(q)$ of two polynomials given by[4]

$$\begin{aligned} S_n(q) &= \sum_{\substack{d|n \\ d \equiv 1 \pmod 2 \\ d > \sqrt{2n}}} \frac{q^{n+(d-2n/d-1)/2} - q^{n+(2n/d-d-1)/2}}{q-1}, \\ R_n(q) &= \sum_{\substack{d|n \\ d \equiv 1 \pmod 2 \\ d < \sqrt{2n}}} \frac{q^{n+(2n/d-d-1)/2} - q^{n+(d-2n/d-1)/2}}{q-1}. \end{aligned}$$

Applying Lemmas 2 and 3, the coefficients of $S_n(q)$ and $R_n(q)$ are non-negative integers. Using the expansion $\frac{q^n-1}{q-1} = 1 + q + q^2 + \cdots + q^{n-1}$, it follows from the explicit formulae for $S_n(q)$ and $R_n(q)$ that the coefficients from (1) can be expressed as $a_{n,i} = a_{n,i}^+ - a_{n,i}^-$, where

$$\begin{aligned} a_{n,i}^+ &= \#\left\{d|n : \quad d \text{ odd}, \, d < \sqrt{2n}, \, i \leq \frac{1}{2}\left(\frac{2n}{d} - d - 1\right)\right\}, \\ a_{n,i}^- &= \#\left\{d|n : \quad d \text{ odd}, \, d > \sqrt{2n}, \, i \leq \frac{1}{2}\left(d - \frac{2n}{d} - 1\right)\right\}. \end{aligned}$$

Notice that the functions $\mathbb{Z}_{\geq 0} \longrightarrow \mathbb{Z}_{\geq 0}$, given by $i \mapsto a_{n,i}^+$ and $i \mapsto a_{n,i}^-$, are both weakly decreasing[5]. Hence, condition (ii) holds provided that $d < \sqrt{2n}$ for each odd divisor d of n, because in this case, $a_{n,i}^- = 0$ for all i.

Suppose that $d_0 > \sqrt{2n}$ for a fixed odd divisor d_0 of n. On the one hand, $a_{n,i_0}^- > a_{n,i_0+1}^-$, where $i_0 := \frac{1}{2}\left(d_0 - \frac{2n}{d_0} - 1\right)$. On the other hand, $a_{n,i_0}^+ = a_{n,i_0+1}^+$, because the equality $\frac{1}{2}\left(d_0 - \frac{2n}{d_0} - 1\right) = \frac{1}{2}\left(\frac{2n}{d} - d - 1\right)$ is impossible[6] for any odd

[4]Notice that, if $d = \sqrt{2n}$, for some integer d, then d is even.
[5]A sequence $s_1, s_2, \ldots, s_n$ is *weakly decreasing* if $s_1 \geq s_2 \geq \cdots \geq s_n$.
[6]This equality would imply that the product $d\,d_0 = 2n$ is even, while both d_0 and d are odd.

divisor d of n. So, $a_{n,i_0} < a_{n,i_0+1}$. Hence, condition (ii) does not hold provided that there is at least one odd divisor d of n satisfying $d > \sqrt{2n}$.

In virtue of Lemma 1, we conclude that conditions (iii) and (ii) are equivalent. The equivalence between (i) and (iii) follows by Lemma 1. $\square$

3. Final Remarks

Consider the symmetric Dyck word [1] $\langle\langle n \rangle\rangle := w_1 w_2 \cdots w_k \in \{+,-\}^*$, whose letters are given by

$$w_i := \begin{cases} +, & \text{if } u_i \in D_n \setminus (2D_n)\,; \\ -, & \text{if } u_i \in (2D_n) \setminus D_n; \end{cases}$$

where D_n is the set of divisors of n, $2D_n := \{2d : \quad d \in D_n\}$ and $u_1, u_2, \ldots, u_k$ are the elements of the symmetric difference $D_n \triangle 2D_n$ written in increasing order. This word encodes the non-zero coefficients of $(q-1)E\left(\widetilde{X}^{[n]}; q\right)$. Theorem 1 admits the language-theoretical reformulation: condition (iii) is equivalent to $\langle\langle n \rangle\rangle = \underbrace{+ + \cdots +}_{\text{s } times}\underbrace{- - \cdots -}_{\text{s } times}$, for some $s \in \mathbb{Z}_{\geq 1}$. For details, see [1].

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