ON A RESULT OF BAMBAH AND CHOWLA

Peter Shiu
353 Fulwood Road, Sheffield, S10 3BQ, United Kingdom
p.shiu@yahoo.co.uk

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Abstract

In 1947, R. P. Bambah and S. Chowla proved that if $\alpha > 2\sqrt{2}$ then, for all large integers n, there are integers u, v such that $n \le u^2 + v^2 < n + \alpha n^{\frac{1}{4}}$. We pose a hypothesis on the distribution of the fractional parts of numbers involving the square root function, and show that, on the assumption of the hypothesis, the same result holds for any $\alpha > 0$. The case $n \le u^2 + v^2 < n + 2\sqrt{2}n^{\frac{1}{4}} - a$, with $a \ge 0$, is also considered.

1. Introduction

The best known result for the upper bound for the gaps between numbers that are sums of two squares is due to R. P. Bambah and S. Chowla [1]. Their theorem states that if $\alpha > 2\sqrt{2}$ then there exists $n_0 = n_0(\alpha)$ with the property that if $n \ge n_0$ then there are integers u, v such that $n \le u^2 + v^2 < n + \alpha n^{\frac{1}{4}}$. Their argument is based on geometric considerations and, when simplified in terms of arithmetic (see [3]), the proof amounts to the following. Let $n \ge 1$ and set

$$u = \lfloor \sqrt{n} \rfloor, \qquad v = \lceil \sqrt{n - u^2} \rceil, \tag{1}$$

so that $\sqrt{n - u^2} \le v < \sqrt{n - u^2} + 1$, that is $n - u^2 \le v^2 < n - u^2 + 2\sqrt{n - u^2} + 1$, and thus

$$\begin{aligned} n \le u^2 + v^2 &< n + 2\sqrt{n - u^2} + 1 < n + 2\sqrt{n - (\sqrt{n} - 1)^2} + 1 \\ &< n + 2\sqrt{2}n^{\frac{1}{4}} + 1, \end{aligned}$$

which is a tiny bit better than the stated result. However, efforts in trying to reduce the 'Bambah-Chowla coefficient' $2\sqrt{2}$, never mind the reduction of the exponent $\frac{1}{4}$, are faced with insurmountable difficulties, and the problem has remained open for over seventy years now. Indeed, as we shall see, even the very modest improvement of replacing $+1$ by -3 in the bound above is already a formidable challenge.

When formulating the Diophantine requirement for the problem we find that the kernel lies in the distribution of the fractional parts of numbers involving the square root function. Unfortunately, although the requirement is very plausible, it appears that its establishment is just beyond our present knowledge on uniform distributions, and methods on the estimation of exponential sums. As a consequence we only have a hypothetical reduction of the said coefficient.

We use Roman and Greek letters for integers and real numbers, respectively, and we write $\{\phi\} = \phi - \lfloor\phi\rfloor$, the fractional part of ϕ, and

$$G = 2up + v^2 - p^2, \qquad 1 \le p \le v \le \sqrt{2u}. \tag{2}$$

Hypothesis H. *Let $0 < \alpha < \beta < 2\sqrt{2}$. Then, corresponding to each large v, there exists $p \le v$ such that*

$$\frac{\beta - \alpha}{2\sqrt{2}} \cdot \frac{1}{\sqrt{p}} < \{\sqrt{G}\} < \frac{\beta}{2\sqrt{2}} \cdot \frac{1}{\sqrt{p+1}}. \tag{3}$$

Theorem 1. *Suppose that Hypothesis H is true. Then, corresponding to each $\alpha > 0$, there exists $n_1 = n_1(\alpha)$ with the property that if $n \ge n_1$ then there are u, v such that $n \le u^2 + v^2 < n + \alpha n^{\frac{1}{4}}$.*

As a corollary to Theorem 1, under the assumption of Hypothesis H, we have the following weaker conclusion: Corresponding to any $a \ge 0$, there exists $n_2 = n_2(a)$ with the property that if $n \ge n_2$ then there are u, v such that

$$n \le u^2 + v^2 < n + 2\sqrt{2}n^{\frac{1}{4}} - a. \tag{4}$$

There is now the interesting problem of establishing such a result unconditionally. Indeed, the case $a = 1$ was established by S. Uchiyama [4] in 1964, and very recently G. J. O. Jameson [2] has dealt with $a = 2$. The case $a = 3$ turns out to be difficult.

For $n = 1, 2, \dots$, there is a sequence of *BC-points*, lattice points $(u, v) = (u_n, v_n)$ defined by (1). Write

$$h_n = u_n^2 + v_n^2 - n, \qquad \Phi_n = 2\sqrt{2}n^{\frac{1}{4}}, \tag{5}$$

and our earlier argument shows that $h_n < \Phi_n + 1$. Partition the set of positive integers into the intervals

$$\mathcal{I}_u = \{n : u^2 \le n < (u+1)^2\}, \qquad u = 1, 2, \dots ; \tag{6}$$

we establish Theorem 1 for each $n \in \mathcal{I}_u$ with u large; note that the abscissa of all the associated BC-point is u, and that $\sqrt{u} \le n^{\frac{1}{4}}$.

2. Proof of Theorem 1

Let $\alpha > 0$, and u be sufficiently large, depending on α. We show that, corresponding to each $n \in \mathcal{I}_u$, there is a lattice point (u, v) with a gap value $h = u^2 + v^2 - n$ satisfying

$$0 \le h \le \alpha\sqrt{u} + 1. \tag{7}$$

Consider the BC-point $(u, v) = (u_n, v_n)$, and define $\beta = \beta(n)$ by

$$h_n = u^2 + v^2 - n = \beta\sqrt{u}.$$

We may assume that $\beta > \alpha$, since otherwise (7) holds for the BC-point already. We proceed to replace the said point with the new lattice point

$$(\tilde{u}, \tilde{v}) = (u - p, v + q), \tag{8}$$

where p, q are positive integers to be chosen so that

$$0 \le (u - p)^2 + (v + q)^2 - n \le \alpha\sqrt{u} + 1.$$

On replacing $u^2 + v^2 - n$ with $\beta\sqrt{u}$, the condition becomes

$$(\beta - \alpha)\sqrt{u} - 1 \le f(p, q) \le \beta\sqrt{u},$$

where $f(p, q) = 2up - p^2 - q(q + 2v)$, so that, by (2),

$$f(p, q) = G - (q + v)^2. \tag{9}$$

For a fixed p, the equation $f(p, q) = A$ has the solution $q = \sqrt{G - A} - v$, which is an integer if and only if $G - A$ is a square. We now write

$$\sqrt{G} = w + \theta, \qquad \text{where} \qquad \theta = \{\sqrt{G}\}, \tag{10}$$

and set

$$A = 2\theta w + \theta^2, \qquad \text{so that} \qquad G - A = w^2. \tag{11}$$

Thus $f(p, q) = A$, with $q = w - v$, and it remains to show that

$$(\beta - \alpha)\sqrt{u} - 1 \le A \le \beta\sqrt{u}. \tag{12}$$

It follows from (2) that $\sqrt{2up} \le \sqrt{G} \le \sqrt{2u(p + 1)}$ so that, by (10),

$$\sqrt{2up} \le w + \theta \le \sqrt{2u(p + 1)}.$$

By (3) in Hypothesis H, there exists $p \le v$ such that

$$\frac{\beta - \alpha}{2\sqrt{2}} \cdot \frac{2}{\sqrt{p}} < 2\theta < \frac{\beta}{2\sqrt{2}} \cdot \frac{2}{\sqrt{p + 1}},$$

so that

$$(\beta - \alpha)\sqrt{u} - \theta^2 < 2\theta w + \theta^2 < \beta\sqrt{u}.$$

The required condition (12) now follows from (11) and $\theta^2 < 1$. Thus, for any fixed $\alpha > 0$, there are integers p, q with the property that the lattice point $(\tilde{u}, \tilde{v})$ in (8) is a replacement with gap value satisfying (7). Since $\alpha > 0$ is arbitrary, the term $+1$ can be ignored, and the theorem is proved. □

3. Ladders Formed by the Gaps

A useful observation is that, as n runs through $\mathcal{I}_u$ in (6), the sequence (h_n) forms *ladders*. Take, for example, $u = 9$, so that there are 19 BC-points, each having the form $(9, v)$ with $0 \le v \le 5$, corresponding to $n = 81, 82, \ldots, 99$, and that

$$h_n = 0, 0, 2, 1, 0, 4, 3, 2, 1, 0, 6, 5, 4, 3, 2, 1, 0, 8, 7;$$

after the first two zeros, the terms form four descending ladders, with the top-steps taking the values $2, 4, 6, 8$, and the largest gap $h_{98} = 8$ is the value of the top-step of the last incomplete ladder.

To see this in general, let the BC-point at $n \in \mathcal{I}_u$ be (u, v_n), and write

$$n = u^2 + m^2 + \ell, \qquad 0 \le m \le \sqrt{2u}, \quad 0 \le \ell \le 2m.$$

Then $v_n = \lceil m^2 + \ell \,\rceil$, so that $v_n = m$, and $h_n = 0$, for $\ell = 0$; and

$$v_n = m + 1, \qquad h_n = 2m - (\ell - 1), \qquad \text{for} \quad 1 \le \ell \le 2m.$$

Thus, after $0, 0$, the values h_n form $M = \lceil \sqrt{2u}\,\rceil - 1$ ladders, with descending unit steps. Note that h_n is small when n is near u^2, whereas $h_n = 2m$ at the top of a ladder is large when n is near $(u+1)^2$. We state the observation as a lemma.

Lemma. *As n runs through $\mathcal{I}_u$, the values for h_n take $0, 0$ and then form $M = \lceil \sqrt{2u}\,\rceil - 1$ ladders, with descending unit steps. The top-step of each ladder is attained at*

$$n = u^2 + m^2 + 1, \quad \textit{with} \quad v_n = m + 1, \; h_n = 2m, \quad \textit{for} \quad m = 1, 2, \ldots, M, \qquad (13)$$

and it follows the zero bottom step of the previous ladder. The last ladder is incomplete unless $2u = v_n^2$. □

4. The Weaker Bound $\Phi_n - a$

The dichotomy between $a = 2$ and $a = 3$ in (4) is revealed in (i) and (ii) in the following theorem. The proof of (i) is given in [2]; our proof here hinges on (13), which also leads to (ii).

Theorem 2. *Let (u_n, v_n) be the BC-point at n, and h_n and Φ_n be defined by* (5).
(i) *For $n \geq 1$, we have*

$$\min\{h_n, (u_n+1)^2 - n\} < \Phi_n - 2.$$

(ii) *There are infinitely many n such that*

$$h_n = (u_n+1)^2 - n > \Phi_n - 3.$$

Proof. (i) Consider first those $n \in \mathcal{I}_u$ having the form

$$n = u^2 + m^2 + 1, \qquad \text{with} \qquad m^2 = 2u - k, \qquad 1 \leq k < 2u. \tag{14}$$

Then $4u^2 = m^4 + 2km^2 + k^2$, so that $4n = m^4 + 2(k+2)m^2 + k^2 + 4$, and hence

$$\Phi_n^4 = 64n > \begin{cases} (2m+2)^4 & \text{if} \quad k \geq 2m, \\ (2m+1)^4 & \text{if} \quad k \geq m. \end{cases} \tag{15}$$

By (13), we have $h_n + 2 = 2m + 2$, and hence $h_n < \Phi_n - 2$, when $k \geq 2m$.

For $1 \leq k < 2m$, we consider the lattice point $(u+1, 0)$ which has the gap value

$$h = (u+1)^2 - n = 2u - m^2 = k.$$

From $\Phi_n = 2\sqrt{2}n^{\frac{1}{4}} > 2\sqrt{2u} = 2\sqrt{m^2+k} > 2m$ it follows that $h = k < \Phi_n - 2$ if $k < m$, and if $k \geq m$ then (15) delivers $\Phi_n > 2m + 1 \geq k + 2 = h + 2$. Thus $(u+1)^2 - n < \Phi_n - 2$ for $1 \leq k < 2m$, as required.

It remains to deal with the numbers $n \in \mathcal{I}_u$ *not* having the form (14). According to the lemma, the BC-point corresponding to any $n \in \mathcal{I}_u$ has a gap value h_n which is a step of a ladder. If it is at the top of the m-th ladder, then n is given by (14); otherwise the gap value h_n is *smaller*, whereas the corresponding bound Φ_n is *larger*, because the value for n has been *increased* from that corresponding to the top step. The same remark applies to the bound for $(u_n+1)^2 - n$. The required result is proved.

(ii) Let m be an even number, and define n by

$$n = u^2 + m^2 + 1, \qquad \text{where} \qquad 2u + 1 = (m+1)^2.$$

Then $n \in \mathcal{I}_u$ and $u = u_n$, so that $h_n = (u_u+1)^2 - n = 2m$. On elimination of u in terms of m in n, we find that

$$\Phi_n^4 = 64n = 16(m^4 + 4m^3 + 8m^2 + 4) < (2m+3)^4, \tag{16}$$

so that $2m > \Phi_n - 3$ for all such n. The theorem is proved. □

From (15) we find that $16m^4 < \Phi_n^4$, so that every BC-point already satisfies (4) if $a = 0$. For $a = 1$, we have $(h_n+1)^4 < \Phi_n^4$ when $k \geq m$, and we need the lattice point $(u+1, 0)$ only for $1 \leq k < m$, which then delivers $(u+1)^2 - n = k < m < \Phi_n - 1$.

From (16) we have the expansion

$$\Phi_n = 2m\Big(1 + \frac{4}{m} + \frac{8}{m^2} + \frac{4}{m^4}\Big)^{\frac{1}{4}} = 2m + 2 + \frac{1}{m} + O\Big(\frac{1}{m^2}\Big), \quad \text{as} \quad m \to \infty,$$

so that Φ_n just exceeds $2m+2$ for such n in (ii).

5. The Case $a = 3$

By (ii) in Theorem 2, there are arbitrarily large values of n for which both gap values of the BC-point and the lattice point $(u+1, 0)$ do not satisfy (4) when $a = 3$, so that a replacement lattice point of the form (8) is necessary for such n. At present there is no guarantee for the existence of such a replacement without a hypothesis, albeit somewhat weaker than Hypothesis H; moreover, it seems likely that any method which can be used to establish such a weaker hypothesis could also be deployed on Hypothesis H itself. In other words, it appears that if (4) can be established unconditionally for $a = 3$ then the same method should also deliver the result in Theorem 1 unconditionally.

For $m = 2862$ in (ii), we find that

$$u = 4098384, \qquad n = 16796759602501, \qquad \Phi_n = 5726{\cdot}000348796\cdots,$$

and that, for this value of n, there is no suitable replacement lattice point $(\tilde{u}, \tilde{v}) = (u - p, v + q)$ with $p < 40$ which delivers a solution to (4) when $a = 3$. For $(p, q) = (40, 15469)$, we find that $\tilde{u}^2 + \tilde{v}^2 - n = 2059 < \Phi_n/2 < \Phi_n - 3$.

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