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# ALTERNATIVE SOLUTIONS TO LINEAR RECURRENCE EQUATIONS

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**Abstract**
Lagrange has shown that the solution of a $k$-th order linear recurrence relation can be expressed in terms of the distinct solutions of its characteristic equation and their multiplicities. This paper shows that such a solution can also be expressed in terms of just one solution of the characteristic equation and the solution of a $(k-1)$-th order linear recurrence relation. Using this, an explicit solution to an arbitrary third, and the most general case of an arbitrary fourth, order linear recurrence relation are obtained.

## 1. Introduction

The function satisfying a general $k$-th order linear recurrence relation can be defined as follows.

**Definition 1.1.** *For* $0 \le n < k$,

$$w_{n,k}(a_0, a_1, \ldots, a_{k-1}; p_1, p_2, \ldots, p_k) = a_n, \tag{1.1}$$

*and for* $n \ge k-1$,
$w_{n+1,k}(a_0, \ldots, a_{k-1}; p_1, \ldots, p_k) =$

$$p_1 w_{n,k}(a_0, \ldots, a_{k-1}; p_1, \ldots, p_k) + \ldots + p_k w_{n+1-k,k}(a_0, \ldots, a_{k-1}; p_1, \ldots, p_k). \tag{1.2}$$

If $a_0, \ldots, a_{k-1}, p_1, \ldots, p_k$ remain constant, $w_{n,k}(a_0, \ldots, a_{k-1}; p_1, \ldots, p_k)$ will usually be written as $w_{n,k}$.

Note that $a_0, \ldots, a_{k-1}, p_1, \ldots, p_k$ can be arbitrary complex numbers.

An explicit formulation for $w_{n,k}$ can be found using the following theorem due to Lagrange ([5] and [6]). For more recent work see Cull [2] or Elaydi [3].

**Theorem 1.** *If* $\gamma_1, \ldots, \gamma_r$*, the distinct roots of the "characteristic equation"*

$$\gamma^k - p_1\gamma^{k-1} - \ldots - p_k = 0, \tag{1.3}$$

*have multiplicities* $m(1), \ldots, m(r)$, *(so that* $\sum_{i=1}^r m(i) = k$*), the general solution of (1.2) is*

$$w_{n,k} = c_1(n)\gamma_1^n + c_2(n)\gamma_2^n + \ldots + c_r(n)\gamma_r^n,$$

*where for each* $i$, $c_i(n)$ *is a polynomial in* $n$ *of order less than* $m(i)$ *which can be determined using (1.1).*

In this paper we provide a method of solving the general linear recurrence relation of Definition 1.1 that requires only one solution of (1.3) and solutions of a $(k-1)$-th order linear recurrence relation. More precisely, if values of $w_{m,k-1}(a'_0, \ldots, a'_{k-2}; p'_1, \ldots, p'_{k-1})$ are tabulated, or easily found for $m < n$, $w_{n,k}(a_0, \ldots, a_{k-1}; p_1, \ldots, p_k)$ can be easily determined. This process can be continued to give an expression for $w_{n,k}$ in terms of $w_{m,2}(a_0^*, a_1^*; p_1^*, p_2^*)$ (often called a Horadam function) for some values of $a_0^*, a_1^*, p_1^*, p_2^*$.

We do this below for $k = 3$ and for the most general case of $k = 4$.

Even when $k > 4$, and a solution of (1.3) cannot be found explicitly, a good approximation of the smallest solution of (1.3), found numerically, will give, by this method, a good approximate expression for $w_{n,k}$.

In what follows we can assume $p_k \neq 0$ as $w_{0,k}(a_0, \ldots, a_{k-1}; p_1, \ldots, p_{k-1}, 0) = a_0$ and, for $n > 0$,

$$w_{n,k}(a_0, \ldots, a_{k-1}; p_1, \ldots, p_{k-1}, 0) = w_{n-1,k-1}(a_1, \ldots, a_{k-1}; p_1, \ldots, p_{k-1}).$$

## 2. Representing $w_{n+1,k}$ in Terms of $w_{i,k-1}$

The following theorem allows us to represent $w_{n+1,k}$ in terms of $w_{i,k-1}$ for $0 \leq i \leq n$.

**Theorem 2.** *If* $\gamma$ *is a solution of (1.3), where* $p_k \neq 0$ *and for* $0 \leq n < k-1$, $a'_n = a_{n+1} - \gamma a_n$ *and* $p'_{n+1} = (\sum_{j=0}^n \gamma^j p_{n+1-j}) - \gamma^{n+1}$, *then*

*(i)* $w_{n+1,k}(a_0, \ldots, a_{k-1}; p_1, \ldots, p_k)$
$= w_{n,k-1}(a'_0, \ldots, a'_{k-2}; p'_1, \ldots, p'_{k-1}) + \gamma w_{n,k}(a_0, \ldots, a_{k-1}; p_1, \ldots, p_k)$,

*(ii)* $w_{n+1,k}(a_0, \ldots, a_{k-1}; p_1, \ldots, p_k)$
$= \gamma^{n-j+1}a_j + \sum_{i=0}^{n-j} \gamma^i w_{n-i,k-1}(a'_0, \ldots, a'_{k-2}; p'_1, \ldots, p'_{k-1})$
*for* $j \leq n-1$ *and* $0 \leq j < k$.

*Proof.* Let, for any $\gamma \neq 0$, $w'_{n,k} = w_{n+1,k} - \gamma w_{n,k}$, then for $0 \leq n < k-1$, let

$$a'_n = w'_{n,k} = a_{n+1} - \gamma a_n$$

and

$$p'_{n+1} = \Big(\sum_{j=0}^{n} \gamma^j p_{n+1-j}\Big) - \gamma^{n+1}.$$

The latter gives:

$$p_1 = \gamma - p_2/\gamma - p_3/\gamma^2 - \ldots - p_{k-1}/\gamma^{k-2} + p'_{k-1}/\gamma^{k-2}.$$

If $\gamma$ is a solution of (1.3), as $p_k \neq 0$, we have that $\gamma \neq 0$, as required above, and so $p_k = -\gamma p'_{k-1}$.

Now for $n \geq k-1$,

$$w_{n+1,k} = p_1 w_{n,k} + p_2 w_{n-1,k} + \ldots + p_k w_{n-k+1,k}$$

$$= (p'_1 + \gamma) w_{n,k} + (p'_2 - \gamma p'_1) w_{n-1,k} + \ldots + (p'_{k-1} - \gamma p'_{k-2}) w_{n-k+2,k} - \gamma p'_{k-1} w_{n-k+1,k}$$

and so,

$$w'_{n,k} = p'_1 w'_{n-1,k} + \ldots + p'_{k-1} w'_{n-k+1,k}.$$

So we have proved

$$w'_{n,k} = w_{n,k-1}(a'_0, \ldots, a'_{k-2}; p'_1, \ldots, p'_{k-1})$$

and for $j \leq n+1$ and $0 \leq j \leq k-1$,

$$w_{n+1,k}(a_0, \ldots, a_{k-1}; p_1, \ldots, p_k)$$

$$= w_{n,k-1}(a'_0, \ldots, a'_{k-1}; p'_1, \ldots, p'_{k-1}) + \gamma w_{n,k}(a_0, \ldots, a_{k-1}; p_1, \ldots, p_k)$$

$$= \gamma^{n-j+1} a_j + \sum_{i=0}^{n-j} \gamma^i w_{n-i,k-1}(a'_0, \ldots, a'_{k-2}; p'_1, \ldots, p'_{k-1}).$$

□

**Remark.** Any solution $\gamma$ of (1.3), repeated or not, real or complex, can be used to give the result in (i) or (ii). If the values of $w_{n,k-1}$ have been tabulated and such a $\gamma$ is at hand, $w_{n,k}$ can be evaluated much more easily than by Lagrange's method (i.e., Theorem 1 above).

The following theorem gives a further connection between $p_1, \ldots, p_k$ and $p'_1, \ldots, p'_{k-1}$.

**Theorem 3.** *If $\gamma$ is a solution of (1.3), where $p_k \neq 0$, then the other solutions of (1.3) are those of*

$$x^{k-1} - p'_1 x^{k-2} - \ldots - p'_{k-1} = 0 \tag{2.1}$$

*where $p'_1 = p_1 - \gamma, p'_2 = p_2 + p'_1\gamma, \ldots, p'_{k-1} = p_{k-1} + p'_{k-2}\gamma$ and $p'_{k-1} = -p_k/\gamma$.*

*Proof.* If $p'_1 = p_1 - \gamma, p'_2 = p_2 + p'_1\gamma, \ldots, p'_{k-1} = p_{k-1} + p'_{k-2}\gamma$ and $p'_{k-1} = -p_k/\gamma$, where $\gamma$ is a solution of (1.3),

$$x^k - p_1x^{k-1} - \ldots - p_k = (x-\gamma)(x^{k-1} - p'_1x^{k-2} - \ldots - p'_{k-1}),$$

so the remaining solutions of (1.3) are the solutions of (2.1). □

In Section 4 we will use Theorem 2 to give explicit general formulas for $w_{n+1,3}$ and for the most general case of $w_{n+1,4}$. First we need some properties of $w_{n,2}$.

## 3. Sums Involving Horadam Functions

In Section 4 we require Lemma 1 below and for that we require the following explicit formulas for $w_{n,2}$. (Proofs can be found in Horadam [4] and Bunder [1].)

**Theorem 4.** *(i) If* $p_1^2 \neq -4p_2, \alpha = (p_1 + \sqrt{p_1^2+4p_2})/2$ *and* $\beta = (p_1 - \sqrt{p_1^2+4p_2})/2$,

$$w_{n,2}(a_0,a_1;p_1,p_2) = \Big(\frac{a_1-a_0\beta}{\alpha-\beta}\Big)\alpha^n - \Big(\frac{a_1-a_0\alpha}{\alpha-\beta}\Big)\beta^n.$$

*(ii)* $w_{n,2}(a_0,a_1;p_1,-p_1^2/4) = na_1(p_1/2)^{n-1} - (n-1)a_0(p_1/2)^n$.

The definitions of $\alpha$ and $\beta$ above will also be used below.

**Lemma 1.** *(i) If* $\gamma \neq \alpha, \gamma \neq \beta$ *and* $n \geq 2$,

$$\sum_{i=0}^{n-2}\gamma^i w_{n-i,2} = \Big(\frac{\gamma^n(p_1a_1+p_2a_0) + \gamma^{n-1}p_2a_1 - \gamma w_{n+1,2} - p_2w_{n,2}}{\gamma^2 - p_1\gamma - p_2}\Big).$$

*(ii) If* $\gamma = \alpha \neq \beta$ *or* $\gamma = \beta \neq \alpha$, *and* $n \geq 2$,

$$\sum_{i=0}^{n-2}\gamma^i w_{n-i,2} = n\Big(\frac{a_1 - a_0(p_1-\gamma)}{2\gamma-p_1}\Big)\gamma^n + \frac{a_1\gamma^{n-1}(p_1-\gamma)}{2\gamma-p_1} - \frac{w_{n+1,2}}{2\gamma-p_1}.$$

*(iii) If* $\gamma = \alpha = \beta \neq 0$ *and* $n \geq 2$,

$$\sum_{i=0}^{n-2}\gamma^i w_{n-i,2} = (n-1)\Big(2w_{n+1,2}/p_1 + a_1(p_1/2)^{n-1}\Big)/2.$$

*Proof.* `Case 1.` $p_1^2 \neq -4p_2$ (i.e., $\alpha \neq \beta$). By Theorem 4(i),

$$\sum_{i=0}^{n-2}\gamma^i w_{n-i,2} = \sum_{i=0}^{n-2}\Big(\frac{a_1-a_0\beta}{\alpha-\beta}\Big)\gamma^i\alpha^{n-i} - \Big(\frac{a_1-a_0\alpha}{\alpha-\beta}\Big)\gamma^i\beta^{n-i}.$$

(i) If $\gamma \neq \alpha$ and $\gamma \neq \beta$,

$$\sum_{i=0}^{n-2} \gamma^i w_{n-i,2} = \alpha^2\Big(\frac{a_1 - a_0\beta}{\alpha-\beta}\Big)\Big(\frac{\gamma^{n-1}-\alpha^{n-1}}{\gamma-\alpha}\Big) - \beta^2\Big(\frac{a_1-a_0\alpha}{\alpha-\beta}\Big)\Big(\frac{\gamma^{n-1}-\beta^{n-1}}{\gamma-\beta}\Big)$$

$$= \frac{\gamma^{n-1}((a_1-a_0\beta)\alpha^2(\gamma-\beta)-(a_1-a_0\alpha)\beta^2(\gamma-\alpha))-(a_1-a_0\beta)\alpha^{n+1}(\gamma-\beta)}{(\alpha-\beta)(\gamma-\alpha)(\gamma-\beta)}$$

$$+ \frac{(a_1-a_0\alpha)\beta^{n+1}(\gamma-\alpha)}{(\alpha-\beta)(\gamma-\alpha)(\gamma-\beta)} = \frac{\gamma^n(p_1a_1+p_2a_0)+\gamma^{n-1}p_2a_1-\gamma w_{n+1,2}-p_2w_{n,2}}{\gamma^2-p_1\gamma-p_2}.$$

(ii) If $\gamma = \alpha \neq \beta$, then $\beta = p_1 - \gamma$ and $\alpha - \beta = 2\gamma - p_1$. If $\gamma = \beta \neq \alpha$, then $\alpha = p_1 - \gamma$ and $\alpha - \beta = p_1 - 2\gamma$. In either case:

$$\sum_{i=0}^{n-2} \gamma^i w_{n-i,2} = \sum_{i=0}^{n-2} \Big(\frac{a_1-a_0(p_1-\gamma)}{2\gamma-p_1}\Big)\gamma^n - \Big(\frac{a_1-a_0\gamma}{2\gamma-p_1}\Big)\Big(\frac{\gamma}{p_1-\gamma}\Big)^i(p_1-\gamma)^n$$

$$= (n-1)\Big(\frac{a_1-a_0(p_1-\gamma)}{2\gamma-p_1}\Big)\gamma^n - \Big(\frac{a_1-a_0\gamma}{2\gamma-p_1}\Big)\Big(\frac{\gamma^{n-1}-(p_1-\gamma)^{n-1}}{2\gamma-p_1}\Big)(p_1-\gamma)^2$$

$$= n\Big(\frac{a_1-a_0(p_1-\gamma)}{2\gamma-p_1}\Big)\gamma^n - \Big(\frac{a_1-a_0(p_1-\gamma)}{(2\gamma-p_1)^2}\Big)\gamma^{n+1} + \Big(\frac{a_1-a_0(p_1-\gamma}{(2\gamma-p_1)^2}\Big)\gamma^n(p_1-\gamma)$$

$$-\Big(\frac{a_1-a_0\gamma}{(2\gamma-p_1)^2}\Big)(p_1-\gamma)^2\gamma^{n-1} + \Big(\frac{a_1-a_0\gamma}{(2\gamma-p_1)^2}\Big)(p_1-\gamma)^{n+1}$$

$$= n\Big(\frac{a_1-a_0(p_1-\gamma)}{2\gamma-p_1}\Big)\gamma^n - \frac{w_{n+1,2}}{2\gamma-p_1} + \frac{a_1\gamma^{n-1}(p_1-\gamma)}{2\gamma-p_1}.$$

Case 2. $p_1^2 = -4p_2$ (i.e., $\alpha = \beta = p_1/2$). By Theorem 4(ii),

$$\sum_{i=0}^{n-2} \gamma^i w_{n-i,2} = \sum_{i=0}^{n-2} \gamma^i(a_1(n-i)(p_1/2)^{n-i-1} - (n-i-1)a_0(p_1/2)^{n-i})$$

$$= (p_1/2)^{n-1}\sum_{i=0}^{n-2}((na_1-(n-1)a_0p_1/2)(2\gamma/p_1)^i - (a_1-a_0p_1/2)i(2\gamma/p_1)^i).$$

(i) If $\gamma \neq p_1/2$,

$$\sum_{i=0}^{n-2} \gamma^i w_{n-i,2} = (p_1/2)(na_1-(n-1)a_0p_1/2)\Big(\frac{\gamma^{n-1}-(p_1/2)^{n-1}}{\gamma-p_1/2}\Big) -$$

$$\gamma(a_1-a_0p_1/2)p_1/2\Big(\frac{(n-2)\gamma^{n-1}-(n-1)p_1/2\gamma^{n-2}+(p_1/2)^{n-1}}{(\gamma-p_1/2)^2}\Big)$$

$$-\frac{\gamma(p_1/2)^n((n+1)a_1-na_0p_1/2-na_1p_1/2+(n-1)a_0(p_1/2)^2)}{(\gamma-p_1/2)^2}$$

$$= \Big(\frac{\gamma^n(p_1a_1+p_2a_0)+\gamma^{n-1}p_2a_1-\gamma w_{n+1,2}-p_2w_{n,2}}{\gamma^2-p_1\gamma-p_2}\Big).$$

(iii) If $\gamma = p_1/2$,

$$\sum_{i=0}^{n-2}\gamma^i w_{n-i,2} = (p_1/2)^{n-1}[\sum_{i=0}^{n-2}(na_1-(n-1)a_0p_1/2)-(a_1-a_0p_1/2)i]$$

$$= (p_1/2)^{n-1}[(n-1)(na_1-(n-1)a_0p_1/2)-(a_1-a_0p_1/2)(n-2)(n-1)/2]$$
$$= (n-1)((n+2)a_1(p_1/2)^{n-1}-na_0(p_1/2)^n)/2$$
$$= (n-1)\Big(2w_{n+1,2}/p_1+a_1(p_1/2)^{n-1})\Big)/2.$$ □

Note that (i) and (ii) also hold if $\gamma = 0$.

## 4. The Third Order Recurrence

The following theorem allows us to represent $w_{n+1,3}$ in terms of $w_{n+1,2}$ and $w_{n,2}$.

**Theorem 5.** *If $\gamma$ is a solution of $\gamma^3 - p_1\gamma^2 - p_2\gamma - p_3 = 0$, then,*
$a'_0 = a_1 - \gamma a_0, a'_1 = a_2 - \gamma a_1, p'_1 = p_1 - \gamma, p'_2 = -p_3/\gamma, \alpha' = (p'_1 + \sqrt{p'^2_1 + 4p'_2})/2,$ $\beta' = (p'_1 - \sqrt{p'^2_1 + 4p'_2})/2$ *and*
*(i) if $\gamma \neq \alpha'$ and $\gamma \neq \beta'$,*
$w_{n+1,3}(a_0, a_1, a_2; p_1, p_2, p_3)) =$

$$\frac{\gamma^{n+1}(a_2 - a_1p'_1 - a_0p'_2) - \gamma w_{n+1,2}(a'_0, a'_1; p'_1, p'_2) - p'_2 w_{n,2}(a'_0, a'_1; p'_1, p'_2)}{\gamma^2 - p'_1\gamma - p'_2}.$$

*(ii) if $\gamma = \alpha' \neq \beta'$ or $\gamma = \beta' \neq \alpha'$,*
$w_{n+1,3}(a_0, a_1, a_2; p_1, p_2, p_3) =$

$$\frac{\gamma^n(n(a'_1 - a'_0(p'_1 - \gamma)) + a_2 - a_1(p'_1 - \gamma)) - w_{n+1,2}(a'_0, a'_1; p'_1, p'_2)}{2\gamma - p'_1}.$$

*(iii) if $\gamma = \alpha' = \beta'$ (i.e., $p_1 = 3\gamma, p_2 = -3\gamma^2$ and $p_3 = \gamma^3$),*
$w_{n+1,3}(a_0, a_1, a_2; p_1, p_2, p_3) =$

$$(p'_1/2)^{n-1}a_2 + \frac{n-1}{2}\Big(\frac{2w_{n+1,2}(a'_0, a'_1; p'_1, p'_2)}{p'_1} + a'_1(p'_1/2)^{n-1}\Big).$$

*Proof.* By Theorem 2(ii), with $k = 3$ and $j = 2$, for $n \geq 1$,

$$w_{n+1,3}(a_0, a_1, a_2; p_1, p_2, p_3) = \gamma^{n-1}a_2 + \sum_{i=0}^{n-2}\gamma^i w_{n-i,2}(a'_0, a'_1; p'_1, p'_2),$$

where $\gamma$ is a solution of $\gamma^3 - p_1\gamma^2 - p_2\gamma - p_3 = 0, a'_0 = a_1 - \gamma a_0, a'_1 = a_2 - \gamma a_1, p'_1 = p_1 - \gamma$ and $p'_2 = -p_3/\gamma = p_2 + p_1\gamma - \gamma^2$.

(i) By Lemma 1(i), if $\gamma \neq \alpha'$ and $\gamma \neq \beta'$,
$w_{n+1,3}(a_0, a_1, a_2; p_1, p_2, p_3)) = \gamma^{n-1}a_2+$

$$\frac{\gamma^n(a'_1p'_1 + a'_0p'_2) + \gamma^{n-1}a'_1p'_2 - \gamma w_{n+1,2}(a'_0, a'_1; p'_1, p'_2) - p'_2 w_{n,2}(a'_0, a'_1; p'_1, p'_2)}{\gamma^2 - p'_1\gamma - p'_2}.$$

Now $a'_1 = a_2 - \gamma a_1$ and $a'_0 = a_1 - \gamma a_0$ give the result.

(ii) If $\gamma = \alpha' \neq \beta'$ or $\gamma = \beta' \neq \alpha'$, Lemma 1(ii) gives the result.

(iii) If $\gamma = \alpha' = \beta'$ (i.e. $\gamma = p'_1/2 = p_1/3$, $p'^2_1 = -4p'_2$ so that $p_2 = -3\gamma^2$ and $p_3 = \gamma^3$), Lemma 1(iii) gives the result. □

**Example 1**. $a_0 = 0, a_1 = 1, a_2 = 2, p_1 = p_2 = 1, p_3 = 2$. In this case $\gamma = 2$ satisfies $\gamma^3 - \gamma^2 - \gamma - 2 = 0$, and then gives $a'_0 = 1, a'_1 = 0, p'_1 = p'_2 = -1$, $\alpha' = (-1 + \sqrt{3}i)/2$ and $\beta' = (-1 - \sqrt{3}i)/2$. So by Theorem 5(i),

$$w_{n,3}(0, 1, 2; 1, 1, 2) = (3.2^n - 2w_{n,2}(1, 0; -1, -1) + w_{n-1,2}(1, 0, -1, -1))/7.$$

Given (previously tabulated) $w_{2,2}(1, 0; -1, -1) = -1, w_{3,2}(1, 0; -1, -1) = 1$ and $w_{4,2}(1, 0; -1, -1) = 0$, we have $w_{3,3}(0, 1, 2; 1, 1, 2) = 3$ and $w_{4,3}(0, 1, 2; 1, 1, 2) = 7$.

If we had chosen $\gamma = (-1+\sqrt{3}i)/2$ or $(-1-\sqrt{3}i)/2$ we would need Theorem 5(ii) and require messier arithmetic. The standard method, which uses all three values of $\gamma$ requires even messier arithmetic.

**Example 2**. $a_0 = 1, a_1 = 2, a_2 = 3, p_1 = 1, p_2 = 8, p_3 = -12$. In this case $\gamma = 2$ and $-3$ satisfy $\gamma^3 - \gamma^2 - 8\gamma + 12 = 0$. With $\gamma = 2$, we have $a'_0 = 0, a'_1 = -1, p'_1 = -1, p'_2 = 6, \alpha' = 2 = \gamma$ and $\beta' = -3$. So by Theorem 5(ii),

$$w_{n,3}(1, 2, 3; 1, 8, -12) = [(10 - n)2^{n-1} - w_{n,2}(0, -1; -1, 6)]/5$$

and (previously tabulated) $w_{2,2}(0, -1; -1, 6) = 1, w_{3,2}(0, -1; -1, 6) = -7$ and $w_{4,2}(0, -1; -1, -6) = 13$ give $w_{3,3}(1, 2, 3; 1, 8, -12) = 7$ and $w_{4,3}(1, 2, 3; 1, 8, -12) = 7$.

**Example 3**. $a_0 = 1, a_1 = 2, a_2 = 3, p_1 = 6, p_2 = -12, p_3 = 8$. In this case $\gamma = 2$ satisfies $\gamma^3 - 6\gamma^2 + 12\gamma - 8 = 0$ and gives $a'_0 = 0, a'_1 = -1, p'_1 = 4$, $p'_2 = -4$ and $\alpha' = \beta' = \gamma = 2$. So by Theorem 5(iii),

$$w_{n,3}(1, 2, 3; 6, -12, 8) = 3.2^{n-2} + (n - 2)(w_{n,2}(0, -1; 4, -4) - 2^{n-1})/4$$

and (previously tabulated) $w_{3,2}(0, -1; 4, -4) = -12$ and $w_{4,2}(0, -1; 4, -4) = -32$ imply $w_{3,3}(1, 2, 3; 6, -12, 8) = 2$ and $w_{4,3}(1, 2, 3; 6, -12, 8) = -8$.

## 5. The Fourth Order Recurrence

By Theorem 2(ii), we can express $w_{n+1,4}(a_0, a_1, a_2, a_3; p_1, p_2, p_3, p_4)$ in terms of $\delta$, a solution of a quartic equation, and a sum from $i = 0$ to $i = n-2$ of $\delta^i w_{n-i,3}(a'_0, a'_1, a'_2; p'_1, p'_2, p'_3)$. We can express each $w_{n-i,3}$, by Theorem 5, in terms of $\gamma$, a solution of a cubic equation, and terms $w_{n-i,2}(a''_0, a''_1; p''_1, p''_2)$ and can then perform the summation using Lemma 1. Altogether there are 14 cases to consider, depending on which of $\delta, \gamma, \alpha'' = (p''_1 + \sqrt{p''^2_1 + 4p''_2})/2$ and $\beta'' = (p''_1 - \sqrt{p''^2_1 + 4p''_2})/2$ are equal to each other. We will solve "the statistically most likely" case where all of these are different.

**Theorem 6.** *If*
*(a)* $\delta$ *is a solution of* $\delta^4 - p_1\delta^3 - p_2\delta^2 - p_3\delta - p_4 = 0,\ a'_0 = a_1 - \delta a_0,\ a'_1 = a_2 - \delta a_1,\ a'_2 = a_3 - \delta a_2,\ p'_1 = p_1 - \delta,\ p'_2 = p_2 + p'_1\delta$ *and* $p'_3 = -p_4/\delta$;
*(b)* $\gamma$ *is a solution of* $\gamma^3 - p'_1\gamma^2 - p'_2\gamma - p'_3 = 0,\ a''_0 = a'_1 - \gamma a'_0,\ a''_1 = a'_2 - \gamma a'_1,\ p''_1 = p'_1 - \gamma,\ p''_2 = -p'_3/\gamma,$
*(c)* $\alpha'' = (p''_1 + \sqrt{p''^2_1 + 4p''_2})/2$ *and* $\beta'' = (p''_1 - \sqrt{p''^2_1 + 4p''_2})/2,$
*and*
*(d)* $\delta \neq \gamma, \delta \neq \alpha'', \delta \neq \beta'',\ \gamma \neq \alpha''$ *and* $\gamma \neq \beta''$,
*then*

$$w_{n+1,4}(a_0, a_1, a_2, a_3; p_1, p_2, p_3, p_4) = \delta^{n-1}a_2 + [(p''_2/\delta)w_{n,2}(a''_0, a''_1; p''_1, p''_2) - p''_2\delta^{n-2}a''_1$$
$$+\gamma^2(\delta^{n-1} - \gamma^{n-1})(a'_2 - a'_1p''_1 - a'_0p''_2)/(\delta - \gamma) - (\gamma + p''_2/\delta)[\delta^n(p''_1a''_1 + p''_2a''_0) + \delta^{n-1}p''_2a''_1$$
$$-\delta w_{n+1,2}(a''_0, a''_1; p''_1, p''_2) - p''_2 w_{n,2}(a''_0, a''_1; p''_1, p''_2)]/(\delta^2 - p''_1\delta - p''_2)]/(\gamma^2 - p''_1\gamma - p''_2).$$

*Proof.* By Theorem 2(ii), with $k = 4$ and $j = 2$, for $n \geq 1$,

$$w_{n+1,4}(a_0, a_1, a_2, a_3; p_1, p_2, p_3) = \delta^{n-1}a_2 + \sum_{i=0}^{n-2} \delta^i w_{n-i,3}(a'_0, a'_1, a'_2; p'_1, p'_2, p'_3)$$

where $\delta$ is a solution of $\delta^4 - p_1\delta^3 - p_2\delta^2 - p_3\delta - p_4 = 0,\ a'_0 = a_1 - \delta a_0,\ a'_1 = a_2 - \delta a_1,\ a'_2 = a_3 - \delta a_2,\ p'_1 = p_1 - \delta,\ p'_2 = p_2 + p'_1\delta$ and $p'_3 = -p_4/\delta$.

By Theorem 5(i), if $\gamma$ is a solution of $\gamma^3 - p'_1\gamma^2 - p'_2\gamma - p'_3 = 0,\ a''_0 = a'_1 - \gamma a'_0,\ a''_1 = a'_2 - \gamma a'_1,\ p''_1 = p'_1 - \gamma\ p''_2 = -p'_3/\gamma, \gamma \neq \alpha''$ and $\gamma \neq \beta''$,
$w_{n-i,3}(a'_0, a'_1, a'_2; p'_1, p'_2, p'_3) =$

$$[\gamma^{n-i}(a'_2 - a'_1p''_1 - a'_0p''_2) - \gamma w_{n-i,2}(a''_0, a''_1; p''_1, p''_2) - p''_2 w_{n-i-1,2}(a''_0, a''_1; p''_1, p''_2)]/(\gamma^2 - p''_1\gamma - p''_2).$$

So,
$w_{n+1,4}(a_0, a_1, a_2, a_3; p_1, p_2, p_3) =$

$$\delta^{n-1}a_2 + [(p''_2/\delta)w_{n,2}(a''_0, a''_1; p''_1, p''_2) - p''_2\delta^{n-2}w_{1,2}(a''_0, a''_1; p''_1, p''_2)$$

$$+\gamma^n \sum_{i=0}^{n-2}((\delta/\gamma)^i(a_2'-a_1'p_1''-a_0'p_2'')-(\gamma+p_2''/\delta)\sum_{i=0}^{n-2}\delta^i w_{n-i,2}(a_0'',a_1'';p_1'',p_2'')]/(\gamma^2-p_1''\gamma-p_2'').$$

Now if $\gamma \neq \delta, \delta \neq \alpha''$ and $\delta \neq \beta''$, using Lemma 1(i),

$$w_{n+1,4}(a_0,a_1,a_2,a_3;p_1,p_2,p_3) = \delta^{n-1}a_2 + [(p_2''/\delta)w_{n,2}(a_0'',a_1'';p_1'',p_2'') - p_2''\delta^{n-2}a_1''$$
$$+\gamma^2(\delta^{n-1}-\gamma^{n-1})(a_2'-a_1'p_1''-a_0'p_2'')/(\delta-\gamma) - (\gamma+p_2''/\delta)[\delta^n(p_1''a_1''+p_2''a_0'')-$$
$$\delta w_{n+1,2}(a_0'',a_1'';p_1'',p_2'')+\delta^{n-1}p_2''a_1''-p_2''w_{n,2}(a_0'',a_1'';p_1'',p_2'')]/(\delta^2-p_1''\delta-p_2'')]/(\gamma^2-p_1''\gamma-p_2'').$$

□

## 6. The Lengyel-Marques Functions

Special cases of $w_{n,k}$ are the functions $T_n(k)$ studied by Lengyel and Marques in [7] and [8]. These are given by:

**Definition 6.1.** $T_n(k) = w_{n,k}(0,1,\ldots,1;1,1,\ldots,1)$.

By Theorem 2 we have:

**Theorem 7.** *If $\gamma$ is a solution of $\gamma^k = \gamma^{k-1}+\ldots+\gamma+1$,*

*(i)* $T_n(k) = w_{n-1,k-1}(1,1-\gamma,\ldots,1-\gamma;1-\gamma,1+\gamma-\gamma^2,\ldots,\frac{\gamma^{k-1}-1}{\gamma-1}-\gamma^{k-1})+\gamma T_{n-1}(k)$

*(ii)* $T_n(k) = \gamma^{n-1}+\sum_{i=0}^{n-2}\gamma^i w_{n-1-i,k-1}(1,1-\gamma,\ldots,1-\gamma;1-\gamma,1+\gamma-\gamma^2,\ldots,\frac{\gamma^{k-1}-1}{\gamma-1}-\gamma^{k-1})$.

We can use Theorem 5(i) and Theorem 4 to give $T_n(3)$ in terms of Horadam functions.

**Theorem 8.** *If $\gamma$ is a solution of $\gamma^3=\gamma^2+\gamma+1$, $\alpha' = (1-\gamma+\sqrt{(5-3\gamma)(1+\gamma)})/2$ and $\beta' = (1-\gamma-\sqrt{(5-3\gamma)(1+\gamma)})/2$,*

$$T_n(3) = [\gamma^{n+1}+(1-2\gamma)w_{n,2}(1,1-\gamma;1-\gamma,1+\gamma-\gamma^2)]/((3\gamma+1)(\gamma-1)$$
$$-w_{n+1,2}(1,1-\gamma;1-\gamma,1+\gamma-\gamma^2)/((3\gamma+1)(\gamma-1))$$
$$=\Big(\gamma^{n+1}+\frac{(1-2\gamma-\alpha')(1-\gamma-\beta')}{(\alpha'-\beta')}\alpha'^n\Big)/((3\gamma+1)(\gamma-1)).$$
$$-\Big(\frac{(1-2\gamma-\beta')((1-\gamma-\alpha')}{(\alpha'-\beta')}\beta'^n\Big)/((3\gamma+1)(\gamma-1)).$$

*Proof.* With $a_0=0, a_1=a_2=p_1=p_2=p_3=1$, and $\gamma$ a solution of $\gamma^3=\gamma^2+\gamma+1$, we have $p_1'=1-\gamma$, $p_2'=1+\gamma-\gamma^2$, $\alpha'=(1-\gamma+\sqrt{(5-3\gamma)(1+\gamma)})/2$ and $\beta'=(1-\gamma-\sqrt{(5-3\gamma)(1+\gamma)})/2$.

As $\gamma \neq -1$ or $5/3$, $\alpha' \neq \beta'$. Also $\gamma = \alpha'$ or $\beta'$ is inconsistent with $\gamma^3 = \gamma^2+\gamma+1$, so we can use Theorem 5(i) to give:

$$\begin{aligned}
T_n(3) = {} & [\gamma^{n+1} - \gamma w_{n,2}(1, 1-\gamma; 1-\gamma, 1+\gamma-\gamma^2)]/(3\gamma+1)(\gamma-1) \\
& -[(1+\gamma-\gamma^2)w_{n-1,2}(1, 1-\gamma; 1-\gamma, 1+\gamma-\gamma^2)]/(((3\gamma+1)(\gamma-1)) \\
& = [\gamma^{n+1} + (1-2\gamma)w_{n,2}(1, 1-\gamma; 1-\gamma, 1+\gamma-\gamma^2)]/((3\gamma+1)(\gamma-1) \\
& \quad -w_{n+1,2}(1, 1-\gamma; 1-\gamma, 1+\gamma-\gamma^2)/((3\gamma+1)(\gamma-1)) \\
& = \Big(\gamma^{n+1} + \frac{(1-2\gamma-\alpha')(1-\gamma-\beta')}{(\alpha'-\beta')}\alpha'^n\Big)/((3\gamma+1)(\gamma-1)) \\
& -\Big(\frac{(1-2\gamma-\beta')((1-\gamma-\alpha')}{(\alpha'-\beta')}\beta'^n\Big)/((3\gamma+1)(\gamma-1)). \qquad \square
\end{aligned}$$

As in these representations of $T_n(k)$, $\gamma$ is irrational, they do not help to solve the problem of finding the highest power of two that divides $T_n(k)$, which is the main topic of [7] and [8].

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