CARDINALITY OF A FLOOR FUNCTION SET

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Abstract
Fix a positive integer X. We quantify the cardinality of the set $\{\lfloor X/n \rfloor : 1 \leq n \leq X\}$. We discuss restricting the set to those elements that are prime, semiprime or similar.

1. Introduction

Throughout we will restrict the variables m and n to positive integer values. For any real number X we denote by $\lfloor X \rfloor$ its integer part, that is, the greatest integer that does not exceed X. The most straightforward sum of the floor function is related to the divisor summatory function since

$$\sum_{n \leqslant X} \left\lfloor \frac{X}{n} \right\rfloor = \sum_{n \leqslant X} \sum_{k \leqslant X/n} 1 = \sum_{n \leqslant X} \tau(n),$$

where $\tau(n)$ is the number of divisors of n. From [2, Theorem 2] we infer

$$\sum_{n \leqslant X} \left\lfloor \frac{X}{n} \right\rfloor = X \log X + X(2\gamma - 1) + O\left(X^{517/1648+o(1)}\right),$$

where γ is the *Euler–Mascheroni* constant, in particular $\gamma \approx 0.57722$.

Recent results have generalized this sum to

$$\sum_{n \leqslant X} f\left(\left\lfloor \frac{X}{n} \right\rfloor\right),$$

where f is an arithmetic function (see [1], [3] and [4]).

In this paper we take a different approach by examining the cardinality of the set

$$S(X) := \left\{ m : m = \left\lfloor \frac{X}{n} \right\rfloor \text{ for some } n \leq X \right\}.$$

Our main results are as follows.

Theorem 1. *Let X be a positive integer. We have*

$$|S(X)| = \left\lfloor \sqrt{4X+1} \right\rfloor - 1.$$

Theorem 2. *We have*

$$|S(X)| = 2\sqrt{X} + O(1).$$

2. Proofs of Main Theorems

Throughout let

$$b = \frac{-1+\sqrt{4X+1}}{2}, \tag{1}$$

and note that

$$\frac{X}{b} = b+1.$$

We define 2 sets:

$$S_1(X) = \left\{ m : m = \left\lfloor \frac{X}{n} \right\rfloor, m \leq b \right\} \tag{2}$$

and

$$S_2(X) = \left\{ m : m = \left\lfloor \frac{X}{n} \right\rfloor, n(n-1) \leq X \right\}. \tag{3}$$

We first show that $S(X) = S_1(x) \cup S_2(x)$, which will allow the use of the inclusion-exclusion principle to formulate the cardinality of $S(X)$. It is clear that $S_1(X) \subseteq S(X)$ and $S_2(X) \subseteq S(X)$. So $S_1(X) \cup S_2(X) \subseteq S(X)$. Next, if $m \in S(X)$ then $m = X/n$ with either $n(n-1) \leq X$ or $n(n-1) > X$. If $n(n-1) \leq X$ then $m \in S_2(X)$. If $n(n-1) > X$ then, solving the quadratic equation, we have

$$n > \frac{1+\sqrt{4X+1}}{2} = b+1.$$

Then

$$m = \left\lfloor \frac{X}{n} \right\rfloor \leq \frac{X}{n} = b.$$

So $m \in S_1(X)$ and therefore $S(X) \subseteq S_1(X) \cup S_2(X)$, which proves that $S(X)$ and $S_1(X) \cup S_2(X)$ are equal.

Now, using the inclusion-exclusion principle,

$$|S(X)| = |S_1(X)| + |S_2(X)| - |S_1(X) \cap S_2(X)|. \tag{4}$$

We now calculate the number of elements of $S_1(X)$. Let m be an arbitrary positive integer with

$$m \leq b = \frac{-1+\sqrt{4X+1}}{2}.$$

This means that

$$m^2 + m - X \leq 0,$$

which implies that $m(m+1) \leq X$. Thus

$$\frac{X}{m(m+1)} \geq 1$$

and therefore

$$\frac{X}{m} - \frac{X}{m+1} \geq 1.$$

Since the interval from $\frac{X}{m}$ to $\frac{X}{m+1}$ has length at least 1, there must be an integer n such that

$$\frac{X}{m+1} < n \leq \frac{X}{m}, \tag{5}$$

from which

$$X - n < mn \leq X.$$

In turn this implies that

$$\frac{X}{n} - 1 < m \leq \frac{X}{n}.$$

This means that $m = \lfloor X/n \rfloor$ and so $m \in S_1(X)$. From (2) there are $\lfloor b \rfloor$ possible values of m. From (5) we see that we can always find an n to give us any of these values of m. Therefore the numbers $1, 2, \ldots, \lfloor b \rfloor$ are the only elements of $S_1(X)$ and so

$$|S_1(X)| = \lfloor b \rfloor. \tag{6}$$

We now consider the cardinality of $S_2(X)$. We show that $n(n-1) < X$ implies $\lfloor X/n \rfloor$ and $\lfloor X/(n-1) \rfloor$ are distinct. We have

$$\left\lfloor \frac{X}{n-1} \right\rfloor - \left\lfloor \frac{X}{n} \right\rfloor = \frac{X}{n-1} - \frac{X}{n} - \left\{ \frac{X}{n-1} \right\} + \left\{ \frac{X}{n} \right\},$$

where $\{\cdot\}$ represents, as usual, the fractional part of the real number. So

$$\left\lfloor \frac{X}{n-1} \right\rfloor - \left\lfloor \frac{X}{n} \right\rfloor = \frac{X}{n(n-1)} + t, \tag{7}$$

where $t \in (-1, 1)$. Recalling that $n(n-1) \leq X$, we have

$$\frac{X}{n(n-1)} \geq 1.$$

Substituting into (7) we see that $\left\lfloor \frac{X}{n-1} \right\rfloor - \left\lfloor \frac{X}{n} \right\rfloor > 0$, which implies that $\lfloor X/n \rfloor$ and $\lfloor X/(n-1) \rfloor$ are distinct. Since $n(n-1) \leq X$ we have, solving the quadratic equation,

$$n \leq \frac{X}{b} \tag{8}$$

and so

$$|S_2(X)| = \left\lfloor \frac{X}{b} \right\rfloor = \lfloor b \rfloor + 1. \tag{9}$$

We now consider $|S_1(X) \cap S_2(X)|$. We have seen that

$$S_1(X) = \{1, 2, \cdots, \lfloor b \rfloor\}.$$

From (8) we see that

$$n \leq \frac{X}{b} = b + 1.$$

So the values of n in $S_2(X)$ are $1, 2, \cdots \lfloor b + 1 \rfloor$ and therefore

$$S_2(X) = \left\{ \left\lfloor \frac{X}{\lfloor b+1 \rfloor} \right\rfloor, \left\lfloor \frac{X}{\lfloor b \rfloor} \right\rfloor, \cdots, X \right\}.$$

The set $S_1(X) \cap S_2(X)$ will be non-empty if

$$\left\lfloor \frac{X}{\lfloor b+1-c \rfloor} \right\rfloor = \lfloor b - d \rfloor,$$

for some $c, d \geq 0$. From this we deduce that $\frac{X}{b+1-c} < b + 1 - d$, so that $X < ((b+1) - d)(b + 1 - c)$. Recalling that $X = b(b + 1)$ we have

$$b + 1 - c(b + d + 1) - d(b + 1) > 0,$$

which is only possible if $c = d = 0$. Thus there will be at most one element of $S_1(X) \cap S_2(X)$ and this one element will occur if, and only if,

$$\left\lfloor \frac{X}{\lfloor b+1 \rfloor} \right\rfloor = \lfloor b \rfloor.$$

In fact,

$$|S_1(X) \cap S_2(X)| = 1 - \left(\left\lfloor \frac{X}{\lfloor b+1 \rfloor} \right\rfloor - \lfloor b \rfloor \right).$$

Combining this equation with (4), (6) and (9) and simplifying we obtain

$$|S(X)| = \lfloor b \rfloor + \left\lfloor \frac{X}{\lfloor b+1 \rfloor} \right\rfloor.$$

Removing b, by using 1, we have

$$|S(X)| = \left\lfloor \frac{-1+\sqrt{4X+1}}{2} \right\rfloor + \left\lfloor \frac{X}{\left\lfloor \frac{1+\sqrt{4X+1}}{2} \right\rfloor} \right\rfloor.$$

So to prove Theorem (1) it will suffice to show that

$$\left\lfloor \frac{-1+\sqrt{4X+1}}{2} \right\rfloor + \left\lfloor \frac{X}{\left\lfloor \frac{1+\sqrt{4X+1}}{2} \right\rfloor} \right\rfloor - \left\lfloor \sqrt{4X+1} \right\rfloor + 1 = 0. \tag{10}$$

Let a be the largest square number less than or equal to X. So

$$X = a^2 + s, \quad 0 \le s < a.$$

We show that (10) is true in 2 all-encompassing cases.

For the first case let $\sqrt{a^2+s+1/4} - \frac{1}{2} \ge a$. We have

$$\begin{aligned} \left\lfloor \frac{-1+\sqrt{4X+1}}{2} \right\rfloor &= \left\lfloor \frac{-1+\sqrt{4(a^2+s)+1}}{2} \right\rfloor \\ &= \left\lfloor \sqrt{a^2+s+1/4} - \frac{1}{2} \right\rfloor = a, \end{aligned} \tag{11}$$

and

$$\left\lfloor \frac{1+\sqrt{4X+1}}{2} \right\rfloor = \left\lfloor \frac{1+\sqrt{4(a^2+s)+1}}{2} \right\rfloor = \left\lfloor \sqrt{a^2+s+1/4} + \frac{1}{2} \right\rfloor = a+1.$$

So

$$\left\lfloor \frac{X}{\left\lfloor \frac{1+\sqrt{4X+1}}{2} \right\rfloor} \right\rfloor = \left\lfloor \frac{a^2+s}{a+1} \right\rfloor = \left\lfloor a - \frac{a-s}{a+1} \right\rfloor = a-1. \tag{12}$$

Also,

$$\left\lfloor \sqrt{4X+1} \right\rfloor = \left\lfloor \sqrt{4a^2+4s+1} \right\rfloor = \left\lfloor \sqrt{4a^2(1+s/a+1/a^2)} \right\rfloor = 2a. \tag{13}$$

Substituting (11), (12) and (13) into (10) shows that Theorem (1) is true in this case.

For the second case let $\sqrt{a^2+s+1/4} - \frac{1}{2} < a$. We have

$$\begin{aligned} \left\lfloor \frac{-1+\sqrt{4X+1}}{2} \right\rfloor &= \left\lfloor \frac{-1+\sqrt{4(a^2+s)+1}}{2} \right\rfloor \\ &= \left\lfloor \sqrt{a^2+s+1/4} - \frac{1}{2} \right\rfloor = a-1, \end{aligned} \tag{14}$$

and

$$\left\lfloor \frac{1+\sqrt{4X+1}}{2} \right\rfloor = \left\lfloor \frac{1+\sqrt{4(a^2+s)+1}}{2} \right\rfloor = \left\lfloor \sqrt{a^2+s+1/4}+\frac{1}{2} \right\rfloor = a.$$

So

$$\left\lfloor \frac{X}{\left\lfloor \frac{1+\sqrt{4X+1}}{2} \right\rfloor} \right\rfloor = \left\lfloor \frac{a^2+s}{a} \right\rfloor = \left\lfloor a+\frac{s}{a} \right\rfloor = a. \tag{15}$$

Also,

$$\left\lfloor \sqrt{4X+1} \right\rfloor = \left\lfloor \sqrt{4a^2+4s+1} \right\rfloor = \left\lfloor \sqrt{4a^2(1+s/a+1/a^2} \right\rfloor = 2a. \tag{16}$$

Substituting (14), (15) and (16) into (10) shows that Theorem (1) is true in this case. This completes the proof of Theorem 1. Theorem 2, that is

$$|S(X)| = 2\sqrt{X} + O(1),$$

follows immediately from Theorem 1.

3. Discussion

3.1. Primes in $S(X)$

We can generalize $S(X)$ by considering elements of $S(X)$ that are divisible by some positive integer $d \leq X$. This is interesting in its own right but could also form the basis for calculating something much more interesting: the number of primes, semiprimes or similar in $S(X)$.

Let

$$S_d(X) = \left\{ m : m = \left\lfloor \frac{X}{n} \right\rfloor \text{ for some } n \leq X, d \mid \left\lfloor \frac{X}{n} \right\rfloor \right\}.$$

A standard approach to express $S_d(X)$ would be to follow a path involving an indicator function, differences of floor functions, the ψ function and exponential sums, hoping that we can bound the exponential sums (here $\psi(y) = y - \lfloor y \rfloor - 1/2$). Unfortunately this is not the case here. The process yields the following result.

Lemma 3. *We have*

$$|S_d(X)| = \frac{4X^{1/2}}{3d} + \sum_{r=1}^{\left\lfloor \frac{X}{b} \right\rfloor} \sum_{j=\frac{X-r}{rd}}^{\frac{X}{rd}} \left(\psi\left(\frac{X}{dj+1} \right) - \psi\left(\frac{X}{dj} \right) \right) + O(1),$$

where $a = X/b$.

A proof is given in the Arxiv version of this paper (see [5, Section 5]). Calculating various sums using Maple suggests that the double sum cannot successfully be bound. In fact Maple suggests that the double sum is asymptotically equivalent to $2X^{1/2}/3d$. If this argument is correct then

$$S_d(X) \sim \frac{2X^{1/2}}{d},$$

as one would expect heuristically.

3.2. Trivial Bounds

In the absence of a better approach we outline some trivial bounds on $|S_d(X)|$. The interested reader may wish to improve these bounds.

Theorem 4. *For a real positive X and a positive integer $d \leq X$ with $d \neq 1$ we have*

$$\frac{X^{1/2}}{d} + O(1) \leq |S_d(X)| \leq 2\sqrt{X} + O(1).$$

Proof. The lower bound follows from the fact that $1, 2, \ldots, \lfloor b \rfloor \in S(X)$ (see Section 2). Of these $\lfloor \lfloor b \rfloor / d \rfloor$ will be divisible by d. Recalling that $b = X^{1/2} + O(1)$ the result follows. The upper bound flows directly from Theorem 1. □

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References

[1] O. Bordellès, L. Dai, R. Heyman, H. Pan, and I. E. Shparlinski, On a sum involving the Euler function, *J. Number Theory* **202** (2019), 278–297.

[2] J. Bourgain and N. Watt, Mean square of zeta function, circle problem and divisor problem revisited, preprint, arXiv:1709.04340.

[3] S. Chern, Notes on sums involving the Euler function, *Bull. Aust. Math. Soc.* **100** (2019), 194–200.

[4] A. Goswami, On a partial sum related to the Euler function, preprint, arXiv: 1812.07556.

[5] R. Heyman, Cardinality of a floor function set, preprint, arXiv:1905.00533.