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PARTITION INEQUALITIES AND APPLICATIONS TO SUM-PRODUCT CONJECTURES OF KANADE-RUSSELL

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Abstract

We consider differences of one- and two-variable finite products and provide combinatorial proofs of the nonnegativity of certain coefficients. Since the products may be interpreted as generating functions for certain integer partitions, this amounts to showing a partition inequality. This extends results due to Berkovich-Garvan and McLaughlin. We then apply the first inequality and Andrews' Anti-telescoping Method to give a solution to an "Ehrenpreis Problem" for recently conjectured sum-product identities of Kanade-Russell. That is, we provide significant further evidence for Kanade-Russell's conjectures by showing nonnegativity of coefficients in differences of product-sides as Andrews-Baxter and Kadell did for the product sides of the Rogers-Ramanujan identities.

1. Introduction

A *partition* λ of an integer n is a multi-set of positive integers $\{\lambda_1, \dots, \lambda_\ell\}$, whose *parts* satisfy

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell \geq 1 \qquad \text{and} \qquad \sum_{j=1}^{\ell} \lambda_j = n.$$

We will often use *frequency notation* to refer to a partition, where $(r^{m_r}, \dots, 2^{m_2}, 1^{m_1})$ represents the partition in which the part i occurs m_i times for $1 \leq i \leq r$.

Visually, a partition λ may be represented by its *Ferrer's diagram*, in which parts are displayed as rows of dots. For example, the Ferrer's diagram of the partition $(5, 3, 2^2)$ is the array below.

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For two q-series $f(q) = \sum_{n \geq 0} a_n q^n$ and $g(q) = \sum_{n \geq 0} b_n q^n$, we write $f(q) \succeq g(q)$ if $a_n \geq b_n$ for all n. If $f(q) \succeq 0$, we will say that f is a *nonnegative series.*

We will use the standard q-Pochhammer symbol,

$$(a;q)_n := \prod_{j=0}^{n-1}\left(1-aq^j\right), \quad (a;q)_\infty := \lim_{n\to\infty}(a;q)_n, \quad \text{and}$$
$$(a_1,\ldots,a_r;q)_n := (a_1;q)_n\cdots(a_r;q)_n.$$

By convention, an empty product equals 1.

The study of the type of partition inequality we consider began at the 1987 A.M.S. Institute on Theta Functions with a question of Leon Ehrenpreis about the Rogers-Ramanujan Identities ([3], Cor. 7.6 and Cor. 7.7):

$$\mathcal{RR}_1: \quad \sum_{n\geq 0}\frac{q^{n^2}}{(q;q)_n} = \frac{1}{(q,q^4;q^5)_\infty},$$

$$\mathcal{RR}_2: \quad \sum_{n\geq 0}\frac{q^{n^2+n}}{(q;q)_n} = \frac{1}{(q^2,q^3;q^5)_\infty}.$$

The identity $\mathcal{RR}_1$ may be interpreted as an equality of certain partition generating functions, giving that the number of partitions of n such that the gap between successive parts is at least 2 equals the number of partitions of n into parts congruent to $\pm 1 \pmod 5$. Similarly, $\mathcal{RR}_2$ gives that the number of partitions of n such that the gap between successive parts is at least 2 and 1 does not occur as a part equals the number of partitions of n into parts congruent to $\pm 2 \pmod 5$. Thus, both combinatorially and algebraically, it is easy to see that

$$\sum_{n\geq 0}\frac{q^{n^2}}{(q;q)_n} - \sum_{n\geq 0}\frac{q^{n^2+n}}{(q;q)_n} \succeq 0.$$

Therefore, it also holds that

$$\frac{1}{(q,q^4;q^5)_\infty} - \frac{1}{(q^2,q^3;q^5)_\infty} \succeq 0. \tag{1}$$

Ehrenpreis' Problem was to provide a proof of (1) that did not reference the (heavy-handed and quite nontrivial) Rogers-Ramanujan identities.

Solutions to Ehrenpreis' Problem have been given in various ways. In the course of proving (1), Andrews-Baxter [2] were led to a new "motivated" proof of the Rogers-Ramanujan Identities themselves. A direct combinatorial proof of (1) was provided by Kadell [8], who constructed an injection from the set of partitions of n with parts congruent to $\pm 2 \pmod 5$ to those with parts congruent to $\pm 1 \pmod 5$. Later, Andrews developed the Anti-telescoping Method for showing positivity in differences of products like (1) [1]. This method was used by Berkovich-Grizzell in [5] to prove infinite classes of partition inequalities, such as the following.

Theorem (Theorem 1.2 of [5]). *For any octuple of positive integers* (L, m, x, y, z, r, s, u),

$$\frac{1}{(q^x, q^y, q^z, q^{rx+sy+uz}; q^m)_L} - \frac{1}{(q^{rx}, q^{sy}, q^{uz}, q^{x+y+z}; q^m)_L} \succeq 0.$$

In [4], Berkovich-Garvan generalized (1) to an arbitrary modulus as follows.

Theorem (Theorem 5.3 of [4]). *Suppose* $L \geq 1$ *and* $1 \leq r < \frac{M}{2}$. *Then*

$$\frac{1}{(q, q^{M-1}; q^M)_L} - \frac{1}{(q^r, q^{M-r}; q^M)_L} \succeq 0$$

if and only if $r \nmid (M - r)$.

One can apply Berkovich-Garvan's result to solve similar "Ehrenpreis Problems" for Kanade-Russell's "mod 9 identities" in [10].

The first result in this paper extends the above in a way that is independent of the modulus. We will use this in Section 3 to solve an "Ehrenpreis Problem" for conjectural product-sum identities of Kanade-Russell in [11].

Theorem 1. *Let* a, b, c *and* M *be integers satisfying* $1 < a < b < c$ *and* $1+c = a+b$. *Then if* $a \nmid b$,

$$\frac{1}{(q, q^c; q^M)_L} - \frac{1}{(q^a, q^b; q^M)_L} \succeq 0 \qquad \text{for any } L \geq 0.$$

Note that we do not necessarily assume $a, b, c \leq M$. Translated into a partition inequality, Theorem 1.1 says that there are more partitions of n into parts of the forms $Mj + 1$ and $Mj + c$ than there are partitions of n into parts of the forms $Mj + a$ and $Mj + b$, where throughout $1 \leq j \leq L$.

Partition inequalities with a fixed number of parts were considered by McLaughlin in [13]. Answering two of McLaughlin's questions, we give combinatorial proofs of finite analogues of Theorems 7 and 8 from [13].

Theorem 2. *Let* a, b *and* M *be integers satisfying* $1 \leq a < b < \frac{M}{2}$ *and* $\gcd(b, M) = 1$. *Define* $c(m, n)$ *by*

$$\frac{1}{(zq^a, zq^{M-a}; q^M)_L(1 - q^{LM+a})} - \frac{1}{(zq^b, zq^{M-b}; q^M)_L} =: \sum_{m,n \geq 0} c(m, n) z^m q^n.$$

Then for any $L, n \geq 0$, *we have* $c(m, nM) \geq 0$. *If in addition* M *is even and* a *is odd, then we also have* $c\left(m, nM + \frac{M}{2}\right) \geq 0$ *for every* $n \geq 0$.

Note that we do not necessarily make the assumption $\gcd(a, M) = 1$ that is in [13]. While these partition inequalities hold only for n in certain residue classes (mod M), Theorem 2 is a strengthening of Theorem 1 for these n. The following is a distinct parts analogue.

Theorem 3. *Let a, b and M be integers satisfying $1 \leq a < b < \frac{M}{2}$ and $\gcd(b, M) = 1$. Define $d(m,n)$ by*

$$(-zq^a, -zq^{M-a}; q^M)_L \left(1 + zq^{LM+a}\right) - (-zq^b, -zq^{M-b}; q^M)_L =: \sum_{m,n \geq 0} d(m,n) z^m q^n.$$

Then for any $L, n \geq 0$, we have $d(m, nM) \geq 0$. If in addition M is even and a is odd, then we also have $d\left(m, nM + \frac{M}{2}\right) \geq 0$ for every $n \geq 0$.

Remark. Taking the limit as $L \to \infty$ in Theorems 2 and 3 recovers McLaughlin's original partition inequalities.

In Section 2, we begin by reviewing the M-modular diagram of a partition. Then we provide combinatorial proofs of Theorems 1-3. In Section 3, we apply Theorem 1 and Andrews' Method of Anti-Telescoping (see [1]) to give a solution to an "Ehrenpreis Problem" for recently conjectured sum-product identities of Kanade-Russell [11]. Our concluding remarks in Section 4 ask for Andrews-Baxter style "motivated proofs" of these conjectured identities.

2. Combinatorial Proofs of Theorems 1-3

2.1. Notation

The *M-modular diagram* of a partition $\lambda = \{\lambda_1, \ldots, \lambda_\ell\}$ is a modification of the Ferrer's diagram, wherein each λ_j is first written as $Mq + r$ for $0 \leq r < M$, and then is represented as a row of q M's and a single r at the end of the row. These r's we will refer to as *ends* or *r-ends*. For example, the 10-modular diagram of $\lambda = (53^2, 46, 36, 16, 11, 1)$ has three 6-ends, two 3-ends and two 1-ends:

$$\begin{array}{llllll}
10 & 10 & 10 & 10 & 10 & 3 \\
10 & 10 & 10 & 10 & 10 & 3 \\
10 & 10 & 10 & 10 & 6 & \\
10 & 10 & 10 & 6 & & \\
10 & 6 & & & & \\
10 & 1 & & & & \\
1 & & & & &
\end{array}$$

We will also speak of *attaching* and *removing* a column from an M-modular diagram. These operations are best defined with an example:

$$\begin{array}{ccccc} 10 & 10 & 10 & 10 & 10 \\ 10 & 10 & 10 & 10 & 10 \\ 10 & 10 & 10 & 10 & \\ 10 & 10 & 10 & & \\ 10 & 10 & 10 & & \\ 10 & 10 & 10 & & \\ 10 & & & & \end{array} \quad \underbrace{\cup}_{\substack{\textbf{attach}\\ \textbf{the column}}} \quad \begin{array}{c} 10 \\ 10 \\ 10 \end{array} \longrightarrow \begin{array}{cccccc} 10 & 10 & 10 & 10 & 10 & 10 \\ 10 & 10 & 10 & 10 & 10 & 10 \\ 10 & 10 & 10 & 10 & 10 & \\ 10 & 10 & 10 & & \uparrow & \\ 10 & 10 & 10 & & & \\ 10 & 10 & 10 & & & \\ 10 & & & & & \end{array}$$

$$\begin{array}{ccccc} 10 & 10 & 10 & 10 & 10 \\ 10 & 10 & 10 & 10 & 10 \\ 10 & 10 & 10 & 10 & \\ 10 & 10 & 10 & & \\ 10 & 10 & 10 & & \\ 10 & 10 & 10 & & \\ 10 & & \uparrow & & \end{array} \quad \underbrace{\setminus}_{\substack{\textbf{remove}\\ \textbf{the column}}} \quad \begin{array}{c} 10 \\ 10 \\ 10 \\ 10 \\ 10 \\ 10 \end{array} \longrightarrow \begin{array}{cccc} 10 & 10 & 10 & 10 \\ 10 & 10 & 10 & 10 \\ 10 & 10 & 10 & \\ 10 & 10 & & \\ 10 & 10 & & \\ 10 & 10 & & \\ 10 & & & \end{array}$$

We shall only attach or remove columns consisting entirely of M's, and it is easy to see that these operations preserve M-modular diagrams.

2.2. Proof of Theorem 1

We provide a combinatorial proof via injection that is nearly identical to that of Theorem 5.1 in [4], but we highlight a technical difference that arises in the general version. In keeping with [4], we let $\nu_j = \nu_j(\lambda)$ denote the number of parts of λ congruent to $j \pmod{M}$. (The modulus never varies and will be clear from context.)

Proof. First let $L = 1$. We will prove the general case as a consequence of this one. For each n, we seek an injection

$$\varphi_1 : \left\{(a^k, b^\ell) \vdash n : k, \ell \geq 0\right\} \hookrightarrow \left\{(1^k, c^\ell) \vdash n : k, \ell \geq 0\right\}.$$

Let $d :=\gcd(a, b)$. Explicitly, φ_1 is as follows:

$$\varphi_1\left(a^k, b^\ell\right) = \begin{cases} \left(1^{\ell+a(k-\ell)}, c^\ell\right) & \text{if } k \geq \ell, & \text{(Case 1)} \\ \left(1^{k+b(\ell-k)}, c^k\right) & \text{if } \ell > k \text{ and } \frac{a}{d} \nmid (\ell - k), & \text{(Case 2)} \\ \left(1^{k+1+b(\ell-k-1)-a}, c^{k+1}\right) & \text{if } \ell > k \text{ and } \frac{a}{d} \mid (\ell - k). & \text{(Case 3)} \end{cases}$$

This definition can be motivated by noting that each pre-image consists either of k pairs (a, b) and $k - \ell$ excess a's, or of ℓ pairs (a, b) and $\ell - k$ excess b's. (There can also be no excess.) The pairs are mapped as $(a, b) \mapsto (1, c)$. The excess a's or b's are treated by the following cases.

`Case 1.` For the $k-\ell$ excess a's, $(a) \mapsto (1^a)$.

`Case 2.` For the $\ell-k$ excess b's, $(b) \mapsto (1^b)$.

`Case 3.` For all but the last two excess b's, $(b) \mapsto (1^b)$. For the last two b's, $(b^2) \mapsto (1^{b-a+1}, c)$.

Note that in Case 3 there are at least two excess b's, for if not, $\frac{a}{d} = 1$ and then $a \mid b$, a contradiction. Also, by hypothesis, $b > \frac{c}{2}$, so that $2b > c$.

Let $(1^{\nu_1}, c^{\nu_c})$ be a partition in the image of φ_1. The cases are separated as follows:

`Case 1.` $a \mid (\nu_1 - \nu_c)$,

`Case 2.` $a \nmid (\nu_1 - \nu_c)$ and $b \mid (\nu_1 - \nu_c)$,

`Case 3.` $\nu_1 - \nu_c \equiv -b \pmod{a}$ and $\nu_1 - \nu_c \equiv -a \pmod{b}$.

This concludes the proof for $L = 1$.

Now let $L \geq 2$. Again we define an injection

$$\varphi_L : \{\lambda \vdash n : \lambda_j \in \{a, b, \ldots, LM + a, LM + b\}\}$$
$$\hookrightarrow \{\lambda \vdash n : \lambda_j \in \{1, c, \ldots, LM + 1, LM + c\}\}.$$

Let λ be a partition in the left set. Then λ consists of the triple

$$\left(\lambda_{(a)}, \lambda_{(b)}, (a^k, b^\ell)\right),$$

where $\lambda_{(a)}$ is the M-modular diagram obtained by subtracting a from every part of the form $Mj + a$; $\lambda_{(b)}$ is defined similarly. We apply φ_1 to (a^k, b^ℓ) and reattach the 1-ends and c-ends based on the case into which (a^k, b^ℓ) falls.

`Case 1`: $k \geq \ell$. Attach the 1-ends to $\lambda_{(a)}$ and the c-ends to $\lambda_{(b)}$. The map φ_1 guarantees exactly $\#\lambda_{(b)}$ c-ends. Likewise, there are at least as many 1-ends as there are parts of $\lambda_{(a)}$; any excess 1's are attached as parts to $\lambda_{(a)}$. The required image of λ is then the union of these two partitions.

`Cases 2 and 3`: $\ell > k$. Attach the 1-ends to $\lambda_{(b)}$ and the c-ends to $\lambda_{(a)}$ as before. φ_1 guarantees at least $\#\lambda_{(a)}$ c-ends. In Case 2 we are guaranteed at least $\#\lambda_{(a)}$ 1-ends because $b > 1$ implies

$$k + b(\ell - k) > \ell.$$

In Case 3, $\frac{a}{d} > 1$ implies $\ell - k > 1$, so

$$k + 1 + b(\ell - k - 1) - a = \ell + (b-1)(\ell - k - 1) - a \geq \ell,$$

and we are guaranteed at least $\#\lambda_{(a)}$ 1-ends.

Given the image of λ, we may clearly recover $\lambda_{(a)}$ and $\lambda_{(b)}$ based on its 1-ends and c-ends and the fact that φ_1 is an injection. Thus, φ_L is an injection. $\square$

Remark. The condition $a \nmid b$ in Theorem 1 is necessary to avoid cases like

$$\frac{1}{(q,q^5;q^6)_L} - \frac{1}{(q^2,q^4;q^6)_L},$$

in which the coefficient of q^4 is -1.

Remark. If we had copied the proof of Theorem 5.1 in [4] exactly, then the conditions “$\frac{a}{d} \mid$” and “$\frac{a}{d} \nmid$” would be replaced by “$a \mid$” and “$a \nmid$”. But this is not an injection because Case 2 is only correctly separated from the other two when $\gcd(a,b) = 1$. For example, this direct version of Berkovich-Garvan’s map gives:

$$\begin{cases} 4^7, 6^4 \\ 4^4, 6^6 \end{cases} \longrightarrow (1^{16}, 9^4), \qquad \text{instead of our} \qquad \begin{cases} 4^7, 6^4 \\ 4^4, 6^6 \end{cases} \longrightarrow \begin{cases} 1^{16}, 9^4 \\ 1^7, 9^5 \end{cases}.$$

In the first example, the partitions fall into cases 1 and 2. The second example corrects the overlap and places the partitions into cases 1 and 3.

We demonstrate the injection of Theorem 1 with an example.

Example 1. Here, $(n, M, L, a, b, c) = (52, 10, 2, 4, 6, 9)$. Numbers above arrows indicate the case into which a pre-image falls.

$16^3, 4$	$\xrightarrow{3}$	$11^3, 9^2, 1$
$16^2, 14, 6$	$\xrightarrow{3}$	$19, 11^2, 9, 1^2$
$16^2, 6^2, 4^2$	$\xrightarrow{3}$	$11^2, 9^3, 1^3$
$16^2, 4^5$	$\xrightarrow{1}$	$19^2, 1^{14}$
$16, 14^2, 4^2$	$\xrightarrow{1}$	$19, 11^2, 1^{11}$
$16, 14, 6^3, 4$	$\xrightarrow{3}$	$19, 11, 9^2, 1^4$
$16, 14, 6, 4^4$	$\xrightarrow{1}$	$19, 11, 9, 1^{13}$
$16, 6^6$	$\xrightarrow{2}$	$11, 1^{41}$
$16, 6^4, 4^3$	$\xrightarrow{3}$	$11, 9^4, 1^5$
$16, 6^2, 4^6$	$\xrightarrow{1}$	$19, 9^2, 1^{15}$
$16, 4^9$	$\xrightarrow{1}$	$19, 1^{33}$
$14^3, 6, 4$	$\xrightarrow{1}$	$11^3, 9, 1^{10}$

$14^2, 6^4$	$\xrightarrow{3}$	$19^2, 9, 1^5$
$14^2, 6^2, 4^3$	$\xrightarrow{1}$	$11^2, 9^2, 1^{12}$
$14^2, 4^6$	$\xrightarrow{1}$	$11^2, 1^{30}$
$14, 6^5, 4^2$	$\xrightarrow{3}$	$19, 9^3, 1^6$
$14, 6^3, 4^5$	$\xrightarrow{1}$	$11, 9^3, 1^{14}$
$14, 6, 4^8$	$\xrightarrow{1}$	$11, 9, 1^{32}$
$6^8, 4$	$\xrightarrow{2}$	$9, 1^{43}$
$6^6, 4^4$	$\xrightarrow{3}$	$9^5, 1^7$
$6^4, 4^7$	$\xrightarrow{1}$	$9^4, 1^{16}$
$6^2, 4^{10}$	$\xrightarrow{1}$	$9^2, 1^{34}$
4^{13}	$\xrightarrow{1}$	1^{52}

2.3. Proofs of Theorems 2 and 3

We begin by recalling the main steps in McLaughlin's proof of Theorem 7 from [13]; our proof is based on a combinatorial reading. First, Cauchy's Theorem ([3], Th. 2.1) is used with some algebraic manipulation to write, for fixed m,

$$\sum_{n\geq 0} c(m,n)q^n = \sum_{\substack{0\leq k<\frac{m}{2} \\ M|m-2k}} \frac{q^{kM}}{(q^M;q^M)_{m-k}(q^M;q^M)_k} \times \left(q^{a(m-2k)} + q^{(M-a)(m-2k)} - q^{b(m-2k)} - q^{(M-b)(m-2k)}\right).$$

It then happens that the factor in parentheses is equal to

$$q^{a(m-2k)}\left(1-q^{(b-a)(m-2k)}\right)\left(1-q^{(M-b-a)(m-2k)}\right).$$

But the conditions on a, b and M that lead to the condition $M \mid (m-2k)$ in the sum imply that both factors above are canceled in $\frac{1}{(q^M;q^M)_{m-k}}$. This gives nonnegativity.

The key steps in the proof are the decomposition of the sum over k and the nonnegativity of

$$\frac{(1-q^r)(1-q^s)}{(q;q)_n} \qquad \text{for } 1\leq r<s\leq n.$$

Both of these have simple combinatorial explanations, which we employ with M-modular diagrams to piece together a proof of Theorem 2. Our proof naturally leads to the finite versions with any $L \geq 1$ instead of ∞. The proof of Theorem 3 is then a slight modification.

Proof of Theorem 2. Let $\mathcal{P}(n,m,j,A)$ denote the set of partitions of n into m parts congruent to $\pm j$ modulo M such that the largest part is at most A. (We have suppressed the modulus M from the notation.) Let $\mathcal{P}_k(n,m,j,A)$ be the subset of partitions $\lambda \in \mathcal{P}(n,m,j,A)$ with either $\nu_j(\lambda) = k$ or $\nu_{M-j}(\lambda) = k$.

Clearly, we have the disjoint union $\mathcal{P}(n,m,j,A) = \bigsqcup_{0\leq k\leq \frac{m}{2}} \mathcal{P}_k(n,m,j,A)$. Thus, to show

$$\mathcal{P}(nM,m,b,LM-b) \hookrightarrow \mathcal{P}(nM,m,a,LM+a),$$

we may provide injections

$$\varphi_k : \mathcal{P}_k(nM,m,b,LM-b) \hookrightarrow \mathcal{P}_k(nM,m,a,LM+a)$$

for each $k \in [0, \frac{m}{2}]$.

Each $\lambda \in \mathcal{P}(nM,m,b,LM-b)$ consists of a triple

$$\left(\lambda_{(b)}, \lambda_{(M-b)}, (b^{\nu_b}, (M-b)^{\nu_{M-b}})\right),$$

where $\lambda_{(b)}$ is the M-modular diagram with ν_b nonnegative parts created by removing the b-ends. The M-modular diagram $\lambda_{(M-b)}$ is defined analogously by removing the $(M-b)$-ends.

When $k = \frac{m}{2}$, we simply map

$$\varphi_{\frac{m}{2}}\left(\lambda_{(b)}, \lambda_{(M-b)}, (b^{\frac{m}{2}}, (M-b)^{\frac{m}{2}})\right) := \left(\lambda_{(b)}, \lambda_{(M-b)}, (a^{\frac{m}{2}}, (M-a)^{\frac{m}{2}})\right).$$

The required partition is then obtained by reattaching the a-ends to $\lambda_{(b)}$ and reattaching the $(M-a)$-ends to $\lambda_{(M-b)}$.

Now assume $k < \frac{m}{2}$. Note that

$$0 \equiv nM \equiv b\nu_b(\lambda) - b\nu_{M-b}(\lambda) \pmod{M}, \tag{2}$$

which implies $\nu_b(\lambda) - \nu_{M-b}(\lambda) \equiv 0 \pmod{M}$ because $\gcd(b, M) = 1$. Thus, we assume without loss of generality that $M \mid (m-2k)$.

Let $y := \frac{(b-a)(m-2k)}{M}$ and $z := \frac{(M-b-a)(m-2k)}{M}$. These are positive integers.

`Case 1`: $\nu_{M-b}(\lambda) = k$. There are k pairs of $(b, M-b)$ and $m-2k$ excess b's. We map

$$\varphi_k\left(\lambda_{(b)}, \lambda_{(M-b)}, (b^{m-k}, (M-b)^k)\right) := \left(\lambda_{(b)} \cup \underbrace{\begin{bmatrix} M \\ \vdots \\ M \end{bmatrix}}_{y \text{ rows}}, \lambda_{(M-b)}, (a^{m-k}, (M-a)^k)\right)$$

$$=: \left(\lambda'_{(b)}, \lambda_{(M-b)}, (a^{m-k}, (M-a)^k\right).$$

Here $\lambda'_{(b)}$ is the M-modular diagram formed by attaching the above column to $\lambda_{(b)}$. Note that $a < b < M$ implies $0 < y < m-k$, so that $\lambda'_{(b)}$ is still an M-modular diagram with $m-k$ nonnegative parts.

To obtain the required partition, attach the a-ends to $\lambda'_{(b)}$ and the $(M-a)$-ends to $\lambda_{(M-b)}$. It is evident that there are m parts. Size is preserved, as

$$\begin{aligned}
&|\lambda'_{(b)}| + |\lambda_{(M-b)}| + (m-k)a + k(M-a) \\
&= |\lambda_{(b)}| + My + |\lambda_{(M-b)}| + a(m-2k) + kM \\
&= |\lambda_{(b)}| + (b-a)(m-2k) + |\lambda_{(M-b)}| + a(m-2k) + kM \\
&= |\lambda_{(b)}| + |\lambda_{(M-b)}| + b(m-2k) + kM \\
&= |\lambda|.
\end{aligned}$$

Moreover, it is clear that the operations are reversible, so that, within Case 1, φ_k is an injection.

`Case 2a`: $\nu_b(\lambda) = k$ and $\lambda_{(M-b)}$ does not contain a column of height y.[1] There are k pairs of $(b, M-b)$ and $m-2k$ excess $(M-b)$'s. We map

$$\varphi_k\left(\lambda_{(b)}, \lambda_{(M-b)}, (b^k, (M-b)^{m-k})\right) := \left(\lambda_{(b)}, \lambda_{(M-b)} \cup \underbrace{\begin{bmatrix} M \\ \vdots \\ M \end{bmatrix}}_{z \text{ rows}}, (a^{m-k}, (M-a)^k)\right)$$

$$=: \left(\lambda_{(b)}, \lambda'_{(M-b)}, (a^{m-k}, (M-a)^k\right),$$

where $\lambda'_{(M-b)}$ is defined by attaching the above column. Note again that $b, a < \frac{M}{2}$ implies $0 < z < m-k$, so that $\lambda'_{(M-b)}$ is still an M-modular diagram with $m-k$ nonnegative parts. Furthermore, $b-a \neq M-b-a$, so $\lambda'_{(M-b)}$ still does not contain a column of height y.

To obtain the required partition, attach the a-ends to $\lambda'_{(M-b)}$ and the $(M-a)$-ends to $\lambda_{(b)}$. It is evident that there are m parts. Size is preserved, as

$$\begin{aligned}
&|\lambda_{(b)}| + |\lambda'_{(M-b)}| + (m-k)a + k(M-a) \\
&= |\lambda_{(b)}| + |\lambda_{(M-b)}| + Mz + a(m-2k) + kM \\
&= |\lambda_{(b)}| + |\lambda_{(M-b)}| + (M-b-a)(m-2k) + a(m-2k) + kM \\
&= |\lambda_{(b)}| + |\lambda_{(M-b)}| + (M-b)(m-2k) + kM \\
&= |\lambda|.
\end{aligned}$$

Moreover, it is clear that the operations are reversible, so that, within Case 2a, φ_k is an injection.

`Case 2b`: $\nu_b(\lambda) = k$ and $\lambda_{(M-b)}$ contains a column of height y.[2] In this case we send

$$\left(\lambda_{(b)}, \lambda_{(M-b)}, (b^k, (M-b)^{m-k})\right) \mapsto \left(\lambda_{(b)}, \lambda_{(M-b)} \setminus \underbrace{\begin{bmatrix} M \\ \vdots \\ M \end{bmatrix}}_{y \text{ rows}}, (a^k, (M-a)^{m-k})\right)$$

$$=: \left(\lambda_{(b)}, \lambda'_{(M-b)}, (a^k, (M-a)^{m-k}\right),$$

where $\lambda'_{(M-b)}$ is defined by removing the above column. As before, we still may consider $\lambda'_{(M-b)}$ an M-modular diagram with $m-k$ nonnegative parts.

[1] Or equivalently, the y-th part of $\lambda_{(M-b)}$ equals the $(y+1)$-st part.

[2] Or equivalently, the y-th part of $\lambda_{(M-b)}$ is strictly greater than the $(y+1)$-st part.

To obtain the required partition, attach the a-ends to $\lambda_{(b)}$ and the $(M-a)$-ends to $\lambda'_{(M-b)}$. It is evident that there are m parts. Size is preserved, as

$$\begin{aligned}&|\lambda_{(b)}| + |\lambda'_{(M-b)}| + ka + (m-k)(M-a)\\ &= |\lambda_{(b)}| + |\lambda_{(M-b)}| - My + kM + (M-a)(m-2k)\\ &= |\lambda_{(b)}| + |\lambda_{(M-b)}| - (b-a)(m-2k) + kM + (M-a)(m-2k)\\ &= |\lambda_{(b)}| + |\lambda_{(M-b)}| + (M-b)(m-2k) + kM\\ &= |\lambda|.\end{aligned}$$

Moreover, it is clear that the operations are reversible, so that, within Case 2b, φ_k is an injection.

Let $\left(\lambda_{(a)}, \lambda_{(M-a)}, a^{\nu_a}, (M-a)^{\nu_{M-a}}\right)$ lie in the image of φ_k. Then cases are separated as follows.

`Case 1:` $\nu_a > \nu_{M-a}$ and $\lambda_{(a)}$ contains a column of height y.

`Case 2a:` $\nu_a > \nu_{M-a}$ and $\lambda_{(a)}$ does not contain a column of height y.

`Case 2b:` $\nu_a < \nu_{M-a}$.

Finally, note that in each case φ_k adds at most M to the largest part of what becomes $\lambda_{(a)}$, so indeed φ_k maps $\mathcal{P}_k(nM, m, b, LM-b)$ into $\mathcal{P}_k(nM, m, a, LM+a)$ as required. This completes the proof of the first statement.

When M is even and a is odd, we can use exactly the same injections, assuming because of (2) that $m - 2k \equiv \frac{M}{2} \pmod{M}$. We note that $\gcd(b, M) = 1$ implies that b is also odd, so y and z are still integers. □

Remark. We note that the extra factor $\frac{1}{(1-q^{LM+a})}$ in the left term of Theorem 2 is necessary. For example, in

$$\frac{1}{(zq^2, zq^5; q^7)_2} - \frac{1}{(zq^3, zq^4; q^7)_2},$$

the coefficients of $z^7q^{70}, z^{13}q^{70}, z^{16}q^{70}$, and $z^{18}q^{70}$ are all negative.

The proof of Theorem 3 is similar, but now cases are determined by columns that occur twice.

Proof of Theorem 3. We define injections φ'_k to be the same as φ_k, except that in Cases 2a and 2b we condition on whether or not a partition contains *two* columns of height y. This ensures that φ'_k preserves distinct parts partitions:

`Case 1`: Note that $\lambda_{(b)}$ is a distinct parts partition into $m-k$ nonnegative parts (so 0 occurs at most once). As such, $\lambda_{(b)}$ must contain a column of height y. (Recall that $y < m - k$.) Attaching another such column means that $\lambda'_{(b)}$ still has distinct nonnegative parts. Attaching the ends as above also preserves distinct parts.

Case 2a: Again attaching the column to $\lambda_{(M-b)}$ preserves distinct parts because $z < m - k$. The fact that $M - b - a \neq b - a$ implies that $\lambda'_{(M-b)}$ still does not contain two columns of height y.

Case 2b: Since $\lambda_{(M-b)}$ contains two columns of height y, removing one such column preserves distinct parts.

Cases are separated as follows.

Case 1: $\nu_a > \nu_{M-a}$ and $\lambda_{(a)}$ contains two columns of height y.

Case 2a: $\nu_a > \nu_{M-a}$ and $\lambda_{(a)}$ does not contain two columns of height y.

Case 2b: $\nu_a < \nu_{M-a}$.

This concludes the proof. □

Remark. Unlike in Theorem 2, it appears that the extra factor $\left(1 + q^{LM+a}\right)$ in the left term of Theorem 3 is often not needed for nonnegativity. A computational search up to $M \leq 12$, $L \leq 20$ and $nM \leq 250$ reveals that for

$$\sum_{m,n\geq 0} d'(m,n)z^m q^n := (-zq^a, -zq^{M-a}; q^M)_L - (-zq^b, -zq^{M-b}; q^M)_L,$$

we have some $d'(m, nM) < 0$ only when $(a, b, M) = (1, 2, 5)$.

In fact, we can condition on more than just 2 columns to prove the following new result.

Proposition 1. *Let $d \geq 0$, $1 \leq a < b < \frac{M}{2}$ and $\gcd(b, M) = 1$. Let $p^{(d)}(n, m, j, A)$ denote the number of partitions of n into m parts congruent to $\pm j \pmod{M}$, whose parts are at most A such that the gap between successive parts is greater than dM. Then for all $n, m \geq 0$,*

$$p^{(d)}(nM, m, a, LM + a) \geq p^{(d)}(nM, m, b, LM - b).$$

If in addition a is odd, then we also have

$$p^{(d)}\left(nM + \frac{M}{2}, m, a, LM + a\right) \geq p^{(d)}\left(nM + \frac{M}{2}, m, b, LM - b\right).$$

Substituting $d = 0$ and $d = 1$ gives Theorems 2 and 3 respectively.

Proof. Let $\lambda = \big(\lambda_{(b)}, \lambda_{(M-b)}, (b^{\nu_b}, (M-b)^{\nu_{M-b}})\big)$ be a partition counted by $p^{(d)}(nM, m, b, LM-b)$. Then the M modular diagrams $\lambda_{(b)}$ and $\lambda_{(M-b)}$ are partitions into nonnegative multiples on M such that the difference in successive parts is at least $(d+1)M$. Our injections $\varphi_k^{(d)}$ are the same as before, except that we condition in cases 2 or 3 on whether or not $\lambda_{(M-b)}$ contains $d+2$ columns of height y. □

3. Applications to Kanade-Russell's Conjectures

In [11], Kanade and Russell conjectured several new Rogers-Ramanujan-type product-sum identities—three arising from the theory of affine Lie algebras, and several companions. Bringmann, Jennings-Shaffer and Mahlburg were able to prove many of these [6], and they reduced the sum-sides of the four conjectures below from triple series to a single series. Here, $\mathcal{KR}_j$ is Identity j in [11], and $H_j(x)$ is the sum side as denoted in [6].

$$\begin{aligned}
\mathcal{KR}_4 : &\qquad H_4(1) = \frac{1}{(q, q^4, q^5, q^9, q^{11}; q^{12})_\infty},\\
\mathcal{KR}_{4a} : &\qquad H_5(1) = \frac{1}{(q, q^5, q^7, q^8, q^9; q^{12})_\infty},\\
\mathcal{KR}_6 : &\qquad H_8(1) = \frac{1}{(q, q^3, q^7, q^8, q^{11}; q^{12})_\infty},\\
\mathcal{KR}_{6a} : &\qquad H_9(1) = \frac{1}{(q^3, q^4, q^5, q^7, q^{11}; q^{12})_\infty}.
\end{aligned}$$

The pairs of sum-sides, $(H_4(1), H_5(1))$ and $(H_8(1), H_9(1))$, are composed of two generating functions for partitions that satisfy the same set of gap conditions, but H_5 and H_9 have an additional condition on the smallest part (see [11]). Hence, as with the Rogers-Ramanujan sum-sides, we have the inequalities

$$H_4(1) - H_5(1) \succeq 0 \qquad \text{and} \qquad H_8(1) - H_9(1) \succeq 0,$$

which, *if the conjectures are true*, imply the following result.

Proposition 2. *The following inequalities hold.*

$$\frac{1}{(q, q^4, q^5, q^9, q^{11}; q^{12})_\infty} - \frac{1}{(q, q^5, q^7, q^8, q^9; q^{12})_\infty} \succeq 0, \tag{3}$$

$$\frac{1}{(q, q^3, q^7, q^8, q^{11}; q^{12})_\infty} - \frac{1}{(q^3, q^4, q^5, q^7, q^{11}; q^{12})_\infty} \succeq 0. \tag{4}$$

Proof. (4) is an immediate consequence of Theorem 1, since for every $L \geq 0$,

$$\frac{1}{(q,q^8;q^{12})_L} - \frac{1}{(q^4,q^5;q^{12})_L} \succeq 0.$$

Multiplying both sides by the positive series $\frac{1}{(q^3,q^7,q^{11};q^{12})_\infty}$ and taking the limit as $L \to \infty$ finishes the proof of (4).

Andrews' Anti-telescoping Method [1] works seamlessly to show (3). Define

$$P(j) := (q,q^4,q^{11};q^{12})_j \qquad \text{and} \qquad Q(j) := (q,q^7,q^8;q^{12})_j,$$

and note that the following implies (3):

$$\frac{1}{P(L)} - \frac{1}{Q(L)} \succeq 0 \quad \text{for all } L \geq 0. \tag{5}$$

Now we write

$$\begin{aligned}
\frac{1}{P(L)} - \frac{1}{Q(L)} &= \frac{1}{Q(L)}\left(\frac{Q(L)}{P(L)} - 1\right) \\
&= \frac{1}{Q(L)} \sum_{j=1}^{L} \left(\frac{Q(j)}{P(j)} - \frac{Q(j-1)}{P(j-1)}\right) \\
&= \sum_{j=1}^{L} \frac{1}{\frac{Q(L)}{Q(j-1)}P(j)} \left(\frac{Q(j)}{Q(j-1)} - \frac{P(j)}{P(j-1)}\right),
\end{aligned}$$

whose j-th term is

$$\frac{(1-q^{12j-11})q^{12(j-1)}}{(q^{12j-11},q^{12j-5},q^{12j-4};q^{12})_{L-j+1}(q,q^4,q^{11};q^{12})_j} \times \left(-q^7 - q^8 + q^4 + q^{11}\right)$$

$$= \frac{(1-q^{12j-11})q^{12(j-1)}}{(q^{12j-11},q^{12j-5},q^{12j-4};q^{12})_{L-j+1}(q,q^4,q^{11};q^{12})_j} \times q^4(1-q^3)(1-q^4). \tag{6}$$

The terms $(1-q^4)$ and $(1-q^{12j-11})$ are cancelled in the denominator, and we can write $\frac{1-q^3}{1-q} = 1+q+q^2$. Hence, (6) is nonnegative for every j, proving (5) and then (3). □

Another pair of identities in [11] with an Ehrenpreis Problem set-up is the following.

$$\mathcal{KR}_5: \qquad H_6(1) = \frac{1}{(q^2;q^4)_\infty} \prod_{n\geq 0}\left(1 + q^{4n+1} + q^{2(4n+1)}\right),$$

$$\mathcal{KR}_{5a}: \qquad H_7(1) = \frac{1}{(q^2;q^4)_\infty} \prod_{n\geq 0}\left(1 + q^{4n+3} + q^{2(4n+3)}\right)$$

Both identities were proved in [6], and there is an obvious injection proving

$$\frac{1}{(q^2;q^4)_\infty}\prod_{n\geq 0}\left(1+q^{4n+1}+q^{2(4n+1)}\right)-\frac{1}{(q^2;q^4)_\infty}\prod_{n\geq 0}\left(1+q^{4n+3}+q^{2(4n+3)}\right)\succeq 0,$$

namely, sending each $(4n+3)$ to the pair $(4n+1,2)$.

Finally, we discuss the Ehrenpreis problems among $\mathcal{KR}_1$, $\mathcal{KR}_2$ and $\mathcal{KR}_3$. These were proved in [6], and their respective sum-sides were denoted $H_1(x)$, $H_2(x)$ and $H_3(x)$. Using the methods of [11], we have found slightly different conditions for the partitions enumerated on the sum-side:

1. No consecutive parts are allowed.
2. Odd parts do not repeat.
3. Even parts appear at most twice.
4. We have $(\lambda_j,\lambda_{j+1},\lambda_{j+2})\notin\{(2k,2k,2k+2),(2k,2k,2k+3),(2k+1,2k+3,2k+5),(2k-2,2k,2k)\}$ for any j and k.[3]

Note that our fourth condition is changed slightly from Kanade and Russell's in [11], page 5. The sum-side of $\mathcal{KR}_2$ has the further restriction that the part 1 may not appear, and in the sum-side of $\mathcal{KR}_3$, the parts 1, 2 and 3 may not appear. Hence, $H_1(1)\succeq H_2(1)\succeq H_3(1)$ and it follows from Theorem 1.1 of [6] that

$$\frac{1}{(q,q^4,q^6,q^8,q^{11};q^{12})_\infty}\succeq\frac{(-q^3,-q^9;q^{12})_\infty}{(q^2,q^4,q^8,q^{10};q^{12})_\infty}\succeq\frac{1}{(q^4,q^5,q^6,q^7,q^8;q^{12})_\infty}.$$

The inequality between the far left and right products is a consequence of Theorem 5.3 of [4], but a direct proof of the other two inequalities remains open.

4. Concluding Remarks

As we pointed out in the introduction, (1) was the start of Andrews-Baxter's "motivated" proof of the Rogers-Ramanujan identities [2]. They defined $G_1:=(q,q^4;q^5)_\infty^{-1}$ and $G_2:=(q^2,q^3;q^5)_\infty^{-1}$, and then recursively

$$G_i:=\frac{G_{i-2}-G_{i-1}}{q^{i-2}},\qquad\text{for } i\geq 3. \tag{7}$$

They then observed computationally that $G_i=1+\sum_{n\geq i}g_{i,n}q^n\succeq 0$. Thus, as $i\to\infty$ the coefficient of q^n in G_i is eventually 0. This was their "Empirical Hypothesis," and proving it leads easily to a new proof of the Rogers-Ramanujan identities.

[3] As in [11], we have written parts in increasing order.

Note that, starting from the sum-sides of G_1 and G_2, the recursive definition (7) and the Empirical Hypothesis are completely natural. For example, if $\mathcal{RR}$ denotes the set of gap-2 partitions, then by the Rogers-Ramanujan Identities,

$$G_1 - G_2 = \sum_{\substack{\lambda \in \mathcal{RR} \\ \lambda \ni 1}} q^{|\lambda|} = q \left(1 + \sum_{\substack{\lambda \in \mathcal{RR} \\ \lambda_j \geq 3}} q^{|\lambda|} \right),$$

and so

$$G_2 - G_3 = \sum_{\substack{\lambda \in \mathcal{RR} \\ \lambda_j \geq 2 \\ \lambda_j \ni 2}} q^{|\lambda|} = q^2 \left(1 + \sum_{\substack{\lambda \in \mathcal{RR} \\ \lambda_j \geq 4}} q^{|\lambda|} \right),$$

and so on.

For $\mathcal{KR}_4$, $\mathcal{KR}_{4a}$, $\mathcal{KR}_6$ and $\mathcal{KR}_{6a}$, we can expect the more complicated conditions on the sum-sides to lead to more complicated recurrences. For example, the recurrence below was shown for $\mathcal{KR}_4$ ([6], equation 4.2).

$$H_4(x) = (1 + xq)H_4(xq^2) + xq^2(1 + xq^3)H_4(xq^4) + x^2q^6(1 - xq^4)H_4(xq^6).$$

Combinatorial proofs of the above and the similar recurrences in [6] may give insight into an "Empirical Hypothesis" for $\mathcal{KR}_4$, $\mathcal{KR}_{4a}$, $\mathcal{KR}_6$ and $\mathcal{KR}_{6a}$. Indeed, the techniques for "motivated proofs" have been expanded to accommodate identities with gap-conditions more complicated than those of $\mathcal{RR}$, notably in [7], [9] and [12]. Perhaps further developments will give an answer to the following question: Do there exist "motivated proofs" of $\mathcal{KR}_4$, $\mathcal{KR}_{4a}$, $\mathcal{KR}_6$ and $\mathcal{KR}_{6a}$?

This would be especially interesting, since to our knowledge there have not yet been "motivated proofs" featuring asymmetric products.

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