ON THE COMPLETE TREE OF PRIMITIVE PYTHAGOREAN QUADRUPLES

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Abstract

A primitive Pythagorean triple is a 3-tuple of natural numbers sharing no nontrivial common factors that satisfies the Pythagorean Theorem. Hall (1970) and Price (2008) found distinct perfect infinite ternary trees whose vertex sets are precisely all primitive Pythagorean triples. Using elementary tools, we will construct an infinite tree whose vertex set consists of all nonnegative primitive Pythagorean quadruples—i.e., 4-tuples (d, a, b, c) of natural numbers having no nontrivial common factors that satisfy $d^2 = a^2 + b^2 + c^2$. We will also present some interesting subtrees with curious properties.

1. Introduction

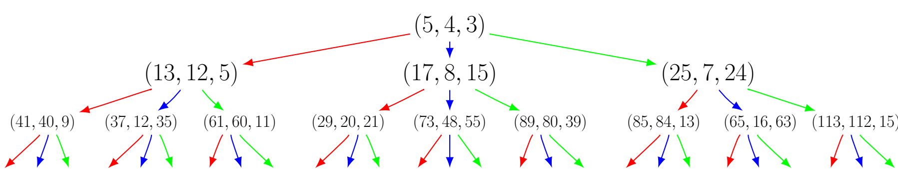
A *Pythagorean triple* is an integer 3-tuple (c, a, b) that satisfies the equation $c^2 = a^2 + b^2$, such as $(15, -12, 9)$ and $(26, 24, 10)$. A triple is said to be a *primitive Pythagorean triple*, or PPT, if c, a, and b share no common nontrivial factors, such as $(5, 4, 3)$ and $(13, 12, -5)$. In his classic work *The Elements*, Euclid characterized all PPTs as follows.

Theorem 1. *Every ordered pair (m, n), $m > n > 0$, of relatively prime integers of* *opposite parity generates a PPT, namely* $(m^2+n^2, 2mn, m^2-n^2)$*, and every PPT corresponds to a unique such ordered pair* (m, n)*.*

We will often depict a PPT as a column vector. That is, the ordered triple (c, a, b) and the column vector $\begin{bmatrix} c \\ a \\ b \end{bmatrix}$ represent the same PPT.

In 1934, Berggren [2] showed that the entire set of PPTs with positive integer entries can be thought of as the vertex set of an infinite perfect ternary tree with root $(5, 4, 3)$. Every vertex in the tree has three children, each formed by multiplying a given PPT by one of three matrices. His results were later rediscovered in 1963 by Barning [1] and in 1970 by Hall [6]. A related complete tree of PPTs was also found in 2008 by Price [7]. Price used the set $\mathcal{P} = \{A, B, C\}$ of matrices given below to construct his PPT tree, which is given in Figure 1.

$$A = \begin{bmatrix} 3 & 1 & -2 \\ 2 & 2 & -2 \\ -1 & 1 & 2 \end{bmatrix}, B = \begin{bmatrix} 3 & -1 & 2 \\ 2 & -2 & 2 \\ 1 & 1 & 2 \end{bmatrix}, C = \begin{bmatrix} 3 & 1 & 2 \\ 2 & 2 & 2 \\ 1 & -1 & 2 \end{bmatrix}.$$



Key: Red: C, Blue: B, Green: A

Figure 1: Price's PPT Tree

We can summarize this information on primitive Pythagorean triples as follows. The well-known methods for generating all PPTs include a simple number theoretic approach (the 2-parameter formulas of Euclid) and a simple matrix algebra approach (the 3-matrix family $\mathcal{P} = \{A, B, C\}$ of Price). Both yield a complete set of nonnegative PPTs - with no duplicates - using elementary tools. One could also say that Price created a nice picture of a nice representative set of PPTs, and he did so in a nice, elegant and yet simple manner.

A *Pythagorean quadruple* is a set of four integers (d, a, b, c) that satisfy the equation $d^2 = a^2 + b^2 + c^2$, such as $(14, 12, -6, 4)$ and $(18, 8, 8, 14)$. The quadruple is said to be a *primitive Pythagorean quadruple*, or PPQ, if d, a, b and c share no common nontrivial divisors such as $(9, -4, 4, 7)$ and $(3, 2, 2, 1)$. There is a not-so-well-known way to generate all PPQs [3] that is similar to Euclid's well-known formulas for generating PPTs.

Theorem 2. *(Carmichael, 1915) Every PPQ is of the form* $(m^2+n^2+p^2+q^2, 2mq+$

$2np, 2nq - 2mp, m^2 + n^2 - p^2 - q^2$) where m, n, p, q are relatively prime nonnegative integers and $m + n + p + q \equiv 1 \pmod 2$.

It should be noted that there is no obvious ordering of the generating 4-tuples (m, n, p, q) used in Carmichael's formulas, unlike the ordering of the pairs of generators (m, n) that appear in Euclid's formulas.

One may contrast these elementary perspectives with the more advanced results (see [4], [5]) that represent the entire family of PPTs as the orbit of $(1, 0, 1)$ under a specific group action. These approaches take advantage of more complicated mathematical machinery, such as the power of Lie algebras. However, these approaches, while they yield the entire set of PPTs, do not differentiate between two or more PPTs that refer to the same geometric object. That is, a triangle with "side lengths described by the 3-tuple" $(5, 4, 3)$ is indistinguishable from the triangle with "side lengths" described by the 3-tuple $(3, 5, -4)$.

These observations led us to ponder the following question: is there an elementary way to generate all nonnegative primitive Pythagorean quadruples? In particular, is it possible—mimicking the results of Price—to easily construct a nice complete tree of nonnegative primitive Pythagorean quadruples? It turns out that the answer to this question is yes, but it is a complicated yes. The complete tree we constructed, while lacking symmetry, possesses many surprising secrets.

The rest of this paper is structured as follows: in the next section, we will present background information of Hall's complete tree of primitive Pythagorean triples. Next, we will provide a new proof of Hall's result. We will then present the proof that our infinite tree of primitive Pythagorean quadruples is complete. Following this, we will describe the construction of our complete infinite tree of all such quadruples. Finally, we will give just a few of the many interesting subtrees hidden in this tree. These special subtrees are characterized by tree structure, or by rate of generation, or by properties of their vertex sets.

2. PPT Background Information

Now that we have defined PPTs and PPQs as well as shown Price's infinite tree of PPTs, it is worth our time to discuss Hall's infinite tree of PPTs. Hall used the set of matrices $\mathcal{H} = \{H_0, H_2, H_3\}$, provided below, to construct his complete infinite ternary tree of PPTs, which appears in Figure 2:

$$H_0 = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}, \; H_2 = \begin{bmatrix} 3 & -2 & 2 \\ 2 & -1 & 2 \\ 2 & -2 & 1 \end{bmatrix}, \; H_3 = \begin{bmatrix} 3 & 2 & -2 \\ 2 & 1 & -2 \\ 2 & 2 & -1 \end{bmatrix}.$$

It is easy to show that any PPT (written as a column vector) multiplied on the

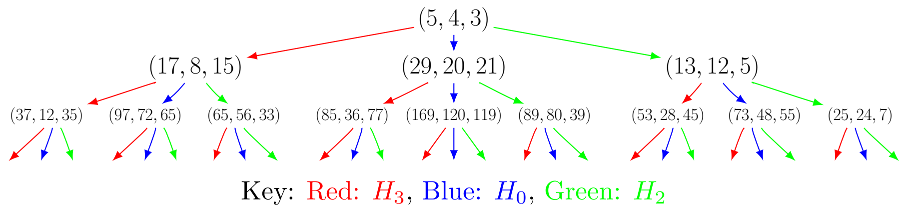


Figure 2: Hall's PPT Tree

left by any of Hall's matrices always yields another PPT. For instance, given an arbitrary PPT (c_1, a_1, b_1), note that

$$H_0 \cdot (c_1, a_1, b_1) = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} c_1 \\ a_1 \\ b_1 \end{bmatrix} = \begin{bmatrix} 3c_1 + 2a_1 + 2b_1 \\ 2c_1 + a_1 + 2b_1 \\ 2c_1 + 2a_1 + b_1 \end{bmatrix} = \begin{bmatrix} c_2 \\ a_2 \\ b_2 \end{bmatrix}.$$

Now observe that

$$\begin{aligned} c_2^2 &= (3c_1 + 2a_1 + 2b_1)^2 \\ &= 9c_1^2 + 4a_1^2 + 4b_1^2 + 8a_1b_1 + 12c_1a_1 + 12c_1b_1 \\ &= (8c_1^2 + c_1^2) + 4a_1^2 + 4b_1^2 + 8a_1b_1 + 12c_1a_1 + 12c_1b_1 \\ &= 8c_1^2 + (a_1^2 + b_1^2) + 4a_1^2 + 4b_1^2 + 8a_1b_1 + 12c_1a_1 + 12c_1b_1 \\ &= 8c_1^2 + 5a_1^2 + 5b_1^2 + 8a_1b_1 + 12c_1a_1 + 12c_1b_1 \\ &= (4c_1^2 + a_1^2 + 4b_1^2 + 8c_1b_1 + 4a_1c_1 + 4a_1b_1) \\ &\quad + (4c_1^2 + 4a_1^2 + b_1^2 + 8c_1a_1 + 4b_1c_1 + 4a_1b_1) \\ &= (2c_1 + a_1 + 2b_1)^2 + (2c_1 + 2a_1 + b_1)^2 \\ &= a_2^2 + b_2^2. \end{aligned}$$

Thus, (c_2, a_2, b_2) is a Pythagorean triple. To show (c_2, a_2, b_2) is primitive, assume that each of a_2, b_2, c_2 is divisible by some prime p. Then p divides any linear combination of a_2, b_2, and c_2. In particular,

$$\begin{aligned} p | 2c_2 + 3a_2 &\Rightarrow p | 2(3c_1 + 2a_1 + 2b_1) + 3(2c_1 + a_1 + 2b_1) \\ &\Rightarrow p | (6c_1 + 4a_1 + 4b_1) - (6c_1 + 3a_1 + 6b_1) \\ &\Rightarrow p | a_1 - 2b_1. \end{aligned}$$

Similarly,

$$\begin{aligned} p | a_2 - b_2 &\Rightarrow p | (2c_1 + a_1 + 2b_1) - (2c_1 + 2a_1 + b_1) \\ &\Rightarrow p | -a_1 + b_1. \end{aligned}$$

But if $p|a_1 - 2b_1$ and $p| - a_1 + b_1$, then p divides their sum; that is, $p|(a_1 - 2b_1) + (-a_1 + b_1)$, or $p| - b_1$, or $p|b_1$. A similar computation shows that $p|a_1$ and hence $p|c_1$.

This is a contradiction, since (c_1, a_1, b_1) is primitive. Therefore, (c_2, a_2, b_2) is primitive and the result follows. The remaining cases for PPTs generated by H_2 and H_3 are similar and left to the curious reader.

It should be noted that the only non-root PPT that occurs in the same location in the complete PPT trees of Hall and Price is $(89, 80, 39)$. In general, most PPTs appear in different levels of the two trees. The remainder of this section will focus on Hall's tree and his generating matrices.

The set of all PPTs can be thought of, more generally, as the orbit of the PPT $(1, 0, 1)$ under the action of H_0 together with the set of all matrices formed by negating and/or swapping columns of H_0. But this action generates up to 48 different but equivalent representatives of each PPT—think $(13, 12, 5)$ vs. $(13, -12, -5)$ vs. $(-12, -13, 5)$, etc.—which motivates the need for an equivalence relation on the set of all PPTs. To this end, given PPTs (c, a, b) and (c', a', b'), we say that

$$(c, a, b) \sim (c', a', b') \text{ if and only if } \{|c|, |a|, |b|\} = \{|c'|, |a'|, |b'|\}.$$

It then follows that any PPT of the form $(c, 0, b)$ is equivalent to the "trivial" PPT $(1, 0, 1)$.

In his proof, Hall applied a descent argument to Euclid's generating formulas to show that the positive integer representation of every PPT equivalence class appears exactly once in his tree. More precisely, he showed that the PPT generated by (m, n) is the parent of the PPTs generated by $(2m - n, m)$, $(2m + n, m)$, and $(m+2n, n)$. He used the inherent structure of all such ordered pairs (m, n) to prove his tree contains each PPT exactly once.

Recall that Carmichael's generating formula for PPQs takes an integer 4-tuple (m, n, p, q) as its argument, whereas Euclid's generating formula for PPTs takes an integer 2-tuple (m, n). We also mentioned that there is no clear ordering to the tuples (m, n, p, q) as there is for the tuples (m, n). So it is probable that an argument similar to the above cannot be applied to PPQs. As a result, a new argument must be found for PPTs and subsequently applied to PPQs.

3. A New PPT Tree Proof

We will now present a different proof of Hall's result, based on the inverses of the elements of $\mathcal{H}$. Unlike Hall, we will include the trivial PPT $(1, 0, 1)$ as the root of our complete PPT tree. To begin, note that

$$H_0^{-1} = \begin{bmatrix} 3 & -2 & -2 \\ -2 & 1 & 2 \\ -2 & 2 & 1 \end{bmatrix},\ H_2^{-1} = \begin{bmatrix} 3 & -2 & -2 \\ 2 & -1 & -2 \\ -2 & 2 & 1 \end{bmatrix},\ H_3^{-1} = \begin{bmatrix} 3 & -2 & -2 \\ -2 & 1 & 2 \\ 2 & -2 & -1 \end{bmatrix}.$$

Definition 1. Given any $x, y \in \{-1, 1\}$ with $x + y \geq 0$, an arbitrary element H_* of $\mathcal{H}$ is denoted

$$H_* = \begin{bmatrix} 3 & 2x & 2y \\ 2 & x & 2y \\ 2 & 2x & y \end{bmatrix}, \text{ and thus } H_*^{-1} = \begin{bmatrix} 3 & -2 & -2 \\ -2x & x & 2x \\ -2y & 2y & y \end{bmatrix}.$$

Given any $i \in \mathbb{N}$, for convenience, we will often denote the PPT (c_i, a_i, b_i) as the vector $\boldsymbol{t}_i$.

Lemma 1. *Given any PPT (c_1, a_1, b_1), if $a_1 \neq 0$, there exists an $H \in \mathcal{H}$ such that $H^{-1} \cdot \boldsymbol{t}_1 = \boldsymbol{t}_2$, where $\boldsymbol{t}_2$ is a nonnegative PPT equivalence class representative and $c_2 < c_1$.*

Proof. Let H_*^{-1} act on $\boldsymbol{t}_1$. This yields

$$\begin{bmatrix} 3 & -2 & -2 \\ -2x & x & 2x \\ -2y & 2y & y \end{bmatrix} \cdot \begin{bmatrix} c_1 \\ a_1 \\ b_1 \end{bmatrix} = \begin{bmatrix} 3c_1 - 2a_1 - 2b_1 \\ x(-2c_1 + a_1 + 2b_1) \\ y(-2c_1 + 2a_1 + b_1) \end{bmatrix} = \begin{bmatrix} c_2 \\ a_2 \\ b_2 \end{bmatrix}.$$

Note that $x^2 = y^2 = 1$. Using the fact $c_1^2 = a_1^2 + b_1^2$, it then follows that

$$\begin{aligned} c_2^2 &= (3c_1 + 2a_1 + 2b_1)^2 \\ &= 9c_1^2 + 4a_1^2 + 4b_1^2 + 8a_1b_1 + 12c_1a_1 + 12c_1b_1 \\ &= (8c_1^2 + c_1^2) + 4a_1^2 + 4b_1^2 + 8a_1b_1 + 12c_1a_1 + 12c_1b_1 \\ &= 8c_1^2 + (a_1^2 + b_1^2) + 4a_1^2 + 4b_1^2 + 8a_1b_1 + 12c_1a_1 + 12c_1b_1 \\ &= 8c_1^2 + 5a_1^2 + 5b_1^2 + 8a_1b_1 + 12c_1a_1 + 12c_1b_1 \\ &= (4c_1^2 + a_1^2 + 4b_1^2 + 8c_1b_1 + 4a_1c_1 + 4a_1b_1) \\ &\quad + (4c_1^2 + 4a_1^2 + b_1^2 + 8c_1a_1 + 4b_1c_1 + 4a_1b_1) \\ &= (-2c_1 + a_1 + 2b_1)^2 + (-2c_1 + 2a_1 + b_{1)})^2 \\ &= (x(-2c_1 + a_1 + 2b_1))^2 + (y(-2c_1 + 2a_1 + b_1))^2 \\ &= a_2^2 + b_2^2. \end{aligned}$$

Thus, (c_2, a_2, b_2) is a Pythagorean triple. To show (c_2, a_2, b_2) is primitive, assume that each of a_2, b_2, c_2 is divisible by some prime p (note the values of x and y do not affect divisibility). Then p divides any linear combination of a_2, b_2, and c_2. In particular,

$$\begin{aligned} p|2c_2 + 3a_2 &\Rightarrow p|2(3c_1 - 2a_1 - 2b_1) + 3(-2c_1 + a_1 + 2b_1) \\ &\Rightarrow p|(6c_1 - 4a_1 - 4b_1) + (-6c_1 + 3a_1 + 6b_1) \\ &\Rightarrow p| - a_1 + 2b_1. \end{aligned}$$

Similarly,

$$p|a_2 - b_2 \Rightarrow p|(-2c_1 + a_1 + 2b_1) - (-2c_1 + 2a_1 + b_1) \Rightarrow p| - a_1 + b_1.$$

But if $p| - a_1 + 2b_1$ and $p| - a_1 + b_1$, then p divides their difference; that is, $p|(-a_1 + 2b_1) - (-a_1 + b_1)$, or $p|b_1$. A similar computation shows that $p|a_1$ and hence $p|c_1$.

This is a contradiction, since (c_1, a_1, b_1) is primitive. Therefore, (c_2, a_2, b_2) is primitive.

Now, suppose $c_2 \geq c_1$. Then

$$\begin{aligned} 3c_1 - 2a_1 - 2b_1 \geq c_1 \Rightarrow 2c_1 \geq 2a_1 + 2b_1 \Rightarrow c_1 \geq a_1 + b_1 &\Rightarrow c_1^2 \geq a_1^2 + 2a_1b_1 + b_1^2 \\ &\Rightarrow c_1^2 \geq 2a_1b_1 + c_1^2 \\ &\Rightarrow 0 \geq 2a_1b_1. \end{aligned}$$

This is a contradiction, since both a_1 and b_1 are positive. Thus, $c_2 < c_1$. □

From the above lemma, it is clear that every positive PPT equivalence class representative that appears in the tree must have a parent in the tree. We shall soon see that every such parent is unique.

Next, we want to show that there is a unique root of the PPT tree, (i.e., a unique PPT with no parent in the tree.) This will prevent the possibility of the existence of multiple PPT trees having disjoint vertex sets. Ideally, all PPTs would have a unique ancestry path back up to a singular root. We formalize this concept of a parent-less PPT in the following definition.

Definition 2. A *terminating PPT* is any PPT (γ, α, β) with the property that, for all $H \in \mathcal{H}$, the action of H^{-1} on (γ, α, β) either leaves (γ, α, β) unchanged or yields a PPT equivalence class representative that contains at least one negative integer.

For instance, $(1, 0, 1)$ is a terminating PPT since

$$H_0^{-1} \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, H_2^{-1} \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, H_3^{-1} \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

It then follows that the process of traveling from a vertex to its parent in Hall's PPT tree via multiplying the PPT by a unique H^{-1} will eventually terminate at a PPT with no parent.

The contrapositive of the last lemma allows us to conclude that $(1, 0, 1)$ is the only terminating PPT. Recall that any PPT of the form $(d, 0, c)$ is equivalent to $(1, 0, 1)$.

Corollary 1. *Given a PPT* (c_1, a_1, b_1)*, if, for every* $H \in \mathcal{H}, H^{-1} \cdot \boldsymbol{t}_1 = \boldsymbol{t}_2$ *implies that either* $c_1 = c_2$ *or* $\boldsymbol{t}_2$ *is not a PPT, then* $a_1 = 0$*. It then follows that* $(1, 0, 1)$ *is the only terminating PPT.*

Lemma 2. *The matrix* H *from Lemma 3.1 is unique.*

Proof. Given an arbitrary non-terminating PPT (c_1, a_1, b_1), consider

$$H_*^{-1} \cdot \begin{bmatrix} c_1 \\ a_1 \\ b_1 \end{bmatrix} = \begin{bmatrix} 3 & -2 & -2 \\ -2x & x & 2x \\ -2y & 2y & y \end{bmatrix} \begin{bmatrix} c_1 \\ a_1 \\ b_1 \end{bmatrix} = \begin{bmatrix} 3c_1 - 2a_1 - 2b_1 \\ x(-2c_1 + a_1 + 2b_1) \\ y(-2c_1 + 2a_1 + b_1) \end{bmatrix} = \begin{bmatrix} c_2 \\ a_2 \\ b_2 \end{bmatrix}.$$

For any real number n, we define

$$sgn(n) = \begin{cases} 0 & \text{if } n = 0 \\ \frac{n}{|n|} & \text{if } n \neq 0. \end{cases}$$

Observe that each of c_2, a_2, and b_2 are nonnegative if and only if the following three statements hold:

$$\begin{aligned} &0 \leq 3c_1 - 2a_1 - 2b_1 \\ &x = sgn(-2c_1 + a_1 + 2b_1) \\ &y = sgn(-2c_1 + 2a_1 + b_1). \end{aligned}$$

This implies that there is a unique ordered pair (x, y) such that (c_2, a_2, b_2) contains only nonnegative integers. Thus, the matrix H from Lemma 3.1 is unique. □

The above result implies that every internal vertex in the tree has a unique parent. That is, given any nonterminating PPT $\boldsymbol{t}$, the set $\left\{H_0^{-1} \cdot \boldsymbol{t}, H_2^{-1} \cdot \boldsymbol{t}, H_3^{-1} \cdot \boldsymbol{t}\right\}$ contains exactly one nonnegative PPT equivalence class representative, and exactly two PPTs with at least one negative entry.

Lemma 3. *If* (c, a, b) *is a PPT with* $(c, a, b) \neq (1, 0, 1)$*, then there exists a unique sequence of matrices* $S_1, S_2, ..., S_n$ *with each* $S_i \in \mathcal{H}$ *such that*

$$S_n^{-1} \cdot \ldots \cdot S_2^{-1} \cdot S_1^{-1} \cdot \begin{bmatrix} c \\ a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

Proof. By Lemma 3.2, given any PPT (c_1, a_1, b_1), there exists a unique $S_1 \in \mathcal{H}$ such that $S_1^{-1} \cdot (c_1, a_1, b_1) = (c_2, a_2, b_2)$, where (c_2, a_2, b_2) is a PPT and $0 < c_2 < c_1$. If $(c_2, a_2, b_2) = (1, 0, 1)$, then we are done.

If not, there exists a unique $S_2 \in \mathcal{H}$ such that $S_2^{-1} \cdot (c_2, a_2, b_2) = (c_3, a_3, b_3)$, where (c_3, a_3, b_3) is a PPT and $0 < c_3 < c_2 < c_1$. If $(c_3, a_3, b_3) = (1, 0, 1)$, then we are done.

Continuing in this manner, since the values of c_i form a strictly decreasing sequence of positive integers, this process must end after a finite number of steps at a minimal value for c_i, namely $c_i = 1$, since $(1, 0, 1)$ is the only terminating PPT. Therefore, there exists an $n \in \mathbb{N}$ such that $S_n^{-1} \cdot (c_n, a_n, b_n) = (c_{n+1}, a_{n+1}, b_{n+1}) = (1, 0, 1)$. □

Since the sequence of inverse matrices that lead a PPT to the root is unique, it follows that every PPT equivalence class appears exactly once in Hall's PPT tree. Combining the results above yields the following.

Theorem 3. *The Hall PPT tree contains every PPT equivalence class exactly once, and each representative is the unique nonnegative PPT in its equivalence class.*

In the next section, we will generalize these ideas to integer 4-tuples, and describe how to construct a complete tree of appropriately-chosen Pythagorean quadruples.

4. PPQ Background Information

Recall that a *Pythagorean quadruple* is an ordered set of four integers (d, a, b, c) that satisfy the equation $d^2 = a^2 + b^2 + c^2$, such as $(14, 6, 12, 4)$. The quadruple is said to be a *primitive Pythagorean quadruple*, or PPQ, if d, a, b and c share no common nontrivial divisors, such as $(7, 3, -6, 2)$.

We will often depict a PPQ as a column vector. That is, the ordered quadruple (d, a, b, c) and the column vector $\begin{bmatrix} d \\ a \\ b \\ c \end{bmatrix}$ represent the same PPQ.

We define an equivalence relation $\sim$ on the set of all PPQs in the following manner: given PPQs (d, a, b, c) and (d', a', b', c'), we say that

$$(d, a, b, c) \sim (d', a', b', c') \text{ if and only if } \{|d|, |a|, |b|, |c|\} = \{|d|, |a'|, |b'|, |c'|\}.$$

It then follows that any PPQ of the form $(d, 0, 0, c)$ is equivalent to $(1, 0, 0, 1)$.

Kac [5] has shown that all variations of the following 4×4 matrix (that is, all versions found by swapping and/or negating columns) can be used to generate all PPQs:

$$Q_0 = \begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}.$$

However, we have found that, among the 384 possible variations of matrix Q_0, only 7 such matrices are required to construct an infinite tree of PPQs. This set of matrices is denoted

$$\mathcal{Q} = \{Q_0, Q_2, Q_3, Q_4, Q_{23}, Q_{34}, Q_{24}\}.$$

Note that the nonzero matrix subscripts denote the columns of Q_0 that are negated. That is,

$$Q_0 = \begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}, Q_2 = \begin{bmatrix} 2 & -1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & -1 & 0 & 1 \\ 1 & -1 & 1 & 0 \end{bmatrix}, Q_3 = \begin{bmatrix} 2 & 1 & -1 & 1 \\ 1 & 0 & -1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & -1 & 0 \end{bmatrix},$$

$$Q_4 = \begin{bmatrix} 2 & 1 & 1 & -1 \\ 1 & 0 & 1 & -1 \\ 1 & 1 & 0 & -1 \\ 1 & 1 & 1 & 0 \end{bmatrix}, Q_{23} = \begin{bmatrix} 2 & -1 & -1 & 1 \\ 1 & 0 & -1 & 1 \\ 1 & -1 & 0 & 1 \\ 1 & -1 & -1 & 0 \end{bmatrix},$$

$$Q_{24} = \begin{bmatrix} 2 & -1 & 1 & -1 \\ 1 & 0 & 1 & -1 \\ 1 & -1 & 0 & -1 \\ 1 & -1 & 1 & 0 \end{bmatrix}, Q_{34} = \begin{bmatrix} 2 & 1 & -1 & -1 \\ 1 & 0 & -1 & -1 \\ 1 & 1 & 0 & -1 \\ 1 & 1 & -1 & 0 \end{bmatrix}.$$

We have found that the following four different types of PPQs are required to describe the structure of our PPQ tree.

Definition 3. The PPQ (d, a, b, c) is called a *twin PPQ* if $a = b$, that is, if the PPQ is of the form (d, a, a, c).

Definition 4. The PPQ (d, a, b, c) is called a *trivial PPQ* if $b = 0$ and hence (d, a, c) is a PPT.

For instance, both $(3, 2, 2, 1)$ and $(9, 4, 4, 7)$ are twin PPQs, whereas $(25, 24, 0, 7)$ and $(13, 12, 0, 5)$ are trivial PPQs.

It can be shown that, for any PPQ (d, a, b, c), it is a child of a trivial PPQ if and only if $d = a + c$. However, the proof of this fact is left to the curious reader. We will use this fact to justify the following definition.

Definition 5. The PPQ (d, a, b, c) is called a *child of a trivial PPQ* if $d = a + c$.

Example 4.1. For instance,

$$Q_0 \cdot \begin{bmatrix} 5 \\ 4 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ 4 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 17 \\ 8 \\ 12 \\ 9 \end{bmatrix}.$$

As a result, $(17, 8, 12, 9)$ is a child of a trivial PPQ, namely a child of $(5, 4, 0, 3)$, and notice that $17 = 8 + 9$, so it satisfies the definition.

Definition 6. The PPQ (d, a, b, c) is called an *ordinary PPQ* if it is not a twin PPQ, trivial PPQ, or a child of a trivial PPQ. Equivalently, the PPQ (d, a, b, c) is called an *ordinary PPQ* if it satisfies each of $d \neq a + c$, $a \neq b$, and $b \neq 0$.

In the next section, we will present a proof that our infinite tree of PPQs is complete. Then, we will describe the construction of our infinite PPQ tree as well as display a picture of the first few layers of the tree.

5. Constructing Our PPQ Tree

This section will begin by presenting our proof that our infinite PPQ tree is complete by mimicking the steps we took to prove that Hall's tree was complete.

Definition 7. Given any $x, y, z \in \{-1, 1\}$ with $x+y+z \geq -1$, an arbitrary element Q_* of $\mathcal{Q}$ is denoted

$$Q_* = \begin{bmatrix} 2 & x & y & z \\ 1 & 0 & y & z \\ 1 & x & 0 & z \\ 1 & x & y & 0 \end{bmatrix}, \text{ and thus } Q_*^{-1} = \begin{bmatrix} 2 & -1 & -1 & -1 \\ -x & 0 & x & x \\ -y & y & 0 & y \\ -z & z & z & 0 \end{bmatrix}.$$

Given any natural number i, for convenience we will often denote the PPQ (d_i, a_i, b_i, c_i) as the vector $\boldsymbol{q}_i$.

Lemma 4. *Given any PPQ (d_1, a_1, b_1, c_1), if $a_1 \neq 0$, there exists a $Q \in \mathcal{Q}$ such that $Q^{-1} \cdot \boldsymbol{q}_1 = \boldsymbol{q}_2$, where $\boldsymbol{q}_2$ is a PPQ and $d_2 < d_1$.*

Proof. Let Q_*^{-1} act on an arbitrary PPQ (d_1, a_1, b_1, c_1) with $a_1 \neq 0$:

$$\begin{bmatrix} 2 & -1 & -1 & -1 \\ -x & 0 & x & x \\ -y & y & 0 & y \\ -z & z & z & 0 \end{bmatrix} \cdot \begin{bmatrix} d_1 \\ a_1 \\ b_1 \\ c_1 \end{bmatrix} = \begin{bmatrix} 2d_1 - a_1 - b_1 - c_1 \\ x(-d_1 + b_1 + c_1) \\ y(-d_1 + a_1 + c_1) \\ z(-d_1 + a_1 + b_1) \end{bmatrix} = \begin{bmatrix} d_2 \\ a_2 \\ b_2 \\ c_2 \end{bmatrix}.$$

Note that $x^2 = y^2 = z^2 = 1$. Using the fact that $d_1^2 = a_1^2 + b_1^2 + c_1^2$, it follows that

$$\begin{aligned}
d_2^2 &= (2d_1 - a_1 - b_1 - c_1)^2 \\
&= 4d_1^2 + a_1^2 + b_1^2 + c_1^2 + 2a_1b_1 + 2a_1c_1 + 2b_1c_1 - 4a_1d_1 - 4b_1d_1 - 4c_1d_1 \\
&= (3d_1^2 + d_1^2) + a_1^2 + b_1^2 + c_1^2 + 2a_1b_1 + 2a_1c_1 + 2b_1c_1 - 4a_1d_1 - 4b_1d_1 - 4c_1d_1 \\
&= (3d_1^2 + a_1^2 + b_1^2 + c_1^2) + a_1^2 + b_1^2 + c_1^2 + 2a_1b_1 + 2a_1c_1 + 2b_1c_1 - 4a_1d_1 - 4b_1d_1 - 4c_1d_1 \\
&= 3d_1^2 + 2a_1^2 + 2b_1^2 + 2c_1^2 + 2a_1b_1 + 2a_1c_1 + 2b_1c_1 - 4a_1d_1 - 4b_1d_1 - 4c_1d_1 \\
&= (d_1^2 + b_1^2 + c_1^2 + 2b_1c_1 - 2b_1d_1 - 2c_1d_1) \\
&\quad + (d_1^2 + a_1^2 + c_1^2 + 2a_1c_1 - 2a_1d_1 - 2c_1d_1) \\
&\quad + (d_1^2 + a_1^2 + b_1^2 + 2a_1b_1 - 2a_1d_1 - 2b_1d_1) \\
&= (x(-d_1 + b_1 + c_1))^2 + (y(-d_1 + a_1 + c_1))^2 + (z(-d_1 + a_1 + b_1))^2 \\
&= a_2^2 + b_2^2 + c_2^2.
\end{aligned}$$

Thus (d_2, a_2, b_2, c_2) is a Pythagorean quadruple. Showing (d_2, a_2, b_2, c_2) is primitive is similar to the PPT case and is left to the curious reader.

Now, suppose $d_2 \geq d_1$. Then

$$\begin{aligned}
2d_1 - a_1 - b_1 - c_1 \geq d_1 &\Rightarrow d_1 \geq a_1 + b_1 + c_1 \\
&\Rightarrow d_1^2 \geq a_1^2 + b_1^2 + c_1^2 + 2a_1b_1 + 2a_1c_1 + 2b_1c_1 \\
&\Rightarrow d_1^2 \geq d_1^2 + 2a_1b_1 + 2a_1c_1 + 2b_1c_1 \\
&\Rightarrow 0 \geq 2a_1b_1 + 2a_1c_1 + 2b_1c_1.
\end{aligned}$$

This is a contradiction, since a_1, b_1, and c_1 are all positive. Thus, $d_2 < d_1$. □

Next, just as in our discussion on the structure of the tree of PPTs, we want to show that there is a unique "terminating PPQ" in our PPQ tree. It will then follow that all PPQs in the tree are descendants of that PPQ, and hence it is a root.

Definition 8. A *terminating PPQ* is any PPQ $(\delta, \alpha, \beta, \gamma)$ with the property that for all $Q \in \mathcal{Q}$, the action of Q^{-1} on $(\delta, \alpha, \beta, \gamma)$ either leaves $(\delta, \alpha, \beta, \gamma)$ unchanged or yields a PPQ equivalence class representative that contains at least one negative integer.

For instance, $(1, 0, 0, 1)$ is a terminating PPQ. This fact can be verified quickly and is left to the reader.

Similar to the PPT case, the contrapositive of the above lemma implies that $(1, 0, 0, 1)$ is the only terminating PPQ.

Corollary 2. *Given a PPQ (d_1, a_1, b_1, c_1), if, for every $Q \in \mathcal{Q}, Q^{-1} \cdot \boldsymbol{q}_1 = \boldsymbol{q}_2$, either $d_1 = d_2$ or $\boldsymbol{q}_2$ is not a PPQ, then $a_1 = 0$. It then follows that $(1, 0, 0, 1)$ is the only terminating PPQ.*

The above corollary is true because of our earlier definition that the PPQ equivalence class representatives with at least one zero entry are of the form $(d, a, 0, c)$—so if $a = 0$, the PPQ is of the form $(d, 0, 0, c)$, and must therefore be $(1, 0, 0, 1)$.

Lemma 5. *Given an arbitrary PPQ* (d_1, a_1, b_1, c_1)*, for all* $Q \in \mathcal{Q}$*,* $Q \cdot \boldsymbol{q}_1 = \boldsymbol{q}_2$ *where* (d_2, a_2, b_2, c_2) *is a PPQ.*

Proof. By Lemma 3.1, we know that:

$$Q^{-1} \cdot \boldsymbol{q}_1 = \boldsymbol{q}_2 \Rightarrow Q \cdot Q^{-1} \cdot \boldsymbol{q}_1 = Q \cdot \boldsymbol{q}_2 \Rightarrow I_4 \cdot \boldsymbol{q}_1 = Q \cdot \boldsymbol{q}_2 \Rightarrow \boldsymbol{q}_1 = Q \cdot \boldsymbol{q}_2$$

Therefore, multiplying an arbitrary PPQ by any element of $\mathcal{Q}$ always results in a PPQ. □

Lemma 6. *The matrix Q from Lemma 5.1 is unique.*

Proof. Given an arbitrary PPQ $(d_1, a_1, b_1, c_1) \neq (1, 0, 0, 1)$, consider

$$Q_*^{-1} \cdot \begin{bmatrix} d_1 \\ a_1 \\ b_1 \\ c_1 \end{bmatrix} = \begin{bmatrix} 2 & -1 & -1 & -1 \\ -x & 0 & x & x \\ -y & y & 0 & y \\ -z & z & z & 0 \end{bmatrix} \begin{bmatrix} d_1 \\ a_1 \\ b_1 \\ c_1 \end{bmatrix} = \begin{bmatrix} 2d_1 - a_1 - b_1 - c_1 \\ x(-d_1 + b_1 + c_1) \\ y(-d_1 + a_1 + c_1) \\ z(-d_1 + a_1 + b_1) \end{bmatrix} = \begin{bmatrix} d_2 \\ a_2 \\ b_2 \\ c_2 \end{bmatrix}.$$

For any real number n, we define

$$sgn(n) = \begin{cases} 0 & \text{if } n = 0 \\ \frac{n}{|n|} & \text{if } n \neq 0. \end{cases}$$

Observe that each of d_2, a_2, b_2, and c_2 are nonnegative if and only if the following four statements hold:

$$0 \leq 2d_1 - a_1 - b_1 - c_1; \quad x = sgn(-d_1 + b_1 + c_1);$$
$$y = sgn(-d_1 + a_1 + c_1); \quad z = sgn(-d_1 + a_1 + b_1).$$

This implies that there is a unique ordered triple (x, y, z) such that (d_2, a_2, b_2, c_2) contains only nonnegative integers. Thus, the matrix Q from Lemma 5.1 is unique. □

Lemma 7. *If* (d, a, b, c) *is a PPT and* $(d, a, b, c) \neq (1, 0, 0, 1)$*, then there exists a unique sequence of matrices* $T_1, T_2, ..., T_n$ *with each* $T_i \in \mathcal{Q}$ *such that*

$$T_n^{-1} \cdot \ldots \cdot T_2^{-1} \cdot T_1^{-1} \cdot \begin{bmatrix} d \\ a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Proof. By Lemma 5.3 given any PPQ (d_1, a_1, b_1, c_1), there exists a unique $T_1 \in \mathcal{Q}$ such that $T_1^{-1} \cdot (d_1, a_1, b_1, c_1) = (d_2, a_2, b_2, c_2)$, where (d_2, a_2, b_2, c_2) is a PPQ and $0 < d_2 < d_1$. If $(d_2, a_2, b_2, c_2) = (1, 0, 0, 1)$, then we are done.

If not, there exists a unique $T_2 \in \mathcal{Q}$ such that $T_2^{-1} \cdot (d_2, a_2, b_2, c_2) = (d_3, a_3, b_3, c_3)$, where (d_3, a_3, b_3, c_3) is a PPQ and $0 < d_3 < d_2 < d_1$. If $(d_3, a_3, b_3, c_3) = (1, 0, 0, 1)$, then we are done.

Continuing in this manner, since the values of d_i form a strictly decreasing sequence of positive integers, this process must eventually end at a minimal value for d_n, namely $d_n = 1$, since $(1, 0, 0, 1)$ is the only terminating PPQ. Therefore, there exists an $n \in \mathbb{N}$ such that $T_n^{-1} \cdot (d_n, a_n, b_n, c_n) = (d_{n+1}, a_{n+1}, b_{n+1}, c_{n+1}) = (1, 0, 0, 1)$. □

By the above, a complete tree of nonnegative PPQ equivalence class representative exists, but it is still unclear as to what this tree would look like. For the remainder of this section, we will present our construction of a complete PPQ tree.

Recall that a perfect ternary tree of all PPTs is generated by the elements of $\mathcal{H}$ applied to $(1, 0, 1)$. Said differently, every nontrivial PPT can be multiplied by all three matrices in $\mathcal{H}$. However, this is not the case with PPQs; there are restrictions on which PPQs can be multiplied by a given element of $\mathcal{Q}$. These restrictions will depend on the type of PPQ, as shown in the following examples.

Example 5.1. Given the PPQ $(11, 2, 6, 9)$, note that

$$Q_{34} \cdot \begin{bmatrix} 11 \\ 2 \\ 6 \\ 9 \end{bmatrix} = \begin{bmatrix} 9 \\ -4 \\ 4 \\ 7 \end{bmatrix}.$$

That is, $Q_{34} \cdot (11, 2, 6, 9)$ is not a nonnegative PPQ representative, so in order to include only nonnegative PPQ representatives in our PPQ tree we cannot multiply $(11, 2, 6, 9)$ by all $Q \in \mathcal{Q}$.

Example 5.2. Consider the children of the PPQ $(9, 4, 4, 7)$ generated by Q_2 and Q_3:

$$Q_2 \cdot \begin{bmatrix} 9 \\ 4 \\ 4 \\ 7 \end{bmatrix} = \begin{bmatrix} 25 \\ 20 \\ 12 \\ 9 \end{bmatrix}, \; Q_3 \cdot \begin{bmatrix} 9 \\ 4 \\ 4 \\ 7 \end{bmatrix} = \begin{bmatrix} 25 \\ 12 \\ 20 \\ 9 \end{bmatrix}.$$

Note that $Q_2 \cdot (9, 4, 4, 7)$ and $Q_3 \cdot (9, 4, 4, 7)$ belong to the same PPQ equivalence class, so in order to preserve uniqueness of vertices in our PPQ tree, $(9, 4, 4, 7)$ cannot be multiplied by both Q_2 and Q_3. A similar thing happens for all twin PPQs.

We are now in a position to establish the "rules" that guarantee our PPQ tree will contain each nonnegative equivalence class representative exactly once. Let $\{M_0, M_2, M_3, M_4, M_{23}, M_{24}, M_{34}\}$ be a family of sets of matrices defined as follows:

- $M_0 = \mathcal{Q} \setminus \{Q_{23}, Q_{24}, Q_{34}\} = \{Q_0, Q_2, Q_3, Q_4\}$,
- $M_2 = \mathcal{Q} \setminus \{Q_{23}, Q_{24}\} = \{Q_0, Q_2, Q_3, Q_4, Q_{34}\}$,
- $M_3 = \mathcal{Q} \setminus \{Q_{23}, Q_{34}\} = \{Q_0, Q_2, Q_3, Q_4, Q_{24}\}$,
- $M_4 = \mathcal{Q} \setminus \{Q_{24}, Q_{34}\} = \{Q_0, Q_2, Q_3, Q_4, Q_{23}\}$,
- $M_{23} = \mathcal{Q} \setminus \{Q_{23}\} = \{Q_0, Q_2, Q_3, Q_4, Q_{24}, Q_{34}\}$,
- $M_{24} = \mathcal{Q} \setminus \{Q_{24}\} = \{Q_0, Q_2, Q_3, Q_4, Q_{23}, Q_{34}\}$,
- $M_{34} = \mathcal{Q} \setminus \{Q_{34}\} = \{Q_0, Q_2, Q_3, Q_4, Q_{23}, Q_{24}\}$.

Let $\boldsymbol{q} = (d, a, b, c)$ denote a nonnegative PPQ equivalence class representative distinct from $(1, 0, 0, 1)$. The rules for generating the children of $\boldsymbol{q}$ in our tree are as follows.

- If $\boldsymbol{q}$ is an ordinary PPQ generated by the matrix $Q_{ij} \in \mathcal{Q}$, multiply $\boldsymbol{q}$ by each element in M_{ij}.
- If $\boldsymbol{q}$ is a twin PPQ generated by the matrix $Q_{ij} \in \mathcal{Q}$, multiply $\boldsymbol{q}$ by each element in $M_{ij} \setminus \{Q_3, Q_{34}\}$.
- If $\boldsymbol{q}$ is a trivial PPQ generated by the matrix $Q_{ij} \in \mathcal{Q}$, multiply $\boldsymbol{q}$ by each element in $M_{ij} \setminus \{Q_3, Q_{23}, Q_{24}, Q_{34}\}$.
- If $\boldsymbol{q}$ is a child of a trivial PPQ generated by the matrix $Q_{ij} \in \mathcal{Q}$, multiply $\boldsymbol{q}$ by each element in $M_{ij} \cup \{Q_{24}\}$.

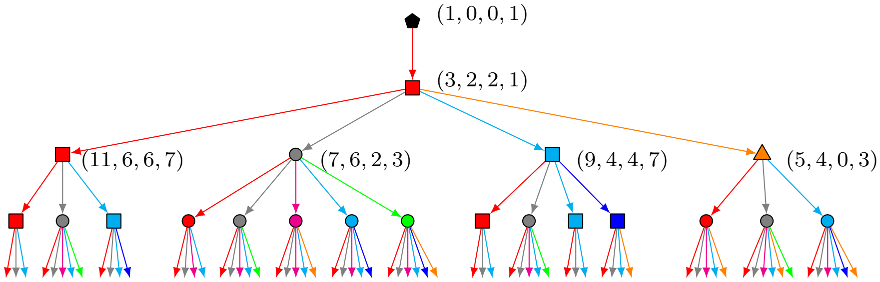
Theorem 4. *A complete PPQ tree, containing each PPQ equivalence class exactly once, is generated by the rules above, and each representative in the tree is nonnegative.*

Our complete tree of all PPQ equivalence classes appears in Figure 3.

In what follows, we will verify five variations of the construction rules listed above; the remaining proofs use similar techniques and are left to the curious reader.

Proof ($Q_{23} \cdot Q_{23}$ Exception). Let $\boldsymbol{q} = (d, a, b, c)$ denote an arbitrary ordinary PPQ generated by Q_{23}. Note that

$$Q_{23} \cdot Q_{23} \cdot \begin{bmatrix} d \\ a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 3 & -2 & -2 & 0 \\ 2 & -1 & -2 & 0 \\ 2 & -2 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} d \\ a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 3d - 2a - 2b \\ 2d - a - 2b \\ 2d - 2a - b \\ -c \end{bmatrix}.$$



Edge Key: Red=Q_0, Gray=Q_2, Pink=Q_3, Cyan=Q_4, Blue=Q_{23}, Orange=Q_{24}, Green=Q_{34}

Vertex Key: Squares denote *twin PPQs* with $a = b$, such as $(9, 4, 4, 7)$. Triangles denote *trivial PPQs* with $b = 0$, like $(5, 4, 0, 3)$. The root, $(1, 0, 0, 1)$, is the only PPQ that is both a twin PPQ and a trivial PPQ.

Figure 3: A Complete PPQ Tree

It follows that $-c < 0$, and so $Q_{23} \cdot Q_{23} \cdot \boldsymbol{q}$ is an equivalence class representative that contains a negative entry. Hence, in order for $\boldsymbol{q}$ to be a PPQ, $Q_{23} \cdot Q_{23} \cdot \boldsymbol{q}$ must contain a negative entry and cannot be included in our tree. Therefore, if a PPQ was generated by Q_{23}, then the resulting PPQ will not be multiplied by Q_{23}. □

Proof ($Q_4 \cdot Q_{34}$ Exception). Consider an arbitrary ordinary PPQ $\boldsymbol{q} = (d, a, b, c)$. Note that

$$Q_{34}^{-1} \cdot Q_4^{-1} \cdot \begin{bmatrix} d \\ a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 3 & 0 & -2 & -2 \\ 0 & -1 & 0 & 0 \\ -2 & 0 & 1 & 2 \\ 2 & 0 & -2 & -1 \end{bmatrix} \cdot \begin{bmatrix} d \\ a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 3d - 2b - 2c \\ -a \\ 2d + b + 2b \\ 2d - 2b - c \end{bmatrix}.$$

It follows that $-a < 0$, and so $Q_{34}^{-1} \cdot Q_4^{-1} \cdot \boldsymbol{q}$ is an equivalence class representative that contains a negative entry. Hence, in order for $\boldsymbol{q}$ to be a PPQ, $Q_{34}^{-1} \cdot Q_4^{-1} \cdot \boldsymbol{q}$ must contain a negative entry and cannot be included in our tree. Therefore, if a PPQ was generated by Q_4, then the resulting PPQ will not be multiplied by Q_{34}. □

Proof (Twin PPQ Exception). Let (d, a, a, c) be a twin PPQ, and let W be a function that satisfies

$$W(x, y, z) = \begin{bmatrix} 2 & x & y & z \\ 1 & 0 & y & z \\ 1 & x & 0 & z \\ 1 & x & y & 0 \end{bmatrix} \cdot \begin{bmatrix} d \\ a \\ a \\ c \end{bmatrix} = \begin{bmatrix} 2d + a(x + y) + cz \\ d + ay + cz \\ d + ax + cz \\ d + a(x + y) \end{bmatrix}.$$

for any $x, y, z \in \{-1, 1\}$.

Note that for any value of z, $W(-1,1,z)$ and $W(1,-1,z)$ represent the same PPQ equivalence class, as they have the same entries (e.g.: $(25,20,12,9)$ and $(25,12,20,9)$). Thus, $W(-1,1,z) \sim W(1,-1,z)$. Hence, multiplying a twin PPQ by Q_2 is equivalent to multiplying by Q_3, and multiplying by Q_{24} is equivalent to multiplying by Q_{34}. Consequently, we ignore the children of twin PPQs generated by Q_3 and Q_{34}, because these children are already generated by Q_2 and Q_{23} and we only want each equivalence class representative in our tree once. □

Proof (Trivial PPQ Exception). Let $(d,a,0,c)$ be a trivial PPQ, and let R be a function that satisfies

$$R(x,y,z) = \begin{bmatrix} 2 & x & y & z \\ 1 & 0 & y & z \\ 1 & x & 0 & z \\ 1 & x & y & 0 \end{bmatrix} \cdot \begin{bmatrix} d \\ a \\ 0 \\ c \end{bmatrix} = \begin{bmatrix} 2d+ax+cz \\ d+cz \\ d+ax+cz \\ d+ax \end{bmatrix},$$

for any $x,y,z \in \{-1,1\}$

Note that, for all x and z, $R(x,-1,z) \sim R(x,1,z)$, since there are no y's in the output of R. So for a trivial PPQ, multiplying by Q_0 is equivalent to multiplying by Q_3, multiplying by Q_2 is equivalent to multiplying by Q_{23}, and multiplying by Q_4 is equivalent to multiplying by Q_{34}. Consequently, we ignore the children of trivial PPQs generated by Q_3, Q_{23}, and Q_{34}; doing so forces each equivalence class representative to appear exactly once in our tree. We also never multiply a trivial PPQ by Q_{24} because all trivial PPQs are generated by Q_{24}, and we do not want to multiply its parent by $Q_{24} \cdot Q_{24}$. □

Proof (Child of a Trivial PPQ Exception). Let (d,a,b,c) be a PPQ, and define (d_2,a_2,b_2,c_2) as follows

$$Q_{24} \cdot Q_* \cdot \begin{bmatrix} d \\ a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 3d+2ax+2cz \\ 2d+ax+2cz \\ -yb \\ 2d+2ax+cz \end{bmatrix} = \begin{bmatrix} d_2 \\ a_2 \\ b_2 \\ c_2 \end{bmatrix}.$$

Note that $Q_* \neq Q_{24}$ because (see rules.)

Assume $y=1$ and $b>0$. Then $\boldsymbol{q}_2$ contains a negative entry and therefore cannot be a nonnegative PPQ equivalence class representative.

Assume $y=-1$ and $b>0$. Then $Q_* \in \{Q_3, Q_{23}, Q_{34}\}$, and for any $Q \in \{Q_3, Q_{23}, Q_{34}\}$, one can multiply by Q then by Q_{24} because (see rules.) So $\boldsymbol{q}_2$ is a nonnegative PPQ equivalence class representative.

Assume $b=0$. Then $b_2=0$ and the value of $\boldsymbol{q}_2$ does not depend on y, so y could be either -1 or 1. So

$$Q_{24} \cdot Q_3 \cdot \boldsymbol{q} = Q_{24} \cdot Q_0 \cdot \boldsymbol{q}; \quad Q_{24} \cdot Q_{23} \cdot \boldsymbol{q} = Q_{24} \cdot Q_2 \cdot \boldsymbol{q}; \quad Q_{24} \cdot Q_{34} \cdot \boldsymbol{q} = Q_{24} \cdot Q_4 \cdot \boldsymbol{q}.$$

In addition, for any $Q \in \{Q_3, Q_{23}, Q_{34}\}$, one can multiply by Q then by Q_{24} because (see rules.) So for all $Q` \neq Q_{24}$, $\boldsymbol{q}_2 = Q_{24} \cdot Q` \cdot \boldsymbol{q}$ is a nonnegative PPQ equivalence class representative.

So, if $\boldsymbol{q}$ is not trivial, the multiplication of its children is described by the rules. Otherwise, if $\boldsymbol{q}$ is trivial, then its children can be multiplied by Q_{24} as well and remain in the tree of nonnegative PPQ equivalence class representatives. □

We will now briefly define and prove two interesting results concerning the generation of trivial and twin PPQs in our tree.

Theorem 5. *Any nonterminating trivial PPQ is generated by* Q_{24}.

Proof. Let $(d, a, 0, c)$ be a nonterminating PPQ. Then there exists a unique parent of $(d, a, 0, c)$ in our tree, namely the PPQ

$$Q_* \cdot \begin{bmatrix} d \\ a \\ 0 \\ c \end{bmatrix} = \begin{bmatrix} 2d - a - c \\ x(-d + c) \\ y(-d + a + c) \\ z(-d + a) \end{bmatrix} = \begin{bmatrix} d_2 \\ a_2 \\ b_2 \\ c_2 \end{bmatrix}.$$

Since $\boldsymbol{q}_2$ is a PPQ, all of its entries are nonnegative. So $x(-d + c) \geq 0$ and $z(-d + a) \geq 0$. It follows that

$$x(-d + c) \geq 0 \Rightarrow -dx + cx \geq 0 \Rightarrow dx \leq cx \Rightarrow x = -1,$$

since $d \geq c$. A similar process with $z(-d + a) \geq 0$ implies that $z = -1$. Since there is no matrix in $\mathcal{Q}$ with $x, y, z = -1$, it must follow that $y = 1$. Thus, in order for $\boldsymbol{q}_2$ to be a PPQ, it must be true that $Q_* = Q_{24}$. □

Theorem 6. *Any nonterminating twin PPQ is a child of another twin PPQ.*

Proof. Consider an arbitrary twin PPQ. Then there exists a unique parent of (d, a, a, c) namely the PPQ

$$\begin{bmatrix} 2 & -1 & -1 & -1 \\ -x & 0 & x & x \\ -y & y & 0 & y \\ -z & z & z & 0 \end{bmatrix} \begin{bmatrix} d \\ a \\ a \\ c \end{bmatrix} = \begin{bmatrix} 2d - a - a - c \\ x(-d + a + c) \\ y(-d + a + c) \\ z(-d + 2a) \end{bmatrix} = \begin{bmatrix} d_2 \\ a_2 \\ b_2 \\ c_2 \end{bmatrix}.$$

Since $\boldsymbol{q}_2$ is a PPQ, its entries must be nonnegative. Note that when $y = x$, it follows that $a_2 = b_2$, and so, by definition, $\boldsymbol{q}_2$ is a twin PPQ.

Suppose $y \neq x$, and, without loss of generality, let $x = 1$ and $y = -1$. Since the entries of $\boldsymbol{q}_2$ must be nonnegative, $x(-d + a + c) \geq 0$ and $y(-d + a + c) \geq 0$. These imply that $d \geq a + c$ and $d \leq a + c$. So $d = a + c$. But if $d = a + c$ then $a_2 = x(-d + a + c) = 0$, and similarly, $b_2 = 0$. So $\boldsymbol{q}_2 = (1, 0, 0, 1)$. As a result, (d_2, a_2, b_2, c_2) is a twin PPQ.

By contraposition of the above, if $\boldsymbol{q}_2 \neq (1,0,0,1)$, then $y = x$. So by the above, $a_2 = b_2$, and thus $\boldsymbol{q}_2$ is a twin PPQ. Therefore, any nonterminating twin PPQ is always a child of another twin PPQ. □

Now we will briefly touch on the use of Carmichael's formula for the PPQ equivalence class representatives in our tree.

Given an arbitrary PPQ equivalence class representative $\boldsymbol{q} = (d,a,b,c)$, let $C(\boldsymbol{q})$ be the set of all integral 4-tuples (m,n,p,q) such that $(d,a,b,c) = (m^2+n^2+p^2+q^2, 2mq+2np, 2nq-2mp, m^2+n^2-p^2-q^2)$. This comes from Carmichael's formula in Theorem 1.2, but includes all integral (m,n,p,q), even the 4-tuples with negative entries. We have noticed that some representatives $\boldsymbol{q}$ in a given equivalence class have the corresponding set $C(\boldsymbol{q})$ that contains no 4-tuples that are strictly nonnegative, such as $(25,12,16,15)$. However, the set C of another representative in the equivalence class of $(25,12,16,15)$ does have a 4-tuple with strictly nonnegative entries, namely $(25,16,12,15)$. But the latter does not appear in our construction of the tree—so, if we wanted to use Carmichael's formula to generate all of the PPQ equivalence class representatives in our tree, we would need to include negative entries in some of the generating 4-tuples (m,n,p,q).

One way we can see this is in the following theorem. But first, let us introduce some notation—given an integral 4-tuple (m,n,p,q), we will say that a corresponding PPQ equivalence class representative (d,a,b,c) is *Carmichael-generated by* (m,n,p,q) if

$$\begin{bmatrix} m^2+n^2+p^2+q^2 \\ 2mq+2np \\ 2nq-2mp \\ m^2+n^2-p^2-q^2 \end{bmatrix} = \begin{bmatrix} d \\ a \\ b \\ c \end{bmatrix}.$$

And if (d,a,b,c) is Carmichael-generated by (m,n,p,q), we will denote that $\langle m,n,p,q\rangle = (d,a,b,c)$.

Theorem 7. *If (d,a,b,c) is a PPQ equivalence class representative Carmichael-generated by (m,n,p,q), then (d,b,a,c) is Carmichael-generated by $(n,m,-p,q)$ and $(-m,n,q,p)$.*

Proof. Let (d,a,b,c) be a PPQ equivalence class representative Carmichael-generated by (m,n,p,q). Then

$$\begin{bmatrix} d \\ a \\ b \\ c \end{bmatrix} = \langle m,n,p,q\rangle = \begin{bmatrix} m^2+n^2+p^2+q^2 \\ 2mq+2np \\ 2nq-2mp \\ m^2+n^2-p^2-q^2 \end{bmatrix} \sim \begin{bmatrix} m^2+n^2+p^2+q^2 \\ 2nq-2mp \\ 2mq+2np \\ m^2+n^2-p^2-q^2 \end{bmatrix} = \begin{bmatrix} d \\ b \\ a \\ c \end{bmatrix}.$$

Note that

$$\begin{bmatrix} d \\ b \\ a \\ c \end{bmatrix} = \begin{bmatrix} m^2+n^2+p^2+q^2 \\ 2nq-2mp \\ 2mq+2np \\ m^2+n^2-p^2-q^2 \end{bmatrix} = \begin{bmatrix} n^2+m^2+(-p)^2+q^2 \\ 2nq+2m(-p) \\ 2mq-2n(-p) \\ n^2+m^2-(-p)^2-q^2 \end{bmatrix} = \langle n, m, -p, q\rangle,$$

and

$$\begin{bmatrix} d \\ b \\ a \\ c \end{bmatrix} = \begin{bmatrix} m^2+n^2+p^2+q^2 \\ 2nq-2mp \\ 2mq+2np \\ m^2+n^2-p^2-q^2 \end{bmatrix} = \begin{bmatrix} (-m)^2+n^2+q^2+p^2 \\ 2(-m)p+2nq \\ 2np-2(-m)q \\ (-m)^2+n^2-q^2-p^2 \end{bmatrix} = \langle -m, n, q, p\rangle.$$

□

Going back to our earlier example of $(25, 16, 12, 15)$, we see that $(25, 16, 12, 15) = \langle 2, 4, 1, 2\rangle$. By the above theorem, that means that the other PPQ equivalence class representative $(25, 12, 16, 15) = \langle 4, 2, -1, 2\rangle = \langle -2, 4, 2, 1\rangle$.

6. Interesting PPQ Subtrees

The PPQ tree constructed in the last section contains several interesting subtrees. In what follows, for brevity we will omit the root $(1, 0, 0, 1)$. The first subtree of note consists of all trivial PPQs, since every trivial PPQ has exactly three grandchildren that are trivial PPQs, and every trivial PPQ is the grandchild of another trivial PPQ. More precisely, given a trivial PPQ $(d, a, 0, c)$, its three trivial PPQ grandchildren are given by the following:

$$Q_{24} \cdot Q_0 \cdot \begin{bmatrix} d \\ a \\ 0 \\ c \end{bmatrix} = \begin{bmatrix} 3 & 2 & 0 & 2 \\ 2 & 1 & 0 & 2 \\ 0 & 0 & -1 & 0 \\ 2 & 2 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} d \\ a \\ 0 \\ c \end{bmatrix} = \begin{bmatrix} 3d+2a+2c \\ 2d+a+2c \\ 0 \\ 2d+2a+c \end{bmatrix},$$

$$Q_{24} \cdot Q_2 \cdot \begin{bmatrix} d \\ a \\ 0 \\ c \end{bmatrix} = \begin{bmatrix} 3 & -2 & 0 & 2 \\ 2 & -1 & 0 & 2 \\ 0 & 0 & -1 & 0 \\ 2 & -2 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} d \\ a \\ 0 \\ c \end{bmatrix} = \begin{bmatrix} 3d-2a+2c \\ 2d-a+2c \\ 0 \\ 2d-2a+c \end{bmatrix},$$

and

$$Q_{24} \cdot Q_4 \cdot \begin{bmatrix} d \\ a \\ 0 \\ c \end{bmatrix} = \begin{bmatrix} 3 & 2 & 0 & -2 \\ 2 & 1 & 0 & -2 \\ 0 & 0 & -1 & 0 \\ 2 & 2 & 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} d \\ a \\ 0 \\ c \end{bmatrix} = \begin{bmatrix} 3d+2a-2c \\ 2d+a-2c \\ 0 \\ 2d+2a-c \end{bmatrix},$$

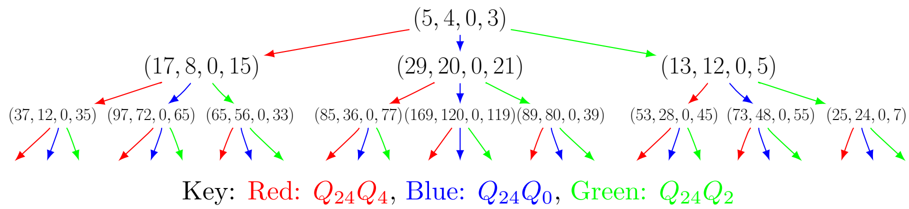


Figure 4: The subtree of all trivial PPQs

which are identical to the children of a PPT (d, a, c) generated by the set of matrices $\mathcal{H} = \{H_0, H_2, H_3\}$, but as quadruples with a zero entry inserted.

The subtree of all nonterminating trivial PPQs appears in Figure 4.

Another interesting subtree contains all nonterminating children of trivial PPQs—that is, all PPQs of the form (d, a, b, c) where $d = a + c$. Note that this perfect ternary tree appears "between" the trivial subtree.

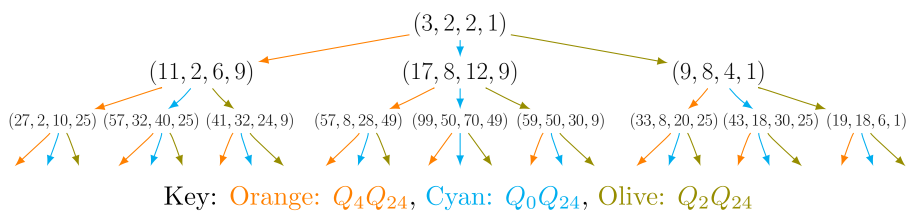


Figure 5: The subtree of all children of trivial PPQs

In what follows, the coloring of edges will be identical to the complete PPQ tree that appears in Figure 3.

The subtree containing all twin PPQs, on the other hand, is not a perfect tree, but it does have a beautiful structure. The first few rows can be seen below in Figure 6.

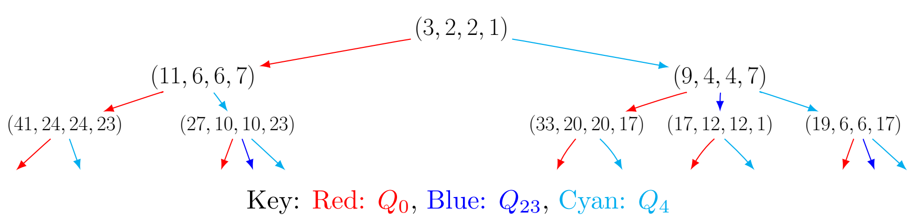


Figure 6: The subtree of all twin PPQs

It should be noted that the perfect binary subtree of all twin PPQs generated by $\{Q_0, Q_4\}$ does not contain all twin PPQs, so we need to consider the matrix Q_{23}

as well. It was shown previously that twin PPQs are only children of another twin PPQ, and it can be shown that they are only generated by one of $\{Q_0, Q_4, Q_{23}\}$.

More precisely, the subtree containing all twin PPQs is constructed as follows: let $\boldsymbol{q} = (d, a, a, c)$ denote a nonterminating twin PPQ. Then:

- if $\boldsymbol{q}$ is generated by Q_0 or Q_{23}, multiply $\boldsymbol{q}$ by each element in $\{Q_0, Q_4\}$;
- if $\boldsymbol{q}$ is generated by Q_4, multiply $\boldsymbol{q}$ by each element in $\{Q_0, Q_4, Q_{23}\}$.

One interesting feature of the subtree of all such twin PPQs is that the number of PPQs in the n^{th} level of the subtree is given by P_n, the n^{th} Pell number. Pell numbers are defined recursively by $P_{n+1} = 2P_n + P_{n-1}$, with $P_0 = 0$, $P_1 = 1$. This connection is due to the following facts.

1. In any given row, each PPQ is multiplied by both Q_0 and Q_4.
2. For each PPQ generated by Q_4, one also multiplies by Q_{23}.
3. The number of PPQs generated by Q_4 in a given row is identical, via (1), to the number of PPQs in the previous row.

In a similar manner, the n^{th} row of the subtree of twin PPQs generated by just $\{Q_4, Q_{23}\}$ contains F_n twin PPQs, where F_n denotes the n^{th} Fibonacci number.

Two other interesting subtrees of PPQs are those generated by $\{Q_0, Q_2, Q_4\}$ and $\{Q_0, Q_3, Q_4\}$. Each is a perfect ternary tree and contains the same PPQ equivalence classes, but they contain different sets of nonnegative equivalence class representatives. These trees can be seen in Figures 7 and 8.

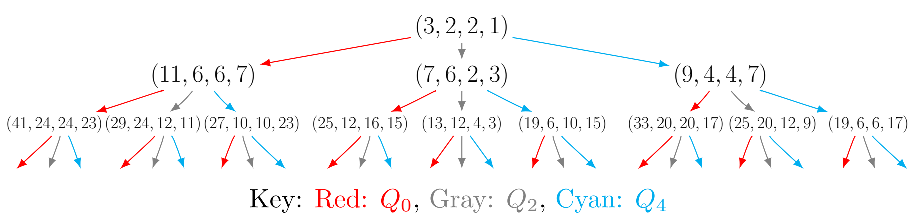


Figure 7: The subtree generated by $\{Q_0, Q_2, Q_4\}$

Notice that, for an arbitrary PPQ (d, a, b, c) in Figure 6, the PPQ in the same "place" in Figure 7 has entries (d, b, a, c). Also notice that the tree generated by $\{Q_0, Q_3, Q_4\}$ is not actually a subtree of our PPQ tree, since, for example, we do not multiply $(3, 2, 2, 1)$ by Q_3.

Another intriguing subset of the PPQ tree consists of all twin PPQs of the form $(d, a, a, 1)$. These particular 4-tuples are solutions to $d^2 - 2a^2 = 1$, a specific version of Pell's equation (not to be confused with the Pell numbers). Consequently, these 4-tuples have the property that the corresponding ratios $\frac{d}{a}$ are rational approximations

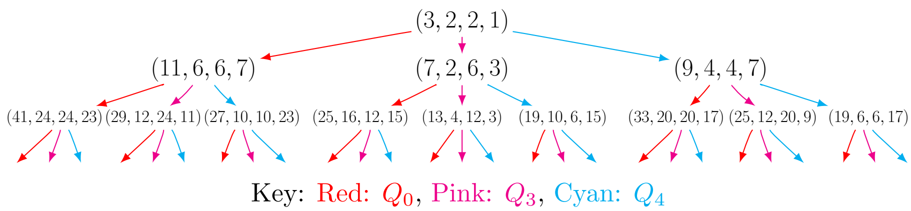


Figure 8: The tree generated by $\{Q_0, Q_3, Q_4\}$

to $\sqrt{2}$, and, in fact, these are the very approximations generated by the simple continued fraction algorithm. It is easy to show that all such 4-tuples are generated by powers of $Q_{23} \cdot Q_4$, applied to $(3, 2, 2, 1)$. So, this "subtree" is simply a subset of the vertices belonging to one branch of infinite length inside of the PPQ tree.

Open Problem 1. We encourage the curious reader to look for other interesting subtrees in our complete PPQ tree. We also encourage brave readers to attempt to build a complete tree of all solutions to the quintary version of the Pythagorean Theorem, namely $h^2 = a^2 + b^2 + c^2 + d^2$. Be aware that there are $2^5 \cdot 5! = 3840$ variations of an order 5 matrix from which to choose that generate all primitive Pythagorean quintuples, so this problem may require a great deal of patience and stamina.

Open Problem 2. Construct an algorithm that determines the location of a specific PPQ in our tree.

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