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A BIJECTION BETWEEN THE TRIANGULATIONS OF CONVEX POLYGONS AND ORDERED TREES

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Abstract

Much has been written about triangulations of convex polygons. A well known result going back to Euler, is that the number of triangulations of a convex $(n+2)$-gon is the Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$. In this paper, we give a direct bijection between triangulations of a convex $(n+2)$-gon and ordered trees. This bijection extends to results involving Fine numbers and Schröder numbers.

1. Introduction

The bijection between triangulations of convex polygons and complete planted binary trees is very well known and elegant; for example see [1], [2], [7] or [9]. In this paper, we introduce a bijection from triangulations of $(n+2)$-gons to ordered trees with n edges. In the second section, we present the bijection along with some examples. In the third and fourth sections, we look at variations that involve the Fine numbers and the little Schröder numbers. In our research several sequences were found which are in the *Online Encyclopedia of Integer Sequences (OEIS)*[5]; the A-numbers refer to this source.

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2. The Bijection Between Triangulations of Convex Polygons and Ordered Trees

Definition 1. Given a triangulation of a convex polygon, a triangle defined by 3 consecutive vertices is called an *ear*. We will also call the middle vertex an *ear*.

Given a polygon with $n+2$ vertices, we fix the lower left vertex and label it 1. The remaining vertices will be labeled 2 through $n+2$ in the clockwise direction. Let $\mathbf{T}$ be a triangulation of the polygon. Note that if $k-1, k, k+1$ is a triangle in $\mathbf{T}$, then k would be an ear. We now label the vertices (clockwise) on the edges of $\mathbf{T}$ that are incident to vertex 1 as $v_1, v_2, \ldots, v_k$, where $v_1 = 2$ and $v_k = n+2$. For $1 \leq i \leq k-1$, let d_i be the edge $(1, v_i)$. We use the edges d_i to subdivide the polygon into $k-1$ subpolygons each involving vertex 1. To start the construction of our bijection, we let vertex 1 correspond to the root of the ordered tree. Every edge of the triangulation that is incident to vertex 1 starts a subpolygon, except the edge $(1, n+2)$, which we delete. If 1 is an ear, then the degree of the root for the corresponding ordered tree is 1 and vertex 2 is the root of the subtree corresponding to the $(n+1)$-gon with vertices $2, 3, \ldots, n+2$. Otherwise, the degree of vertex 1 is greater than or equal to 3. Hence, if the number of edges incident to vertex 1 is k and $k \geq 3$, then the degree of the root of the corresponding tree is $k-1$. Vertex v_1 is the root of the corresponding subtree of the subpolygon from vertex v_1 to vertex v_2, v_2 is the root of the corresponding subtree of the subpolygon from vertex v_2 to vertex v_3, …, and vertex v_k is the root of the corresponding subtree of the subpolygon from vertex v_{k-1} to vertex v_k. This gives us a partial tree that has root labeled 1, and recursively, an ordered set of subtrees with roots $v_1, v_2, \ldots, v_{k-1}$. Since ordered trees are defined recursively, our bijection is established.

Following are some examples.

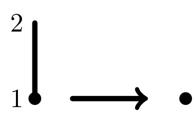

Figure 1: $n = 0$

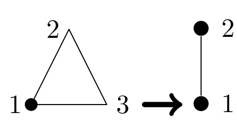

Figure 2: n=1

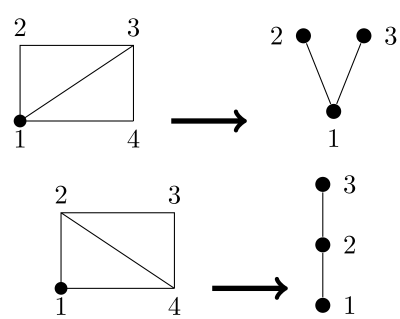

Figure 3: $n = 2$

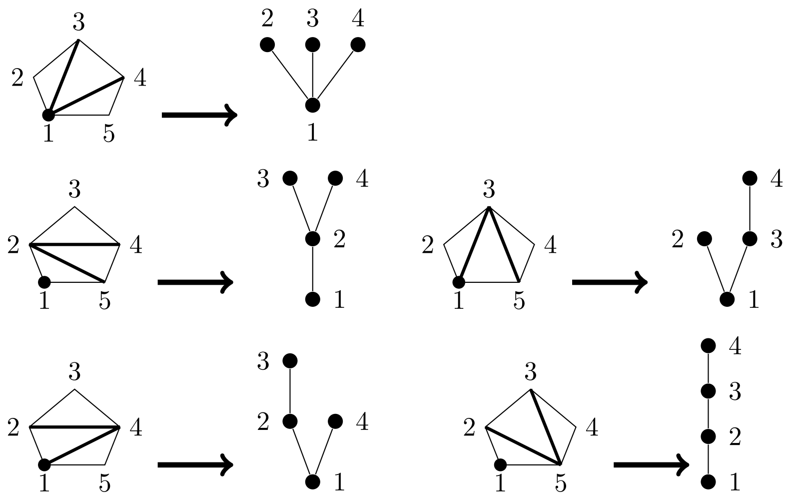

Figure 4: $n = 3$

Here is a more substantial example worked in stages.

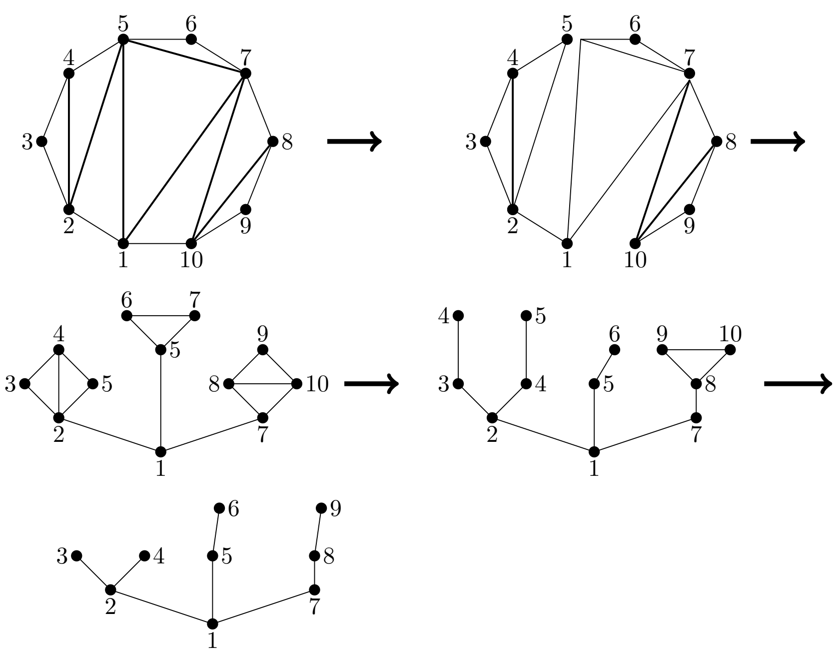

Figure 5: Example with 10 vertices.

Note that an ear, other than $(n+2,1,2)$, will lead to a leaf in the associated tree and a subsequence of the form UD in the preorder traversal bijection to Dyck paths.

The number of ordered trees with n edges is $C_n = \frac{1}{n+1}\binom{2n}{n}$, the n^{th} Catalan number. Recall that the generating function for the Catalan numbers is $C(z) = \sum_{n=0}^{\infty} C_n z^n$. It is well known that

$$\begin{aligned} C(z) &= 1 + zC^2(z) \\ &= \frac{1}{1 - zC(z)} \\ &= \frac{1-\sqrt{1-4z}}{2z} \\ &= 1 + z + 2z^2 + 5z^3 + 14z^4 + 42z^5 + \cdots (A000108, [5]). \end{aligned}$$

Since the Catalan numbers count the number of ordered trees, we get the following theorem.

Theorem 1. *The number of triangulations of a convex $(n+2)$-gon is the Catalan number C_n.*

Using this bijection, we can translate results about ordered trees directly to triangulations of an $(n+2)$-gon.

Corollary 1. *The number of internal diagonals at vertex 1 is one less than the root degree of the associated tree.*

Definition 2. Given an ordered tree, if a vertex v is a child of the root and has no children, then we call the edge from the root to v a *stump*.

Definition 3. Let $\mathbf{T}$ be a triangulation of an $(n+2)$-gon. If the triangle $(1, i, i+1)$ is a triangle in $\mathbf{T}$, then we call it a *wedge*.

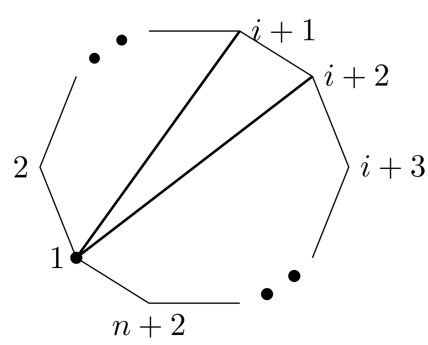

Figure 6: Wedge $(1, i+1, i+2)$

Note that using the bijection described in this paper, every wedge corresponds to a stump. This, in turn, corresponds to a hill in the preorder traversal bijection to Dyck paths, where a hill is a subsequence UD (up, down) starting and ending on the x-axis.

3. Triangulations and the Fine Numbers

The *Fine number* sequence $(F_n)_{n\geq 0}$ is a close companion of the Catalan sequence and counts several things, such as

- Dyck paths without hills,
- Ordered trees with no leafs at height 1,
- Ordered trees where the root has even degree.

The generating function is given by

$$F(z) = \sum_{n=0}^{\infty} F_n z^n = 1 + z^2 + 2z^3 + 6z^4 + 18z^5 + \cdots, (A000957, [5]).$$

The identities

$$\begin{aligned} C(z) &= \frac{F(z)}{1 - zF(z)} \\ \text{and} & \\ F(z) &= \frac{C(z)}{1 + zC(z)} \\ &= \frac{1}{z} \cdot \frac{1 - \sqrt{1-4z}}{3 - \sqrt{1-4z}} \end{aligned}$$

relate the Fine numbers and the Catalan numbers. For more on the Fine numbers see [3].

Theorem 2. *The number of triangulations of an $(n+2)$-gon with an odd number of diagonals at vertex 1 is the Fine number F_n.*

Proof. By Corollary 1, the associated tree of an $(n+2)$-gon with an odd number of diagonals at the vertex 1 is a tree with even root degree. These are precisely the trees counted by the Fine numbers. □

Theorem 3. *The number of triangulations with no consecutive vertices v_k, v_{k+1} incident to vertex 1 is counted by the Fine numbers.*

Proof. This corresponds first to an ordered tree with no leafs at height 1 and thus to Dyck paths with no hills. □

Theorem 4. *The number of triangulations of an $(n+2)$-gon with an even number of diagonals at vertex 1 is given by the generating function*

$$\frac{zC(z)}{1-z^2C^2(z)} = zC(z)F(z) = z + z^2 + 3z^3 + 8z^4 + 24z^5 + 75z^6 + \cdots, (A000958, [5]).$$

Proof. All ordered trees have either even or odd degree at the root. Let $E(z)$ be the generating function counting those with even root degree and $O(z)$ the generating function counting those with odd root degree. Then

$$\begin{aligned} C(z) &= E(z) + O(z) \\ &= F(z) + O(z). \end{aligned}$$

Thus

$$\begin{aligned} O(z) &= C(z) - F(z) \\ &= \frac{F(z)}{1 - zF(z)} - F(z) \\ &= F(z)\frac{zF(z)}{1 - zF(z)} \\ &= zF(z)C(z). \end{aligned}$$

□

Theorem 5. *The number of triangulations of an $(n+2)$-gon with the lowest numbered vertex k that has an edge from vertex 1 is $C_{k-3}C_{n+2-k}$, where $k \geq 3$.*

Proof. Let vertex k be the lowest numbered vertex other than 2 with diagonal incident with vertex 1. Hence, $3 \leq k \leq n+1$. Because of the triangulation, there is a diagonal $(2, k)$. The diagonal $(1, k)$ cuts the polygon into two subpolygons. The subpolygon to the left of $(1, k)$ that contains vertices 2 and k can be triangulated C_{k-3} ways and the subpolygon to the left of $(1, k)$ that contains vertices 1 and k can be triangulated C_{n+2-k} ways.

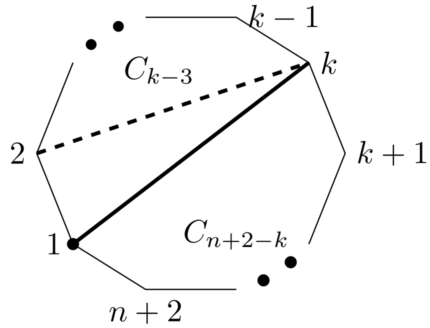

Figure 7: Subdivision by the first internal diagonal.

When vertex 1 is an ear, then $k = n+2$. And there are C_{n-1} triangulations in which vertex 1 is an ear; see Figure 8.

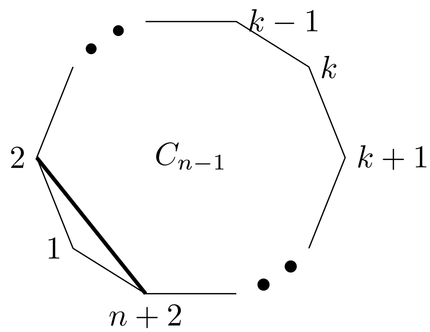

Figure 8: When vertex 1 is an ear.

□

Example 1. The following chart gives the number of triangulations of an $(n+2)$-gon that has vertex k as the first vertex incident to vertex 1, where $k \geq 3$.

	$k=3$	4	5	6	7	8	9
$n=1$	1						
2	1	1					
3	2	1	2				
4	5	2	2	5			
5	14	5	4	5	14		
6	42	14	10	10	14	42	
7	132	42	28	25	28	42	132

The distribution of the rows approaches a U-shaped distribution similar to the arcsine distribution; see [4].

Theorem 6. *Let k be minimal with respect to $(1,k)$ being an internal diagonal. The probability for this is*

$$\frac{C_{k-3}C_{n+2-k}}{C_n}.$$

We now are interested in the distribution of the first diagonal edge that is incident with vertex 1.

Theorem 7. *As n gets large, the probability that $(1,k)$ is the first internal diagonal approaches $\frac{C_k}{4^k}$ (here k is fixed as $n \to \infty$).*

Proof. Recall that

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

It is well known that

$$\frac{C_n}{C_{n+1}} = \frac{n+2}{4n+2} \to \frac{1}{4} \text{ as } n \to \infty.$$

Thus

$$C_k\left(\frac{C_{n-k}}{C_{n-k+1}}\right)\left(\frac{C_{n-k+1}}{C_{n-k+2}}\right)\dots\left(\frac{C_n}{C_{n+1}}\right) \to C_k\left(\frac{1}{4}\right)\left(\frac{1}{4}\right)\dots\left(\frac{1}{4}\right) = \frac{C_k}{4^k}.$$

□

Corollary 2. *As n increases, the probability that vertex 3 is the lowest numbered vertex of an $(n+2)$-gon with an internal diagonal from vertex 1 approaches $\frac{1}{4}$. This is the same as the probability that vertex 1 has no internal diagonals.*

Corollary 3. *As n increases, the probability that vertex 4 is the lowest numbered vertex of an $(n+2)$-gon with an internal diagonal from vertex 1 approaches $\frac{1}{16}$. This is the same as the probability that vertex $n+1$ is the lowest numbered vertex with an internal diagonal from vertex 1.*

We can also use triangulations to get results about ordered trees. For example, each triangulation of an $(n+2)$-gon has $n-1$ internal diagonals. So the total number of internal diagonals is $(n-1)\frac{1}{n+1}\binom{2n}{n} = (n-1)C_n$. Thus the average number of internal diagonals incident to v is $\frac{2(n-1)C_n}{(n+2)C_n} = \frac{2(n-1)}{n+2}$. Adding the two boundary edges, the average number of edges incident to any vertex (except vertex 1) is $2 + \frac{2n-2}{n+2} = \frac{4n+2}{n+2}$. For vertex 1, we disregard the edge $(1, n+2)$, so that the average number of edges incident to it is $1 + \frac{2n-2}{n+2} = \frac{3n}{n+2}$. This corresponds to the well known result that the average degree of the root of ordered trees goes to 3 as $n \to \infty$.

Definition 4. Given a triangulation **T** of an $(n+2)$-gon, if vertices $i, i+1, i+2, \ldots i+k$ are all incident with vertex 1, then $(1, i, i+1, \ldots, i+k)$ is called a $k-fan$.

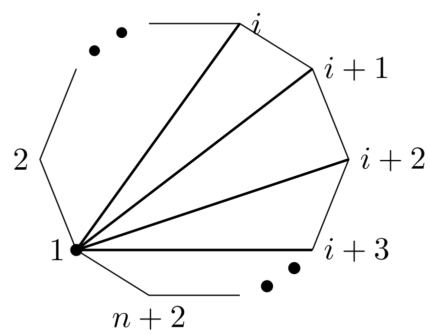

Figure 9: 3-fan

Theorem 8. *The number of triangulations of an* $(n+2)$*-gon without fans is the Fine number* F_n.

Theorem 9. *The number of wedges in the triangulations of an* $(n+2)$*-gon is the Catalan number* C_n.

Proof. Each wedge corresponds to a leaf of height one in the associated tree, and thus a hill in the corresponding Dyck path. The generating function counting these is $C(z)zC(z) = C(z) - 1$. □

See bijective exercise 2 in [8]. In Figure 4, the triangulations have, respectively, 3, 0, 1, 1, and 0 wedges.

Theorem 10. *The number of 2 consecutive wedges in the triangulation of an* $n+2$*-gon is* C_{n-1}.

Proof. Consecutive hills are counted by the generating function

$$C(z)z^2C(z) = z(C(z) - 1).$$

□

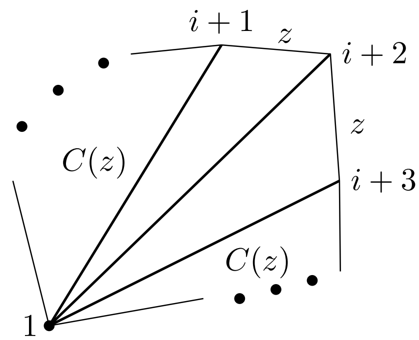

Figure 10: $z^2C^2 = z^2 + 2z^3 + 5z^4 + 14z^5 + ...$

Definition 5. If $(1, i+1), (i+1, i+2), (i+2, i+3), (1, i+3)$, and $(i+1, i+3)$ are edges in a triangulation of an $(n+2)$-gon, then it is called a *kite*.

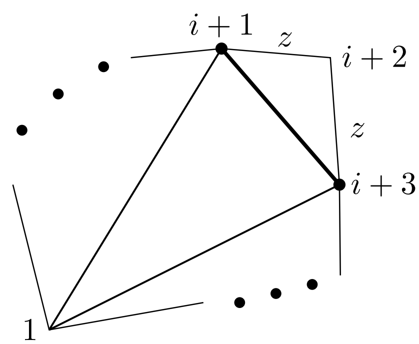

Figure 11: Kite in an $(n+2)$-gon.

Note that in our bijection, kites are associated with ordered trees that have stem height 2, and will be an UUDD starting and ending on the x-axis in a Dyck path.

Theorem 11. *The generating function counting the number of ways to triangulate an $(n+2)$-gon without kites is*

$$\frac{C(z)}{1+z^2C(z)} = 1 + z + z^2 + 3z^3 + 10z^4 + 31z^5 + 98z^6 + \cdots, (A114487, [5]).$$

Proof. Let $N(z)$ be the generating function for Dyck paths with no UUDD subpaths on the x-axis. Then

$$\begin{aligned} C(z) &= N(z) + N(z)z^2N(z) + N(z)(z^2N(z))^2 + \cdots \\ &= \frac{N(z)}{1 - z^2N(z)}, \end{aligned}$$

where $N(z)(z^2N(z))^k$ counts paths with k $UUDD$ $x-$axis based subpaths. Solving for $N(z)$ gives the result. □

4. Triangulations and Schröder Numbers

The little Schröder numbers s_n count many objects, such as the number of ways to insert some pairs of matching parentheses into a string of n distinct letters. The number s_n also counts the number of lattice paths in the Cartesian plane that start at $(0, 0)$, end at $(2n, 0)$, do not go below the x-axis, are composed only of steps $(1, 1)$ (up), $(1, -1)$ (down), and $(2, 0)$ (horizontal), and have no horizontal steps on the x-axis. Such paths are called *little Schröder paths*. These numbers appear in the classic 1870 paper by Schröder; see [6]. The generating function for the *little Schröder Numbers* is

$$\begin{aligned} s(z) &= \frac{1 + z - \sqrt{1 - 6z + z^2}}{4z} \\ &= 1 + z + 3z^2 + 11z^3 + 45z^4 + 197z^5 + \cdots (A001003, [5]). \end{aligned}$$

We begin our investigation between triangulations and the little Schröder numbers with the following definition.

Definition 6. A *partial triangulation* of an $(n+2)$-gon is a triangulation with some or all of the internal diagonals removed.

Example 2. If $n = 3$, we get 11 triangulations and partial triangulations, which are given in Figure 12.

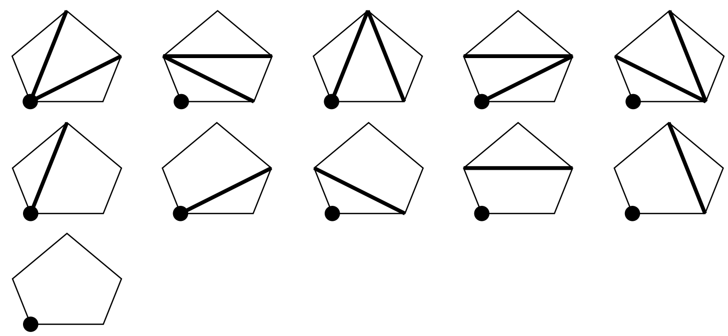

Figure 12: Triangulations and partial triangulations when $n = 3$.

Suppose the partial triangulation has no internal edges. Then we can consider the following bijection between such partial triangulations and little Schröder paths that contain one up step, followed by n level steps, followed by a down step (see Figure 13).

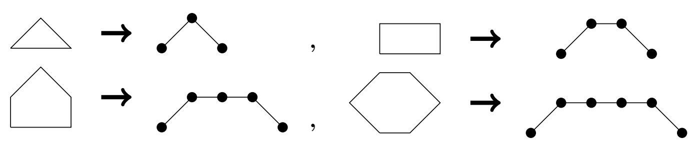

Figure 13: Mapping for partial triangulation with no internal edges.

Using a recursive definition, a bijection is established between the triangulations and partial triangulations and little Schröder paths. Hence, we get the following theorem.

Theorem 12. *There is a bijection between the triangulations and partial triangulations of an* $(n+2)$*-gon and the little Schröder paths.*

Example 3. Recall that $s_3 = 11$. The following figure gives the mapping between partial triangulations of a polygon to little Schröder paths when $n = 3$. For the other 5 cases when we have triangulations, we simply use Figure 4 and the usual preorder traversal from ordered trees to Dyck paths.

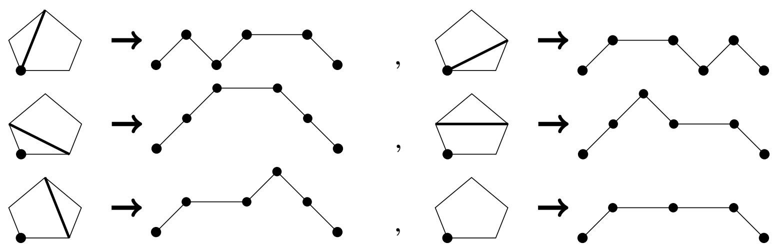

Figure 14: Mapping from partial triangulations to little Schröder paths when $n = 3$.

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References

[1] R. Brualdi, *Introductory Combinatorics*, 5th ed., Prentice-Hall, Upper Saddle River, 2009.

[2] L. Comtet, *Advanced Combinatorics*, D. Reidel Publishing Company, Dordrecht, 1974.

[3] E. Deutsch and L. Shapiro, A survey of the Fine numbers, *Discrete Math.* **241** (2001), 241-265.

[4] P. Flajolet and R. Sedgewick, *Analysis of Algorithms*, 2nd ed., Addison-Wesley, Boston, 2013.

[5] *Online Encyclopedia of Integer Sequences*, http://oeis.org/.

[6] E. Schröder, Vier combinatorische probleme, *Z. für Math. Phys.* **15** (1870), 361-376.

[7] R. Stanley, *Enumerative Combinatorics: Volume 2*, Cambridge University Press, Cambridge, 1999.

[8] R. Stanley, *The Catalan Numbers*, Cambridge University Press, Cambridge, 2015.

[9] J. van Lint and R. Wilson, *A Course in Combinatorics*, Cambridge University Press, Cambridge, 2nd ed., 2001.