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THE PRODUCT OF PARTS OR "NORM" OF A PARTITION

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Abstract

In this article we study the "norm" of an integer partition, which we define to be the product of the parts. This partition-theoretic statistic has appeared here and there in the literature of the last century or so, and is at the heart of current research by both authors. We survey known results and give new results related to this all-but-overlooked object, which, it turns out, plays a comparable role in partition theory to the size, length, and other standard partition statistics.

1. Introduction: A Multiplicative Statistic on (Additive) Partitions

The theory of integer partitions is a rich source of identities, bijections, and interrelations at the confluence of number theory, combinatorics, algebra, analysis, and the physical sciences. Let

$$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$$

denote a generic partition, with integer parts $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r \geq 1$, and let $\emptyset$ denote the *empty partition*. Alternatively, it is often useful to notate partitions using classical "frequency superscript notation", viz.,

$$\lambda = \langle 1^{m_1}\ 2^{m_2}\ 3^{m_3} \cdots \rangle$$

where $m_j = m_j(\lambda)$ is the *frequency* of occurrence (or *multiplicity*) of j as a part in the partition, noting only finitely many m_j are nonzero, with the conventions that if $m_j = 1$ then it may be omitted in the superscript, and j^{m_j} is usually omitted if $m_j = 0$.

Many famous identities are related to the statistic $p(n)$ called the *partition function*, counting the number of partitions of a natural number n, like Euler's seminal *partition generating function*.

Theorem 1 (Euler). *For $q \in \mathbb{C}, |q| < 1$ we have that*

$$\prod_{n=1}^{\infty} \frac{1}{1-q^n} = \sum_{n=0}^{\infty} p(n) q^n.$$

Other statistics about partitions also feature heavily into partition theory, such as the *size* $|\lambda| := \lambda_1 + \lambda_2 + \cdots + \lambda_r$ of partition λ (sum of the parts), the *length* $\ell(\lambda) := r$ of λ (number of parts), the *largest part* $\lg(\lambda) := \lambda_1$, Dyson's *rank* $rk(\lambda) := \lg(\lambda) - \ell(\lambda)$, etc.

Here we will study another, all-but-overlooked statistic that plays a comparable role in partition theory to the size, length, and others listed above.

Definition 2. Let $N(\lambda)$ denote the product of the parts, or *norm*, of the partition λ:

$$N(\lambda) := \lambda_1 \lambda_2 \lambda_3 \cdots \lambda_r,$$

with $N(\emptyset) := 1$ (the empty product). Equivalently, we have $N(\lambda) = 1^{m_1} 2^{m_2} 3^{m_3} \cdots$.

The defining characteristic of the set $\mathcal{P}$ of partitions is that one *adds* the parts together, so this multiplicative norm perhaps feels a little artificial. On the other hand, if we view a partition purely as a multiset of whole numbers, then multiplying the elements together is just as natural an operation as adding them. Likewise, one can express the size $|\lambda|$ in terms of the norm:

$$|\lambda| = N(\lambda) \sum_{\lambda_i \in \lambda} \frac{1}{N(\lambda/\lambda_i)}, \tag{1}$$

where "$\lambda_i \in \lambda$" indicates $\lambda_i \in \mathbb{N}$ is a part of λ, and we let $\lambda/\lambda_i \in \mathcal{P}$ denote the partition obtained by deleting λ_i from λ. (This identity follows instantly from considering the ratio $|\lambda|/N(\lambda)$.)

MacMahon's partial fraction decomposition of the generating function for partitions of length at most n may be the first explicit appearance of the partition norm in the literature (notated below in the conventions of this paper) [8, 9].

Theorem 3 (MacMahon). *For q not equal to a kth root of unity, $1 \leq k \leq n$, we have that*

$$\prod_{j=1}^{n} \frac{1}{1-q^j} = \sum_{\lambda \vdash n} \frac{1}{N(\lambda)\ m_1!\ m_2!\ m_3! \cdots (1-q)^{m_1}(1-q^2)^{m_2}(1-q^3)^{m_3} \cdots}$$

where "$\lambda \vdash n$" on the right side means the sum is taken over the partitions of n.

The partition norm features centrally in the first author's work (e.g., [15, 16, 17]) on partition zeta functions and partition analogs of classical arithmetic functions, and the second author independently studied the product of parts in his own work [20] on MacMahon's partial fractions.

Immediately, there are a number of questions one might ask about this partition statistic. For example, does it have a product-sum generating function interpretation? Does the norm admit a natural combinatorial (or probabilistic) interpretation? What are its maximum, minimum and average values over the partitions of n? Does the norm obey any nice asymptotics? Does it connect to other areas of partition theory or, more broadly, of mathematics?

2. Generating Functions and Dotted Young Diagrams

Let us note that a generating function $\sum_{\nu} P(\nu)q^{\nu}$, where $P(\nu)$ is the number of partitions of norm ν, is not possible, as there are infinitely many partitions of any fixed norm $\nu \geq 1$: adjoining arbitrarily many 1's to a partition gives a new partition of the same norm. Moreover, we cannot control multiplication in the exponent of q via product generating functions in the same way we generate partitions in the exponent. We can get close, though, if we consider norms of partitions with no 1's and relax our expectations for a power series generating function.

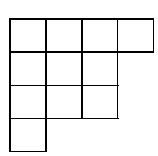
Following [17, Appendix A], let a *nuclear partition* be a partition in which all parts are greater than 1 (thus $|\mu| \leq N(\mu)$ for μ a nuclear partition). Then the (finite) number of nuclear partitions of fixed norm ν (which is equivalent to the number of multiplicative partitions of ν) has the following "non-power series" generating function.

Theorem 4. *Let $\widetilde{P}(\nu)$ denote the number of nuclear partitions of fixed norm $\nu \geq 1$. Then for $x \in \mathbb{R},\ 0 < x < e^{-1}$, we have*

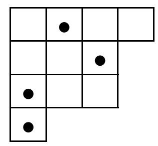
$$\prod_{n=2}^{\infty} \frac{1}{1 - x^{\log n}} = \sum_{\nu=1}^{\infty} \widetilde{P}(\nu) x^{\log \nu}. \tag{2}$$

Proof. Observing for any partition λ that $\log \lambda_1 + \log \lambda_2 + \cdots \log \lambda_r = \log N(\lambda)$, then as the product starts with index $n = 2$, classical generating function ideas yield the identity. For justification that the product and sum converge for $0 < x < e^{-1}$, we refer the reader to the proof of Theorem 26 below. □

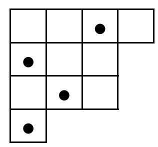
We now offer a combinatorial interpretation of $N(\lambda)$. Recall the *Young diagram* for a partition $(\lambda_1, \lambda_2, \ldots, \lambda_r)$, in which the ith part is pictured as the ith row of the diagram consisting of λ_i squares, e.g., the Young diagram for $\lambda = (4, 3, 3, 1)$ is:



Let us impose further structure on this diagram by placing a dot in one of the squares of each horizontal row, and call the resulting diagram a *dotted Young diagram* of partition λ:



This pattern of dots is not unique; here is another dotted Young diagram of λ:



The different dot patterns for a given Young diagram are enumerated by the norm.

Theorem 5. *The number of dotted Young diagrams of a partition λ is $N(\lambda)$.*

Proof. There are λ_1 different ways to place a dot in row one, λ_2 ways to dot row two, λ_3 ways to dot row three, etc., yielding $\lambda_1\lambda_2\lambda_3\cdots\lambda_r = N(\lambda)$ different dotted Young diagrams of λ. $\square$

Remark 6. More generally, if we place k dots in each row, the number of k-tuple dotted Young diagrams of $\lambda = \langle 1^{m_1}2^{m_2}\cdots i^{m_i}\cdots\rangle$ is $\prod_{i=1}^{\infty}\binom{i}{k}^{m_i}$, with binomial coefficients $:= 0$ when $i < k$.

In the context of dotted Young diagrams, the norm admits the following generating function interpretation.

Theorem 7. *Let $\dot{p}(n)$ denote the number of dotted Young diagrams of size n. Then*

$$\dot{p}(n) = \sum_{\lambda\vdash n} N(\lambda).$$

For $|q| < 1$ we have the generating function

$$\prod_{n=1}^{\infty}\frac{1}{1-nq^n} = \sum_{n=0}^{\infty}\dot{p}(n)q^n.$$

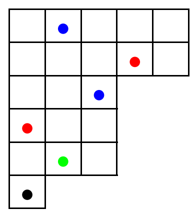
Proof. The first equation of the theorem is an immediate corollary of Theorem 5. The generating function for $\dot{p}(n)$ follows naturally from this corollary together with [16, Corollary 4.3]:

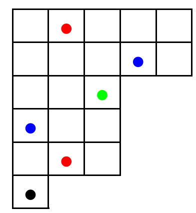
$$\prod_{n=1}^{\infty}\frac{1}{1-nq^n} = \sum_{\lambda\in\mathcal{P}} N(\lambda)q^{|\lambda|} = \sum_{n=0}^{\infty} q^n \sum_{\lambda\vdash n} N(\lambda). \tag{3}$$

$\square$

We may define a yet more general object. For a fixed dotted Young diagram of λ, if i appears as a part with frequency $m_i > 1$, we will color each of the dots differently over the m_i rows of i squares (that is, we give each dot one of m_i distinct colors). Let us call such a diagram a *multicolor dotted Young diagram* of partition λ.

Here are two different colorings of the same dotted Young diagram of $\lambda = (5, 5, 3, 3, 3, 1)$:





Theorem 8. *The number of multicolor dotted Young diagrams of a partition λ is*

$$N(\lambda)\ m_1!\ m_2!\ m_3! \cdots m_i! \cdots .$$

Proof. There are $N(\lambda)$ different dotted Young diagrams of λ, and $m_i!$ ways to permute m_i colors among the rows of length i in each dotted diagram. □

Thus the probability of picking a particular multicolor dotted Young diagram of a fixed partition λ is

$$\frac{1}{N(\lambda)\ m_1!\ m_2!\ m_3! \cdots m_i! \cdots}. \tag{4}$$

In [20] the second author refers to these fractions as *MacMahon coefficients* of the partial fraction decomposition in Theorem 3, and in [21] shows by the following result of N. J. Fine [5, p. 38, Eq. (22.2)] that, if each partition of n occurs with the probability equal to its MacMahon coefficient, this is a discrete probability distribution.

Theorem 9 (Fine). *We have that*

$$\sum_{\lambda \vdash n} \frac{1}{N(\lambda)\ m_1!\ m_2!\ m_3! \cdots} = 1.$$

This identity can be viewed as the $q = 0$ case of Theorem 3. Numerous identities involving MacMahon coefficients arise naturally from the classical Faà di Bruno's formula (see, e.g., [17, Appendix D]), like the following result stemming from Euler's partition generating function.

Theorem 10. *Let $p(n)$ denote the partition function. Then we have*

$$p(n) = \sum_{\lambda \vdash n} \frac{\sigma(1)^{m_1} \sigma(2)^{m_2} \sigma(3)^{m_3} \cdots}{N(\lambda)\ m_1!\ m_2!\ m_3! \cdots},$$

where $\sigma(n) := \sum_{d|n} d$*.*

Proof. Setting $a(n) \equiv c$ identically, $c \geq 0$, in Prop. D.1.1 of [17] gives for $|q| < 1$ the formula

$$\prod_{n=1}^{\infty} \frac{1}{(1-q^n)^{\pm c}} = \sum_{n=0}^{\infty} q^n \sum_{\lambda \vdash n} (\pm c)^{\ell(\lambda)} \frac{\sigma(1)^{m_1} \sigma(2)^{m_2} \sigma(3)^{m_3} \cdots}{N(\lambda)\ m_1!\ m_2!\ m_3! \cdots},$$

where "±" represents the same sign, positive or negative, on both the left- and right-hand side. Letting $c = 1$ with "± = plus" gives the theorem, by comparison with Theorem 1. □

Seen in a certain light, the norm is a component of the partition function $p(n)$. We would like to find combinatorial interpretations for formulas like Theorem 10 arising from Faà di Bruno's formula, as well.

3. Maximum, Minimum and Average Values of the Norm

Another immediate question one asks about a statistic such as the norm is, "How big is it?" Then it is natural that much of the literature related to the product of the parts of partitions seems to focus on the magnitude of the product; we survey some of these results, and record a few of our own.

For instance, the following theorem appears as an exercise in a few sources, e.g., [6, pp. 30–31, 188], [10, p. 5, prob. 15]; to whom to attribute the result is unclear.

Theorem 11 (Halmos, Newman, et al.). *Among all partitions of* $n \geq 1$*, the partition with maximum norm is:*

i. $\langle 3^{n/3} \rangle$ *if* $n \equiv 0 \pmod 3$,

ii. $\langle 3^{(n-4)/3} 4 \rangle$ *as well as* $\langle 2^2 3^{(n-4)/3} \rangle$ *if* $n \equiv 1 \pmod 3$ *and* $n > 1$,

iii. $\langle 2\ 3^{(n-2)/3} \rangle$ *if* $n \equiv 2 \pmod 3$,

iv. $\langle 1 \rangle$ *if* $n = 1$.

Remark 12. The sequence $a(n)$ = "maximum norm over all partitions of n" is A000792 in the OEIS [22].

More recently, Došlić [4, Theorem 4.1] gives an analogous result for partitions into odd parts.

Theorem 13 (Došlić). *Among all partitions of* $n \geq 3$ *into odd parts, the partition with maximum norm is:*

i. $\langle 3^{n/3} \rangle$ *if* $n \equiv 0 \pmod 3$,

ii. $\langle 1\ 3^{(n-1)/3} \rangle$ *if* $n \equiv 1 \pmod 3$,

iii. $\langle 3^{(n-5)/3} 5 \rangle$ *if* $n \equiv 2 \pmod 3$.

Another result [4, Theorem 3.1] of Došlić handles the partitions into distinct parts via a connection to triangular numbers.

Theorem 14 (Došlić). *Let $T_k := k(k+1)/2$, the kth triangular number. Among the partitions of $n \geq 2$ into distinct parts, the partition $\Delta^{max} = \Delta^{max}(n)$ with maximum norm is as follows: given that n can be expressed uniquely as $T_k + j$ for some $-1 \leq j \leq k-2$, then*

$$\Delta^{max} = (k+1, k, k-1, \ldots, k-j+1, k-j-1, k-j-2, \ldots, 3, 2),$$

i.e., the partition in which the parts are one copy each of all integers 2 through $k+1$ inclusive, with the exception of $k-j$. The norm of this partition is $N(\Delta^{max}) = \frac{(k+1)!}{k-j}$.

Remark 15. The sequence $a(n)$ = "maximum norm over partitions of n into distinct parts" is A034893 in OEIS [22]. We note further connections exist in the literature between partitions and triangular numbers (see, e.g., [3, 13]).

Based on these examples, it seems that the norms of other interesting subclasses of partitions may yield analogous results. Here we give another example, which does not seem to have appeared previously in the literature; we are interested to identify further such subclasses.

Recall that $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ is a *Rogers–Ramanujan partition* if $\lambda_i - \lambda_{i+1} \geq 2$ for $i = 1, 2, \ldots, r-1$ (see [1, 19]).

Theorem 16. *Let $D_k := k(k+1)$, and write $n = D_k + j$ where $0 \leq j < 2k+2$; also, set $j' := j - k$ if $j > k$. Among all Rogers–Ramanujan partitions of size n, let $\rho^{max} = \rho^{max}(n)$ denote the one of maximum norm.*

i. $\rho^{max} = (2k, 2k-2, 2k-4, \ldots, 6, 4, 2)$ *with* $N(\rho^{max}) = 2^k k!$, *if* $j = 0$,

ii. $\rho^{max} = (2k+1, 2k-1, 2k-3, \ldots, 2k-2j+3, 2k-2j, 2k-2(j+1), \ldots, 6, 4, 2)$ *with* $N(\rho^{max}) = \frac{2^{k-2j}(k-j)!(k-j+1)!(2k+2)!}{(k+1)!(2(k-j)+2)!}$, *if* $1 \leq j < k$,

iii. $\rho^{max} = (2k+1, 2k-1, 2k-3, \ldots, 7, 5, 3)$ *with* $N(\rho^{max}) = \frac{(2k+2)!}{2^{k+1}(k+1)!}$, *if* $j = k$,

iv. $\rho^{max} = (2k+2, 2k, 2k-2, \ldots, 2k-2j'+4, 2k-2j'+1, 2k-2(j'+1)+1, \ldots, 7, 5, 3)$ *with* $N(\rho^{max}) = \frac{\left(2(k-j')+2\right)!(k+1)!}{2^{k-2j'+1}(k-j'+1)!^2}$, *if* $k < j < 2k$,

v. $\rho^{max} = (2k+2, 2k, 2k-2, \ldots, 8, 6, 4)$ *with* $N(\rho^{max}) = 2^{k+1}(k+1)!$, *if* $j = 2k$, *vi.* $\rho^{max} = (2k+3, 2k, 2k-2, 2k-4, \ldots, 8, 6, 4)$ *with* $N(\rho^{max}) = 2^{k-1}(2k+3)k!$, *if* $j = 2k+1$.

Proof. The theorem (as well as, morally, the preceding results) follows from the simple fact that for $a, b, c \in \mathbb{N}$, if $a < b$ then the magnitude of the product ab is more significantly enlarged by increasing the smaller factor a by an additive constant $c \geq 1$ than by increasing b by an equal amount, as $(a+c)b = ab + bc > ab + ac = a(b+c)$. By the same token, if $a + b = n,\ 1 < a < b$, and we wish to vary the summands while keeping n constant, the product ab is increased when we borrow $c < b - a$ from the larger summand b to increase the smaller summand a, and is decreased by borrowing $c < a$ from the smaller summand to increase the larger, as $(a+c)(b-c) = ab + c(b-a) - c^2 \ \geq\ ab \ \geq\ ab - c(b-a) - c^2 = (a-c)(b+c)$.

The same holds for the sum versus the product of natural numbers $a_1, a_2, \ldots, a_r$ such that $a_1 \geq a_2 \geq \cdots \geq a_r > 1$: borrowing from larger summands of $a_1 + a_2 + \cdots + a_r$ to increase smaller summands (or create smaller summands greater than 1), while maintaining their relative "$\geq$" ordering, generally increases the product $a_1 a_2 a_3 \cdots$ and the opposite action generally reduces the product. (Summands $a_i = 1$ break this rule: they increase size but fix the norm, viz. $1 \cdot b < 1 + b$.) Noting that partitions of n represent exactly such sums $a_1 + a_2 + \cdots + a_r = n$, then any partition of n might be transformed into a partition of n of greater norm by reducing larger parts to increase (or create new) smaller parts accordingly, so long as the relative ordering of the existing parts is not violated. Restrictions on type of integers used or ordering of the parts (e.g., differences of a specified kind) limit the transformations possible.

If we seek a Rogers–Ramanujan partition (distinct parts with differences at least 2) of $n = D_k = k(k+1) = 2 \cdot T_k$, then by the preceding "borrowing from larger parts to increase smaller parts" principle, it is clear in the partition $\alpha := (2k, 2k-2, 2k-4, \ldots, 6, 4, 2)$ that no part greater than $2k$ may be increased (or a new part created) without violating the distinctness and difference restrictions. Moreover, any other allowed partition of $n = D_k$ must be formed by borrowing from *smaller* parts of α to increase other summands, decreasing the norm from $N(\alpha)$. Thus α is the Rogers-Ramanujan partition of D_k with greatest norm.

For a Rogers–Ramanujan partition of $n = D_k + j,\ 0 < j < 2k+2$, we want to stay as close in minimal shape to α above by distributing the quantity j between the parts of α. But no part of α may be increased unless the preceding part is first increased without violating the order restriction, so by the "borrowing from larger to increase smaller" rule, the partition of largest norm is achieved by adding 1 to each of the largest j parts of α in the case $j < k$, to yield partition $(2k+1, 2k-2+1, 2k-4+1, \ldots, 2k-2(j-1)+1, 2k-2j, \ldots, 6, 4, 2)$ of maximum norm. If $j = k$ we "use up" all k of the 1's by this process to yield partition $\beta := (2k+1, 2k-2+1, 2k-4+1, \ldots, 7, 5, 3)$. For $k < j < 2k+1$, restart the process of adding 1 to each of the largest $j' = j - k$ parts of β to yield partition $(2k+2, 2k-2+2, 2k-4+2, \ldots, 2k-2(j'-1)+2, 2k-2j'+1, \ldots, 7, 5, 3)$ of maximum norm. If $j = 2k$ then having "used up" $2k$ of the 1's in the preceding steps, we arrive at partition $\gamma := (2k+2, 2k, 2k-2, \ldots, 8, 6, 4)$. For $j = 2k+1$ we again restart the process for a final move, adding the remaining 1 to the largest part of γ.

In each of these cases, the value of the norm is immediate from standard factorial manipulations. □

Remark 17. The preceding proof of Theorem 16 is sufficiently general that it could be used to prove analogous results for the maximum norm of other restricted classes of partitions of size n, e.g., the *Göllnitz–Gordon partitions* (partitions with difference at least two between parts and no consecutive even numbers as parts), or the *Schur partitions* (partitions with difference at least three between parts and no consecutive multiples of 3 as parts), etc.; see [1, 19].

Let us now look also at questions of minimality. Clearly the partition of integer $n \geq 1$ of *minimum* norm is $(1, 1, 1, ..., 1)$, with n repetitions. It is also not hard to see that among all partitions of $n \geq 3$ with *distinct* parts, the one with minimum norm is $(n-1, 1)$.

With a slight change of perspective, one might ask instead about partitions of a *fixed norm*, say ν, having minimum or maximum *size*. The maximal result is easy, as any number of 1's can be adjoined to a partition without altering its norm; thus there is no fixed-norm partition of maximum size. The minimum size problem is somewhat less trivial.

Theorem 18. *The minimum possible size of a partition of norm* ν *is*

$$a_1 p_1 + a_2 p_2 + a_3 p_3 + \cdots + a_i p_i + \cdots,$$

where $\nu = p_1^{a_1} p_2^{a_2} p_3^{a_3} \cdots\ p_i^{a_i} \cdots$ *is the prime factorization of* ν ($p_1 = 2, p_2 = 3, p_3 = 5$, *etc., with only finitely many* $a_i \geq 0$ *being nonzero). This minimal size is achieved by norm-*ν *partitions of the shape*

$$\left\langle p_1^{a_1 - 2b}\ p_2^{a_2}\ 4^b\ p_3^{a_3}\ p_4^{a_4} \ldots\ p_i^{a_i} \ldots \right\rangle$$

for every integer b *such that* $0 \leq b \leq \frac{1}{2}a_1$.

Proof. Consider a partition $\gamma = \langle k_1^{m_{k_1}}\ k_2^{m_{k_2}}\ k_3^{m_{k_3}} ...\ k_t^{m_{k_t}} ... \rangle$ with norm $N(\gamma) = \nu$. We exclude partitions with 1 as a part, as some or all of the 1's can be deleted from such a partition, diminishing its size without changing its norm.

Certainly, one partition of norm ν is $\rho = \langle 2^{a_1}\ 3^{a_2}\ 5^{a_3}\ 7^{a_4} ...\ p_i^{a_i} ... \rangle$ consisting of the prime factors of ν including multiplicities. For $\gamma \neq \rho$, since the product of the parts of γ equals ν, each part is the product of some of the factors of ν, i.e., $k_j = p_1^{c_1} p_2^{c_2} p_3^{c_3} \cdots\ p_i^{c_i} \cdots$ with $0 \leq c_i \leq a_i$ for all i. Thus the parts $k_1, k_2, k_3, ...$

essentially represent a regrouping of this set of prime factors into a smaller set of numbers including products of some of the primes.

But since $x_1 + x_2 + ... + x_r \ \leq\ x_1x_2 \cdots x_r$ for $x_i \geq 2$ with equality only in the case $2+2 = 2 \cdot 2$, then $p_1^{c_1} + p_2^{c_2} + p_3^{c_3} + ... \leq k_j$, and by extension, $|\rho| \leq |\gamma|$. In this case, equality occurs when γ is formed by replacing some number b of pairs of 2's in partition ρ by the same number b of 4's, since this replacement changes neither the size nor the norm. □

Turning now to asymptotic-type results, we recall work of Lehmer [7] connecting the reciprocal of the norm to the *Euler–Mascheroni constant* $\gamma = 0.5772\ldots$, which is defined by $\gamma := \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - \log n\right)$.

Theorem 19 (Lehmer). *We have that*

$$\lim_{n\to\infty}\frac{1}{n}\sum_{\lambda\vdash n}\frac{1}{N(\lambda)} = e^{-\gamma}.$$

A similar result holds if we restrict the sum to partitions $\mathcal{D}$ into distinct parts.

Theorem 20 (Lehmer). *Let $\mathcal{D}$ denote the set of partitions into distinct parts. Then*

$$\lim_{n\to\infty}\sum_{\substack{\lambda\vdash n\\ \lambda\in\mathcal{D}}}\frac{1}{N(\lambda)} = e^{-\gamma}.$$

We note that the first of the two theorems above is almost-but-not-quite an average (the sum is taken over the partitions of n, not over $1, 2, 3, ..., n$). Along similar lines, it is natural to want to know the average magnitude of the norm.

Theorem 21. *The expected value of the norm over all the partitions of n is*

$$E[N] = \prod_{i=1}^{n}\sqrt[i]{i}.$$

Proof. It is a result of the second author [21], which can be proved by setting $q_i = 1/i$ in Eq. 14 of [23], that the expected value $E[m_i]$ of the frequency of i obeys

$$E[m_i] = \frac{1}{i}. \tag{5}$$

Then noting $E[N] = 1^{E[m_1]}2^{E[m_2]}3^{E[m_3]}\cdots n^{E[m_n]}$ completes the proof. □

Remark 22. Thus, by (5) the expected length of a partition of n is the nth harmonic number:

$$E[\ell] = E[m_1] + E[m_2] + ... + E[m_n] = 1 + 1/2 + 1/3 + ... + 1/n \sim \log n + \gamma.$$

Let $\gamma_1 = -0.0728\ldots$ denote the first of the *Stieltjes constants* $\gamma_k,\ k \geq 0$, generalizations of the Euler–Mascheroni constant $\gamma = \gamma_0$ defined by the coefficients of the Laurent series expansion of the (analytically continued) Riemann zeta function $\zeta(s)$ around $s = 1$:

$$\zeta(s) = \frac{1}{s-1} + \sum_{k=0}^{\infty}(-1)^k(s-1)^k\frac{\gamma_k}{k!}. \tag{6}$$

Well-known facts about γ_1 (see, e.g., [2]) together with Theorem 21 give an asymptotic for the norm.

Corollary 23. *As $n \to \infty$, the expected value of the norm over partitions of n obeys the estimate*

$$E[N] \ \sim \ e^{-\gamma_1 n}.$$

Remark 24. In other words, $\lim_{n\to\infty}\frac{1}{n}\log E[N] = -\gamma_1 = 0.0728....$

It is interesting to see another connection between the partition norm and the Euler– Mascheroni constant, passing through the Riemann zeta function. In the next section we explore further zeta function connections.

4. Partition Zeta Functions and Analogs of Arithmetic Functions

In [15] the first author introduced a broad class of *partition zeta functions* arising from a fusion of Euler's product formulas for both the partition generating function and the Riemann zeta function, in which the norm $N(\lambda)$ is the pivotal object.

Definition 25. In analogy to the Riemann zeta function $\zeta(s) := \sum_{n=1}^{\infty} n^{-s}$ (convergent for $\mathrm{Re}(s) > 1$), for a proper subset $\mathcal{P}' \subset \mathcal{P}$ and value $s \in \mathbb{C}$ for which the following series converges, we define a *partition zeta function* to be the sum over partitions in $\mathcal{P}'$:

$$\zeta_{\mathcal{P}'}(s) := \sum_{\lambda\in\mathcal{P}'}\frac{1}{N(\lambda)^s}. \tag{7}$$

If we let $\mathcal{P}'$ equal the partitions $\mathcal{P}_{\mathbb{X}}$ whose parts all lie in some subset $\mathbb{X} \subset \mathbb{N},\ 1 \notin \mathbb{X}$, there is also an Euler product convergent for $\mathrm{Re}(s) > 1$:

$$\zeta_{\mathcal{P}_{\mathbb{X}}}(s) = \prod_{n\in\mathbb{X}}\left(1 - \frac{1}{n^s}\right)^{-1}. \tag{8}$$

Right away, equation (8) of Definition 25 connects with Theorem 4.

Theorem 26 (Schneider–Schneider). *For $\mathbb{X} \subset \mathbb{N},\ 1 \notin \mathbb{X},\ 0 < x < e^{-1}, s := -\log x \in \mathbb{R}^+$, we have*

$$\zeta_{\mathcal{P}_{\mathbb{X}}}(s) = \prod_{n\in\mathbb{X}}\frac{1}{1 - x^{\log n}}.$$

Proof. This is an instance of [14, Remark 4.6]; we flesh out the proof sketched there. For $x \in \mathbb{R}$, we have $x^{\log n} = (e^{\log x})^{\log n} = (e^{\log n})^{\log x} = n^{\log x} = n^{-s}$, giving the product side of the identity. Similarly, we can rewrite $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ as

$$\zeta(-\log x) = \sum_{n=1}^{\infty} x^{\log n}. \tag{9}$$

Noting $s = -\log x > 1$ for $0 < x < e^{-1}$, thus $\zeta(s) = \zeta(-\log x)$ converges absolutely, gives convergence in the theorem. $\square$

Remark 27. Theorem 4 is the case $\mathbb{X} = \mathbb{N} \setminus \{1\}$ of Theorem 26.

By the Euler product formula for $\zeta(s)$, the usual Riemann zeta function represents the partition zeta function $\zeta_{\mathcal{P}_{\mathbb{P}}}(s)$, i.e., $\mathbb{X} = \mathbb{P}$.

Partition zeta sums over other proper subsets of $\mathcal{P}$ can yield nice closed- form results of varying natures. To see how choice of subset influences the evaluations, fix $s = 2$ and sum over three unrelated subsets of $\mathcal{P}$: partitions $\mathcal{P}_{\mathbb{P}}$ into prime parts, partitions $\mathcal{P}_{2\mathbb{N}}$ into even parts, and partitions $\mathcal{D}$ into distinct parts.

Theorem 28 (Schneider). *We have the identities*

$$\zeta_{\mathcal{P}_{\mathbb{P}}}(2) = \frac{\pi^2}{6}, \qquad \zeta_{\mathcal{P}_{2\mathbb{N}}}(2) = \frac{\pi}{2}, \qquad \zeta_{\mathcal{D}}(2) = \frac{\sinh \pi}{\pi}.$$

The proofs (see [17]) involve variations of Euler's product formula for $\sin x$.

Notice how different choices of partition subsets induce very different partition zeta values for fixed s. Interestingly, differing powers of π appear in all three examples given. Another, slightly complicated-looking formula involving π arises if we take $s = 3$ (noting the value of the case $\zeta(3)$ is unknown) and sum on nuclear partitions defined above.

Let us denote the set of nuclear partitions (partitions with no 1's) by $\mathcal{N}$, and recall $\widetilde{P}(\nu)$ enumerates nuclear partitions of norm ν, i.e., multiplicative partitions of ν.

Theorem 29. *We have that*

$$\zeta_{\mathcal{N}}(3) = \sum_{\nu=1}^{\infty} \frac{\widetilde{P}(\nu)}{\nu^3} = \frac{3\pi}{\cosh\left(\frac{1}{2}\pi\sqrt{3}\right)}.$$

Proof. That the partition zeta function equals the right-hand value is [15, Corollary 2.3]. Now, using Theorem 26 on the left side of Theorem 4, and noting that $s = -\log x$ gives $x^{\log \nu} = \nu^{-s}$ on the right side, yields $\zeta_{\mathcal{N}}(s) = \sum_{\nu=1}^{\infty} \widetilde{P}(\nu)\nu^{-s}$; setting $s = 3$ (i.e., $x = e^{-3}$) completes the proof. $\square$

These partition formulas for π are interesting, but they look a little too disparate to comprise a *family* like Euler's values $\zeta(2k) = \pi^{2k}\times$ "rational number". There is at least one (non-Riemann) class of partition zeta functions that yields nice evaluations like this.

Definition 30. We define

$$\zeta_{\mathcal{P}}(\{s\}^k) := \sum_{\ell(\lambda)=k} \frac{1}{N(\lambda)^s},$$

with the sum taken over all partitions of fixed length $k \geq 0$, with $\zeta_{\mathcal{P}}(\{s\}^0) := N(\emptyset)^{-s} = 1$.

The $k = 1$ case is $\zeta(s)$. At argument $s = 2$ these partition zeta functions yield explicit values closely related to Euler's even-argument zeta evaluations.

Theorem 31 (Schneider). *For $s = 2,\ k \geq 1$, we have the identity*

$$\zeta_{\mathcal{P}}(\{2\}^k) = \frac{2^{2k-1}-1}{2^{2k-2}}\zeta(2k),$$

and analogous formulas exist for partitions into distinct parts.

So these particular partition zeta values are rational multiples of Euler's zeta values (and of π^{2k}). Note that if we set $k = 0$ and solve the above identity for $\zeta(0)$, we conclude formally that $\zeta(0) = \frac{2^{-2}}{2^{-1}-1}\zeta_{\mathcal{P}}(\{2\}^0) = -1/2$, which is the correct value for $\zeta(0)$ under analytic continuation. This raises the question of analytic continuation for the function $\zeta_{\mathcal{P}}(\{s\}^k)$.

The preceding zeta formulas and numerous others, including general structural relations, are proved in [15]. In [11], the authors prove other facts about partition zeta functions, including a farther-reaching follow-up to Theorem 31.

Theorem 32 (Ono–Rolen–Schneider). *For $m \geq 1, k \geq 1$, we have*

$$\zeta_{\mathcal{P}}(\{2m\}^k) = \pi^{2mk} \times \text{“rational number”}.$$

These zeta sums over partitions of fixed length do indeed form a family like Euler's zeta values for positive even arguments. In [18], we give a closed formula for general $s \in \mathbb{C}$ as a combination of Riemann zeta functions and MacMahon coefficients, via both analytic and combinatorial proofs.

Theorem 33 (Schneider–Sills). *For $\mathrm{Re}(s) > 1, k \geq 1$, we have*

$$\zeta_{\mathcal{P}}(\{s\}^k) = \sum_{\lambda \vdash k} \frac{\zeta(s)^{m_1}\zeta(2s)^{m_2}\zeta(3s)^{m_3}\cdots\zeta(ks)^{m_k}}{N(\lambda)\ m_1!\ m_2!\ m_3!\ \cdots\ m_k!}.$$

Thus $\zeta_{\mathcal{P}}(\{s\}^k)$ inherits analytic continuation as well as trivial zeroes at $s = -2, -4, -6, ...,$ from $\zeta(s)$, has poles at $s = 1, 1/2, 1/3, 1/4, ..., 1/k$ with the order of the pole $s = 1/i$ being $\lfloor k/i \rfloor, 1 \leq i \leq k$, and indeed equals $\pi^{2mk}\times$ "rational number" for $s = 2m, m \geq 1$.

Zeta functions are only the trail-head of many paths connecting partition theory and classical multiplicative number theory, as shown for instance by the first author and his collaborators in [11, 12, 15, 16, 17]. In addition to the zeta function analogs seen already, there are partition-theoretic versions of classical arithmetic functions such as the Möbius function $\mu(n)$, the sum of divisors function $\sigma(n)$, the Euler phi function $\varphi(n)$, etc. *Partition Dirichlet series* are also defined for any function $f : \mathcal{P} \to \mathbb{C}$ defined on partitions (see [11, 17]), viz. for $\mathcal{P}' \subseteq \mathcal{P}$ we set

$$\mathcal{D}_{\mathcal{P}'}(f, s) := \sum_{\lambda \in \mathcal{P}'} \frac{f(\lambda)}{N(\lambda)^s} \tag{10}$$

where convergence depends on f and $s \in \mathbb{C}$ as well as the subset $\mathcal{P}'$.

To give a concrete example, the *partition phi function* $\varphi_{\mathcal{P}}(\lambda)$ is defined in [16] in terms of the norm:

$$\varphi_{\mathcal{P}}(\lambda) := N(\lambda) \prod_{\substack{\lambda_i \in \lambda \\ \text{no repetition}}} \left(1 - \frac{1}{\lambda_i}\right), \tag{11}$$

where the product is taken over the parts λ_i of λ without repetition. This function fits into partition theory in an almost identical manner to $\varphi(n)$ in elementary number theory, as the following pair of identities suggests.

For $\delta, \lambda \in \mathcal{P}$, we say δ is a *subpartition* of λ and write "$\delta|\lambda$" if all the parts of δ are also parts of λ including their frequencies.

Theorem 34 (Schneider). *We have the following identities:*

$$\sum_{\delta|\lambda} \varphi_{\mathcal{P}}(\delta) = N(\lambda), \qquad \sum_{\lambda \in \mathcal{P}_{\mathbb{X}}} \frac{\varphi_{\mathcal{P}}(\lambda)}{N(\lambda)^s} = \frac{\zeta_{\mathcal{P}_{\mathbb{X}}}(s-1)}{\zeta_{\mathcal{P}_{\mathbb{X}}}(s)} \quad (\mathrm{Re}(s) > 2),$$

where in the first sum, "$\delta|\lambda$" means the sum is taken over subpartitions of λ, and the second sum holds for any subset $\mathbb{X} \subset \mathbb{N}$.

The second summation above represents a partition Dirichlet series, and actually converges even for $\mathbb{X} = \mathbb{N}$ since $\varphi_{\mathcal{P}}$ vanishes on partitions with any part $= 1$ (but the ratio of zeta functions on the right-hand side then becomes indeterminate since $\zeta_{\mathcal{P}}(z)$ is infinite for all $z \in \mathbb{C}$). These formulas generalize the classical identities

$$\sum_{d|n} \varphi(n) = n, \qquad \sum_{n=1}^{\infty} \frac{\varphi(n)}{n^s} = \frac{\zeta(s-1)}{\zeta(s)} \quad (\mathrm{Re}(s) > 2). \tag{12}$$

Other well-known objects and identities from multiplicative number theory also represent special cases of partition-theoretic theorems (see [12, 16, 17] for further reading).

We close with a curious identity connecting the nice family of partition zeta functions described in Theorem 31 to another constant of much interest historically, as well as π.

Theorem 35. *Let* $\phi = \frac{1+\sqrt{5}}{2}$ *denote the golden ratio. Then we have*

$$\frac{\phi\ \pi}{5} \ = \ \sum_{k=0}^{\infty} \frac{\zeta_{\mathcal{P}}(\{2\}^k)}{100^k}.$$

Proof. The equation results from comparing Theorem 33 above with the coefficients of $1/100^k$ in the first identity of [17, Proposition D.2.4]:

$$\phi \ = \ \frac{5}{\pi} \sum_{\lambda \in \mathcal{P}} \frac{\zeta(2)^{m_1}\zeta(4)^{m_2}\zeta(6)^{m_3}\zeta(8)^{m_4}\cdots}{N(\lambda)\ 100^{|\lambda|}\ m_1!\ m_2!\ m_3!\ m_4!\ \cdots}, \tag{13}$$

which itself follows from trigonometric facts about the golden ratio, Euler's product formula for the sine function, the Maclaurin series for $-\log(1-x)$ and Faà di Bruno's formula. □

Remark 36. We note that by Theorem 31, the right-hand sum of Theorem 35 may be rewritten in terms of $\zeta(2k)$.

Due to the tantalizing connections we find it to have in the literature, as well as in our research, the partition norm seems worthy of further study in its own right.

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