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# REPRESENTATION NUMBERS OF SPINOR REGULAR TERNARY QUADRATIC FORMS

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**Abstract**

Recently, Earnest and Haensch established that there are exactly twenty-nine (classes of) spinor regular primitive positive-definite integral ternary quadratic forms, which are not regular. In this paper we determine explicit formulas for the representation numbers of the twenty-seven of these ternary quadratic forms, which are alone in their spinor genus. For the remaining two spinor regular forms, which are not alone in their spinor genus, we determine their representation numbers for even positive integers. As a consequence of our formulas we are able to determine exactly which positive integers are represented by the twenty-seven ternary quadratic forms alone in their spinor genus. The integers represented by six of these forms had been found by Lomadze in 1977 and three of them by Berkovich in 2015, one form of which had already been treated by Lomadze. Our method is a new approach and quite different from the methods of said authors.

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## 1. Introduction, Overview and Notation

In this paper a "form" or a "ternary" or a "ternary form" always means a ternary quadratic form, which is integral, primitive, and positive-definite. The study of the representation of positive integers by such forms has its origins in the work of Legendre, Gauss and Dirichlet, who proved that a positive integer $n$ is represented by the form $x^2 + y^2 + z^2$ if and only if $n$ is not of the form $4^k(8l + 7)$ for any non-negative integers $k$ and $l$. Later, Ramanujan in his work on quaternary quadratic forms $ax^2+by^2+cz^2+dt^2$ that represent all positive integers, found that he needed to know which positive integers were represented by ternary forms such as $x^2+y^2+2z^2$ and $x^2 + y^2 + 3z^2$ [21]. The integers represented by these and other ternary forms were determined by Dickson in 1927 [7]. This led Dickson to the concept of a regular form. Regular ternary forms were studied extensively by Dickson as well as by his students Jones and Pall [13].

A ternary form is said to be regular if it represents all the integers represented by its genus. The genus of a form is defined in [22, p. 72]. Due to the work of Jagy, Kaplansky and Schiemann [11], it is now known that there are (up to equivalence) at most 913 ternaries which are regular. Jones [12] proved in 1931 that a regular ternary represents all positive integers except for those that lie in the union of finitely many progressions $\{A^k(Bl + C) \mid k, l = 0, 1, 2, \ldots\}$, where $A$, $B$ and $C$ are positive integers with $C < B$. The excluded progressions for all 913 ternaries were given explicitly in 2019 by Doyle, Muskat, Pehlivan and Williams [8], where references to earlier determinations for some regular forms were provided.

In this paper we extend the work in [8] to the determination of the integers represented by the spinor regular ternaries; that is, those ternaries that represent all the integers represented by their spinor genus. The concept of a spinor genus is defined in [22, p. 104]. Earnest and Haensch [9] established in 2019 that there are exactly 29 such forms that are not regular. These are listed in Table 1.1.

The number in parenthesis after the identification number indicates the position of the ternary in the original table of Earnest and Haensch [9, p. 214]. Of these forms, 27 are alone in their spinor genus, and for each of these forms we obtain explicit formulas for the number of representations of a positive integer by the form. These 27 spinor regular ternaries are not regular and so are not alone in their genus. Thus their representation numbers cannot be determined directly from Siegel's formula (stated here in Proposition 2.2). Therefore, we take a different approach. Our approach involves the new concept of "derivability" introduced by the first and fifth authors in their recent paper [1]. If a ternary form $f$ is derivable from another ternary form $g$, then the representation number of $f$ can be given in terms of the representation number of $g$. In particular, if $g$ is alone in its genus its representation number can be determined by means of Siegel's formula, and hence a formula for the representation number of $f$ can be deduced. We show

| Identification Number | Form | Discriminant $= 2^r$ |
|---|---|---|
| A1 (1) | (2,2,5,2,2,0) | $64 = 2^6$ |
| A2 (4) | (1,4,9,4,0,0) | $128 = 2^7$ |
| A3 (5) | (2,5,8,4,0,2) | $256 = 2^8$ |
| A4 (6) | (4,4,5,0,4,0) | $256 = 2^8$ |
| A5 (12) | (4,9,9,2,4,4) | $1024 = 2^{10}$ |
| A6 (13) | (4,5,13,2,0,0) | $1024 = 2^{10}$ |
| A7 (14) | (5,8,8,0,4,4) | $1024 = 2^{10}$ |
| A8 (17) | (4,8,17,0,4,0) | $2048 = 2^{11}$ |
| A9 (19) | (9,9,16,8,8,2) | $4096 = 2^{12}$ |
| A10 (20) | (4,9,32,0,0,4) | $4096 = 2^{12}$ |
| A11 (21) | (5,13,16,0,0,2) | $4096 = 2^{12}$ |
| A12 (25) | (9,17,32,-8,8,6) | $16384 = 2^{14}$ |
| A13 (26) | (9,16,36,16,4,8) | $16384 = 2^{14}$ |
| Identification Number | Form | Discriminant $= 2^r \cdot 3^s$ |
| B1 (2) | (3,3,4,0,0,3) | $108 = 2^2 \cdot 3^3$ |
| B2 (3) | (3,4,4,4,3,3) | $108 = 2^2 \cdot 3^3$ |
| B3 (7) | (1,7,12,0,0,1) | $324 = 2^2 \cdot 3^4$ |
| B4 (9) | (3,7,7,5,3,3) | $432 = 2^4 \cdot 3^3$ |
| B5 (10) | (4,4,9,0,0,4) | $432 = 2^4 \cdot 3^3$ |
| B6 (11) | (3,4,9,0,0,0) | $432 = 2^4 \cdot 3^3$ |
| B7 (16) | (4,9,12,0,0,0) | $1728 = 2^6 \cdot 3^3$ |
| B8 (18) | (4,9,28,0,4,0) | $3888 = 2^4 \cdot 3^5$ |
| B9 (23) | (9,16,16,16,0,0) | $6912 = 2^8 \cdot 3^3$ |
| B10 (24) | (13,13,16,-8,8,10) | $6912 = 2^8 \cdot 3^3$ |
| B11 (27) | (9,16,48,0,0,0) | $27648 = 2^{10} \cdot 3^3$ |
| B12 (28) | (9,16,112,16,0,0) | $62208 = 2^8 \cdot 3^5$ |
| Identification Number | Form | Discriminant $= 2^r \cdot 7^s$ |
| C1 (8) | (2,7,8,7,1,0) | $343 = 7^3$ |
| C2 (15) | (7,8,9,6,7,0) | $1372 = 2^2 \cdot 7^3$ |
| C3 (22) | (8,9,25,2,4,8) | $5488 = 2^4 \cdot 7^3$ |
| C4 (29) | (29,32,36,32,12,24) | $87808 = 2^8 \cdot 7^3$ |

Table 1.1: Spinor regular positive-definite ternary quadratic forms $(a, b, c, d, e, f) = ax^2 + by^2 + cz^2 + dyz + ezx + fxy$ which are not regular

that the 27 spinor regular ternaries, which are not regular and alone in their spinor genus, are each derivable from a suitable regular ternary. This is how we determine explicit formulas for the representation numbers of the 27 spinor regular ternaries which are not regular and alone in their spinor genus. By examining for which positive integers $n$ the representation number vanishes, we determine exactly which integers are not represented by each of the 27 forms. The excluded integers for each of the 27 spinor regular ternaries lie in the union of finitely many progressions $\{A^k(Bl+C) \mid k,l=0,1,2,\ldots\}$, where $A$, $B$, $C$ are positive integers with $C<B$, and one or more exceptional squareclasses. These are given explicitly in Table A.17 of the Appendix.

For the two remaining spinor regular forms which are not alone in their spinor genus, we determine their representation numbers for even positive integers $n$ by relating these to the representation numbers of a divisor of $n$ (such as $\frac{n}{4}$) by a spinor regular form which is alone in its spinor genus. We now introduce some notation.

As usual we let $\mathbb{N}$, $\mathbb{N}_0$ and $\mathbb{Z}$ denote the set of positive integers, non-negative integers and integers, respectively. The matrix of the ternary quadratic form $ax^2+by^2+cz^2+dyz+ezx+fxy$ is

$$\begin{bmatrix} 2a & f & e \\ f & 2b & d \\ e & d & 2c \end{bmatrix}$$

and its discriminant is

$$\frac{1}{2}\begin{vmatrix} 2a & f & e \\ f & 2b & d \\ e & d & 2c \end{vmatrix} = 4abc + def - ad^2 - be^2 - cf^2.$$

It is convenient to introduce the following notation. For $t \in \{3,4,7,8\}$ we let

$$M_t := \text{set generated by 1 and primes } p \text{ such that } \left(\frac{-t}{p}\right) = 1.$$

Here $\left(\frac{*}{p}\right)$ is the Legendre symbol modulo $p$, if $p$ is an odd prime, and the Kronecker symbol, if $p=2$. Thus, we have

$$\begin{aligned}
M_3 &= \text{set generated by 1 and primes } p \equiv 1 \pmod 3,\\
M_4 &= \text{set generated by 1 and primes } p \equiv 1 \pmod 4,\\
M_7 &= \text{set generated by 1 and primes } p \equiv 1,2,4 \pmod 7, \text{ so that } 2 \in M_7,\\
M_8 &= \text{set generated by 1 and primes } p \equiv 1,3 \pmod 8.
\end{aligned}$$

For $k \in \mathbb{N}$ we set $\omega_k = e^{2\pi i/k}$.

The representation number $r(a,b,c,d,e,f;n)$ of the ternary $ax^2+by^2+cz^2+dyz+ezx+fxy$ is defined for all $n \in \mathbb{N}_0$ by

$$r(a,b,c,d,e,f;n) := \text{card}\{(x,y,z) \in \mathbb{Z}^3 \mid ax^2+by^2+cz^2+dyz+ezx+fxy = n\},$$

so that $r(a,b,c,d,e,f;0)=1$. For $n \notin \mathbb{N}_0$ we define $r(a,b,c,d,e,f;n)=0$.

We denote by $\mathcal{H}$ the complex upper-half plane, namely the set of complex numbers $w$ with $\text{Im}(w)>0$. The complex conjugate of the complex number $w$ is denoted by $\overline{w}$. For a positive-definite integral quadratic form $f = f(x_1, \dots, x_n)$ and $w \in \mathcal{H}$ the theta function of $f$ is defined by

$$\theta(f;w) := \sum_{(x_1,\dots,x_n)\in\mathbb{Z}^n} e^{2\pi i w f(x_1,\dots,x_n)}. \tag{1.1}$$

When $f = x^2$ we write $\theta(w)$ for $\theta(f, w)$, so that

$$\theta(w) := \sum_{x=-\infty}^{\infty} e^{2\pi i w x^2}. \tag{1.2}$$

By a classical theorem of Jacobi, we have

$$\theta(w) = \frac{\eta^5(2w)}{\eta^2(w)\eta^2(4w)}, \qquad w \in \mathcal{H}, \tag{1.3}$$

where $\eta(w)$ is the Dedekind eta function, which is defined for all $w \in \mathcal{H}$ by

$$\eta(w) := e^{\pi i w/12} \prod_{m=1}^{\infty} \left(1 - e^{2\pi i m w}\right). \tag{1.4}$$

If $k$ is a positive integer, we denote the squarefree part of $k$ by $\mathrm{sqf}(k)$. When considering the representability of a positive integer $n$ by a ternary quadratic form of discriminant $\Delta$, it is convenient to define the positive squarefree integer $n^*$ by

$$n^* := \mathrm{sqf}(n\Delta), \tag{1.5}$$

the positive squarefree integer $g$ by

$$g := \text{largest positive integer such that } g \mid \mathrm{sqf}(n) \text{ and } (g, 2\Delta) = 1, \tag{1.6}$$

and the positive integer $h$ by

$$h := \text{largest positive integer such that } (h, 2\Delta) = 1 \text{ and } h^2 \mid n. \tag{1.7}$$

Further we define

$$l(n) := \prod_{p \mid h} \left( \sigma\left(p^{\nu_p(h)}\right) - \left(\frac{-n^*}{p}\right) \sigma\left(p^{\nu_p(h)-1}\right)\right), \tag{1.8}$$

where $p$ runs through all the primes dividing $h$, $\nu_p(h)$ is the exponent of the largest power of $p$ dividing $h$, and for $m \in \mathbb{N}$ $\sigma(m)$ denotes the sum of divisors of $m$. From (1.5) we see that, if $4 \mid n$, then

$$\left(\frac{n}{4}\right)^* = \mathrm{sqf}\left(\frac{n}{4}\Delta\right) = \mathrm{sqf}(n\Delta) = n^*, \tag{1.9}$$

and from (1.8) that

$$l\left(\frac{n}{4}\right) = l(n). \tag{1.10}$$

The class number of the imaginary quadratic field $\mathbb{Q}\left(\sqrt{-n^*}\right)$ is denoted by $h\left(\mathbb{Q}\left(\sqrt{-n^*}\right)\right)$. We denote the modular group by $\Gamma$, that is,

$$\Gamma := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\}.$$

Let $N$ be a positive integer. The congruence subgroup $\Gamma_0(N)$ of $\Gamma$ is defined by

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \mid c \equiv 0 \pmod{N} \right\}.$$

The index of $\Gamma_0(N)$ in $\Gamma$ is

$$N \prod_{p \mid N} \left(1 + \frac{1}{p}\right),$$

see [15, p. 23]. The space of modular forms of weight $k$ and character $\chi$ for the congruence subgroup $\Gamma_0(N)$ is denoted by $M_k(\Gamma_0(N), \chi)$. If $D \equiv 0, 1 \pmod 4$ we write $\chi_D$ for the Dirichlet character $\left(\frac{D}{*}\right)$.

In Theorem 2.1 we give the derivability relations showing that the representation numbers for each of the 27 spinor regular ternaries alone in their spinor genus can be obtained from those of a regular ternary alone in its genus. The representation number of the ternary Am (m=1,2,. . . ,13), Bm (m=1,2,. . . ,12), and Cm (m=1,2,3,4) is given in Theorem 4.m, 5.m, and 6.m, respectively. The formulas for the representation numbers all have the same basic form

$$r(f; n) = k_f(n) l(n) h\left(\mathbb{Q}\left(\sqrt{-n^*}\right)\right),$$

except on one or more exceptional square classes. The integers not represented by each of the spinor regular ternaries are given in Table A.17 of the Appendix.

The code to determine the representation numbers of the 29 spinor regular ternaries in Table 1.1 and the 16 regular ternaries in Table A.0 of the Appendix for $n$ up to $3 \cdot 10^6$ was written in C++ and R. The computation times on two Intel Xeon E5-2683 v4 Broadwell @ 2.1 GHz with 16 cores each ranged from 2 hours (form C4) to 45 hours (form A1) for spinor-regular ternaries and from 9 hours to 107 hours for regular ternaries.

## 2. Derivability Relations

The underlying idea behind our approach to evaluating the representation numbers of spinor regular ternary quadratic forms is the concept of a ternary quadratic form $f$ being derivable from another ternary quadratic form $g$. We now formally define this concept.

**Definition 2.1.** Let $f$ and $g$ be positive-definite integral ternary quadratic forms. We say that *$f$ is derivable from $g$ with reach $R$* if there exist complex numbers $a_j, b_d, c_j$; positive integers $R$ and $S$; integers $r_j$ and $s_j$ with $0 \le r_j \le s_j - 1$; and integers $k_j$ and $d_j$ with $d_j < 0, d_j \equiv 0, 1 \pmod 4$ such that for all $w \in \mathcal{H}$

$$\begin{aligned}\theta(f;w) = &\sum_{j=1}^{R} a_j \theta\left(g; w + \frac{j}{R}\right) + \sum_{1<d\mid R} b_d \theta(g; d^2 w)\\ &+ \sum_{j=1}^{S} c_j \sum_{\substack{m=1\\ m \equiv r_j \pmod{s_j}}}^{\infty} \left(\frac{d_j}{m}\right) m e^{2\pi i m^2 k_j w},\end{aligned} \tag{2.1}$$

where the theta function $\theta(f;w)$ is defined in (1.1) and $\left(\frac{d_j}{*}\right)$ is the Legendre-Jacobi-Kronecker symbol for discriminant $d_j$.

The usefulness of such an identity lies in the fact that it relates the representation number of $f$ to that of $g$.

**Proposition 2.1.** *If $f$ and $g$ are positive-definite integral ternary quadratic forms with $f$ derivable from $g$ as in Definition 2.1, then for all $n \in \mathbb{N}$ we have*

$$r(f;n) = \left(\sum_{j=1}^{R} a_j e^{\frac{2\pi i n j}{R}}\right) r(g;n) + \sum_{1<d\mid R} b_d r\left(g; \frac{n}{d^2}\right) + \sum_{j=1}^{S} c_j e_j(n), \tag{2.2}$$

*where*

$$e_j(n) = \begin{cases} 0 & \text{if } \frac{n}{k_j} \ne \text{ square},\\ 0 & \text{if } \frac{n}{k_j} = \text{ square}, \ \sqrt{\frac{n}{k_j}} \not\equiv r_j \pmod{s_j},\\ \left(\frac{d_j}{\sqrt{\frac{n}{k_j}}}\right)\sqrt{\frac{n}{k_j}} & \text{if } \frac{n}{k_j} = \text{ square}, \ \sqrt{\frac{n}{k_j}} \equiv r_j \pmod{s_j}. \end{cases}$$

*Proof.* We let $q := e^{2\pi i w}$ $(w \in \mathcal{H})$ so that $|q| < 1$. We have

$$\theta(f;w) = \sum_{n=0}^{\infty} r(f;n) q^n,$$

$$\theta(g;w) = \sum_{n=0}^{\infty} r(g;n) q^n,$$

$$\sum_{j=1}^{R} a_j \theta\left(g; w + \frac{j}{R}\right) = \sum_{n=0}^{\infty} r(g;n) \left(\sum_{j=1}^{R} a_j e^{\frac{2\pi i n j}{R}}\right) q^n,$$

$$\sum_{1<d|R} b_d\theta(g;d^2w) = \sum_{n=0}^{\infty}\left(\sum_{1<d|R} b_d r\left(g;\frac{n}{d^2}\right)\right)q^n,$$

and

$$\begin{aligned}
&\sum_{j=1}^{S} c_j \sum_{\substack{m=1\\ m\equiv r_j \pmod{s_j}}}^{\infty} \left(\frac{d_j}{m}\right) me^{2\pi i m^2 k_j w}\\
&\quad= \sum_{j=1}^{S} c_j \sum_{\substack{m=1\\ m\equiv r_j \pmod{s_j}}}^{\infty} \left(\frac{d_j}{m}\right) mq^{k_j m^2}\\
&\quad= \sum_{j=1}^{S} c_j \sum_{\substack{n=1\\ n/k_j=\text{ square}\\ \sqrt{n/k_j}\equiv r_j \pmod{s_j}}}^{\infty} \left(\frac{d_j}{\sqrt{\frac{n}{k_j}}}\right)\sqrt{\frac{n}{k_j}}q^n\\
&\quad= \sum_{j=1}^{S} c_j \sum_{n=1}^{\infty} e_j(n)q^n\\
&\quad= \sum_{n=1}^{\infty}\left(\sum_{j=1}^{S} c_j e_j(n)\right)q^n.
\end{aligned}$$

Equating coefficients of $q^n$ ($n \in \mathbb{N}$) in (2.1), we obtain (2.2). Equating the coefficients of $q^0$ in (2.1), we see that the $a_j$ and $b_d$ must satisfy

$$1 = \sum_{j=1}^{R} a_j + \sum_{1<d|R} b_d$$

as $r(f;0) = r(g;0) = 1$. □

In order to determine a formula for the representation number $r(f;n)$ of a spinor regular ternary quadratic form $f$, which is not regular, we seek a regular ternary quadratic form $g$, which is alone in its genus, such that $f$ is derivable from $g$. As $g$ is alone in its genus, $r(g;n)$ can be given explicitly by means of a special case of a formula of Siegel (see for example [4, pp. 374–378]). If we can find such a ternary $g$, then a formula for $r(f;n)$ follows from Proposition 2.1. Siegel's formula requires the concept of a local density, the existence of which was proved by Siegel.

**Definition 2.2.** Let $g$ be a ternary quadratic form. For $n \in \mathbb{N}$ and $p$ a prime the *local density* $d(g,n,p)$ is defined by

$$d(g,n,p) := \lim_{t\to\infty} \frac{\text{number of solutions of } g(x,y,z) \equiv n \pmod{p^t}}{p^{2t}}.$$

**Proposition 2.2.** (Siegel [4, Appendix B.3, pp. 374–378]) *Let $g(x,y,z) = ax^2 + by^2 + cz^2 + dyz + ezx + fxy$ be a positive-definite primitive integral ternary quadratic form, which is alone in its genus. Then for $n \in \mathbb{N}$*

$$r(g;n) = \frac{2\pi\sqrt{n}}{\sqrt{abc + \frac{1}{4}def - \frac{1}{4}(ad^2 + be^2 + cf^2)}} \prod_p d(g,n,p),$$

*where the product is taken over all primes $p$.*

When $g$ is a diagonal ternary quadratic form, which is alone in its genus, Lomadze [18] has given a formula for $r(g;n)$ (valid for all $n \in \mathbb{N}$) using a different method. In his formulas it is convenient to replace the Dirichlet $L$-series that occur by the class number of an appropriate imaginary quadratic field using Dirichlet's class number formula. When $g$ is not diagonal, Siegel's formula can be used in conjunction with the evaluation of local densities given by Yang [24] to give a formula for $r(g;n)$. The formulas for the representation numbers of ternary forms alone in their genus that we require are given in the Appendix (Propositions A.1–A.16).

Theorem 2.1 gives the identities of type (2.1) which show that each of the 27 spinor regular ternaries $f$ which are alone in their spinor genus is derivable from a regular ternary quadratic form $g$ which is alone in its genus. We require the following expansions involving the Dedekind eta function $\eta(w)$ ($w \in \mathcal{H}$), which was defined in (1.4), all of which can be found in or deduced from the work of Köhler [16],[17].

$$\eta^3(8w) = \sum_{n=1}^{\infty} \left(\frac{-4}{n}\right) n e^{2\pi i n^2 w} \qquad [17, \text{Cor. 1.4, p. 8}]$$

$$\frac{\eta^2(3w)\eta^2(12w)}{\eta(6w)} = \sum_{n=1}^{\infty} \left(\frac{-3}{n}\right) n e^{2\pi i n^2 w} \qquad [16, \text{Theorem (3), p. 147}]$$

$$\frac{\eta^5(24w)}{\eta^2(48w)} = \sum_{\substack{n=1 \\ n \equiv 1 \pmod 2}}^{\infty} \left(\frac{-3}{n}\right) n e^{2\pi i n^2 w} \qquad [16, \text{Theorem (2), p. 147}]$$

$$\frac{\eta^2(12w)\eta^2(48w)}{\eta(24w)} = -\frac{1}{2} \sum_{\substack{n=1 \\ n \equiv 0 \pmod 2}}^{\infty} \left(\frac{-3}{n}\right) n e^{2\pi i n^2 w} \qquad [16, \text{Theorem (3), p. 147}]$$

$$\frac{\eta^9(16w)}{\eta^3(8w)\eta^3(32w)} = \sum_{n=1}^{\infty} \left(\frac{-8}{n}\right) n e^{2\pi i n^2 w} \qquad [16, \text{Theorem (4), p. 147}]$$

$$\frac{\eta^5(6w)}{\eta^2(3w)} = \sum_{\substack{n=1 \\ n \equiv 1 \pmod 2}}^{\infty} \left(\frac{-3}{n}\right) n e^{2\pi i n^2 w} - \sum_{\substack{n=1 \\ n \equiv 0 \pmod 2}}^{\infty} \left(\frac{-3}{n}\right) n e^{2\pi i n^2 w} \qquad [16, \text{Theorem (1), p. 147}]$$

We now state our derivability relations.

**Theorem 2.1.** (i) *Let* $f := 2x^2 + 2y^2 + 5z^2 + 2yz + 2zx$ *(form A1). Then with* $g := x^2 + y^2 + 4z^2$ *we have for all* $w \in \mathcal{H}$

$$\theta(f; w) = \sum_{j=1}^{8} a_j \theta\left(g; w + \frac{j}{8}\right) + \theta(g; 4w) - 2\eta^3(8w),$$

*where for* $j \in \{1, 2, \ldots, 8\}$

$$a_j = \begin{cases} -\frac{1}{8}\left(\frac{-1}{j}\right) i & j \equiv 1 \pmod 2, \\ \frac{1}{8}\left(-1 - \left(\frac{-1}{j/2}\right) i\right) & j \equiv 2 \pmod 4, \\ \frac{1}{8}\left(1 + (-1)^{j/4}\right) & j \equiv 0 \pmod 4. \end{cases}$$

(ii) *Let* $f := x^2 + 4y^2 + 9z^2 + 4yz$ *(form A2). Then with* $g := x^2 + y^2 + 8z^2$ *we have for all* $w \in \mathcal{H}$

$$\theta(f; w) = \sum_{j=1}^{4} a_j \theta\left(g; w + \frac{j}{4}\right) + \theta(g; 4w) - 2\eta^3(16w),$$

*where for* $j \in \{1, 2, 3, 4\}$

$$a_j = \begin{cases} \frac{1}{8}\left(-1 - \left(\frac{-1}{j}\right) i\right) & j \equiv 1 \pmod 2, \\ \frac{1}{8}\left(1 + (-1)^{j/2}\right) & j \equiv 0 \pmod 2. \end{cases}$$

(iii) *Let* $f := 2x^2 + 5y^2 + 8z^2 + 4yz + 2xy$ *(form A3). Then with* $g := x^2 + y^2 + 4z^2$ *we have for all* $w \in \mathcal{H}$

$$\theta(f; w) = \sum_{j=1}^{32} a_j \theta\left(g; w + \frac{j}{32}\right) + \theta(g; 16w) - \eta^3(8w) - 2\eta^3(32w),$$

*where for* $j \in \{1, 2, \ldots, 32\}$

$$a_j = \begin{cases} -\frac{1}{96}\left(\frac{-1}{j}\right) i & j \equiv 1 \pmod 2, \\ \frac{1}{96}\left(-1 - 2\left(\frac{-1}{j/2}\right) i\right) & j \equiv 2 \pmod 4, \\ \frac{1}{96}\left(-1 - 6\left(\frac{-1}{j/4}\right) i\right) & j \equiv 4 \pmod 8, \\ \frac{1}{32}\left(-1 - 2\left(\frac{-1}{j/8}\right) i\right) & j \equiv 8 \pmod{16}, \\ \frac{1}{32}\left(3 + 2(-1)^{j/16}\right) & j \equiv 0 \pmod{16}. \end{cases}$$

(iv) *Let* $f := 4x^2 + 4y^2 + 5z^2 + 4zx$ *(form A4). Then with* $g := x^2 + y^2 + 4z^2$ *we have for all* $w \in \mathcal{H}$

$$\theta(f; w) = -\frac{1}{8}\theta\left(g; w + \frac{1}{2}\right) + \frac{1}{8}\theta(g; w) + \theta(g; 4w) - \eta^3(8w).$$

(v) *Let* $f := 4x^2+9y^2+9z^2+2yz+4zx+4xy$ *(form A5). Then with* $g := x^2+4y^2+4z^2$ *we have for all* $w \in \mathcal{H}$

$$\theta(f;w) = \sum_{j=1}^{8} a_j \theta\left(g; w + \frac{j}{8}\right) + \theta(g;4w) - \eta^3(8w),$$

*where for* $j \in \{1,2,\ldots,8\}$

$$a_j = \begin{cases} \frac{\sqrt{2}}{32}\left(\left(\frac{2}{j}\right) - \left(\frac{-2}{j}\right) i\right) & j \equiv 1 \pmod 2, \\ -\frac{1}{16}\left(\frac{-1}{j/2}\right) i & j \equiv 2 \pmod 4, \\ \frac{1}{16}(-1)^{j/4} & j \equiv 0 \pmod 4. \end{cases}$$

(vi) *Let* $f := 4x^2+5y^2+13z^2+2yz$ *(form A6). Then with* $g := x^2+4y^2+5z^2+4yz$ *we have for all* $w \in \mathcal{H}$

$$\theta(f;w) = -\frac{1}{8}\theta\left(g; w + \frac{1}{2}\right) + \frac{1}{8}\theta(g;w) + \theta(g;4w) - \frac{1}{2}\eta^3(8w).$$

(vii) *Let* $f := 5x^2+8y^2+8z^2+4zx+4xy$ *(form A7). Then with* $g := x^2+y^2+4z^2$ *we have for all* $w \in \mathcal{H}$

$$\theta(f;w) = \sum_{j=1}^{32} a_j \theta\left(g; w + \frac{j}{32}\right) + \theta(g;16w) - \frac{1}{2}\eta^3(8w) - 2\eta^3(32w),$$

*where for* $j \in \{1,2,\ldots,32\}$

$$a_j = \begin{cases} -\frac{1}{96}\left(\frac{-1}{j}\right) i & j \equiv 1 \pmod 2, \\ \frac{1}{96}\left(-1 - 2\left(\frac{-1}{j/2}\right) i\right) & j \equiv 2 \pmod 4, \\ -\frac{1}{96} & j \equiv 4 \pmod 8, \\ \frac{1}{32}\left(1 - \left(\frac{-1}{j/8}\right) i\right) & j \equiv 8 \pmod{16}, \\ \frac{1}{32}\left(1 + (-1)^{j/16}\right) & j \equiv 0 \pmod{16}. \end{cases}$$

(viii) *Let* $f := 4x^2+8y^2+17z^2+4zx$ *(form A8). Then with* $g := x^2+2y^2+16z^2$ *we have for all* $w \in \mathcal{H}$

$$\theta(f;w) = \sum_{j=1}^{4} a_j \theta\left(g; w + \frac{j}{4}\right) + \theta(g;4w) - \frac{\eta^9(16w)}{\eta^3(8w)\eta^3(32w)},$$

*where for* $j \in \{1,2,3,4\}$

$$a_j = \begin{cases} -\frac{1}{8}\left(\frac{-1}{j}\right) i & j \equiv 1 \pmod 2, \\ \frac{1}{8}(-1)^{j/2} & j \equiv 0 \pmod 2. \end{cases}$$

(ix) *Let* $f := 9x^2 + 9y^2 + 16z^2 + 8yz + 8zx + 2xy$ *(form A9). Then with* $g := x^2 + y^2 + 4z^2$ *we have for all* $w \in \mathcal{H}$

$$\theta(f;w) = \sum_{j=1}^{16} a_j \theta\left(g; w + \frac{j}{16}\right) + \theta(g;16w) - \frac{1}{2}\eta^3(8w) - \eta^3(32w),$$

*where for* $j \in \{1, 2, \ldots, 16\}$

$$a_j = \begin{cases} -\frac{1}{96}\left(\frac{-1}{j}\right) i & j \equiv 1 \pmod 2, \\ \frac{1}{384}\left(-4 + 3\left(\frac{2}{j/2}\right)\sqrt{2} - 3\left(\frac{-2}{j/2}\right) i\sqrt{2}\right) & j \equiv 2 \pmod 4, \\ \frac{1}{192}\left(2 - 3\left(\frac{-1}{j/4}\right) i\right) & j \equiv 4 \pmod 8, \\ \frac{1}{192}\left(2 + 3(-1)^{j/8}\right) & j \equiv 0 \pmod 8. \end{cases}$$

(x) *Let* $f := 4x^2 + 9y^2 + 32z^2 + 4xy$ *(form A10). Then with* $g := x^2 + 8y^2 + 8z^2$ *we have for all* $w \in \mathcal{H}$

$$\theta(f;w) = -\frac{1}{8}\theta\left(g; w + \frac{1}{2}\right) + \frac{1}{8}\theta(g;w) + \theta(g;4w) - \frac{1}{2}\eta^3(8w).$$

(xi) *Let* $f := 5x^2 + 13y^2 + 16z^2 + 2xy$ *(form A11). Then with* $g := x^2 + y^2 + 4z^2$ *we have for all* $w \in \mathcal{H}$

$$\theta(f;w) = \sum_{j=1}^{16} a_j \theta\left(g; w + \frac{j}{16}\right) + \theta(g;16w) - \eta^3(32w),$$

*where for* $j \in \{1, 2, \ldots, 16\}$

$$a_j = \begin{cases} -\frac{1}{96}\left(\frac{-1}{j}\right) & j \equiv 1 \pmod 2, \\ \frac{1}{384}\left(-4 - 3\left(\frac{2}{j/2}\right)\sqrt{2} + 3\left(\frac{-2}{j/2}\right) i\sqrt{2}\right) & j \equiv 2 \pmod 4, \\ \frac{1}{96} - \frac{1}{64}\left(\frac{-1}{j/4}\right) i & j \equiv 4 \pmod 8, \\ \frac{1}{192}\left(2 + 3(-1)^{j/8}\right) & j \equiv 0 \pmod 8. \end{cases}$$

(xii) *Let* $f := 9x^2 + 17y^2 + 32z^2 - 8yz + 8zx + 6xy$ *(form A12). Then with* $g := x^2 + y^2 + 4z^2$ *we have for all* $w \in \mathcal{H}$

$$\theta(f;w) = \sum_{j=1}^{128} a_j \theta\left(g; w + \frac{j}{128}\right) + \theta(g;64w) - \frac{1}{4}\eta^3(8w) - \frac{1}{2}\eta^3(32w) - 2\eta^3(128w),$$

*where for* $j \in \{1, 2, 3, \ldots, 128\}$

$$a_j = \begin{cases} -\left(\frac{-1}{j}\right)\frac{i}{384} & j \equiv 1 \pmod 2, \\ \frac{1}{384}\left(-1 - 2\left(\frac{-1}{j/2}\right)i\right) & j \equiv 2 \pmod 4, \\ -\frac{1}{384} & j \equiv 4 \pmod 8, \\ \frac{1}{384}\left(3 - 2\left(\frac{-1}{j/8}\right)i\right) & j \equiv 8 \pmod{16}, \\ \frac{1}{768}\left(2 + 3\left(\frac{2}{j/16}\right)\sqrt{2} - 3\left(\frac{-2}{j/16}\right)i\sqrt{2}\right) & j \equiv 16 \pmod{32}, \\ \frac{1}{384}\left(5 - 3\left(\frac{-1}{j/32}\right)i\right) & j \equiv 32 \pmod{64}, \\ \frac{1}{384}\left(5 + 3(-1)^{j/64}\right) & j \equiv 0 \pmod{64}. \end{cases}$$

(xiii) *Let* $f := 9x^2 + 16y^2 + 36z^2 + 16yz + 4zx + 8xy$ *(form A13). Then with* $g := x^2 + y^2 + 4z^2$ *we have for all* $w \in \mathcal{H}$

$$\theta(f; w) = \sum_{j=1}^{64} a_j \theta\left(g; w + \frac{j}{64}\right) + \theta(g; 16w) - \frac{1}{4}\eta^3(8w) - \eta^3(32w),$$

*where for* $j \in \{1, 2, 3, \ldots, 64\}$

$$a_j = \begin{cases} \frac{1}{192}\left(1 + \left(\frac{-1}{j}\right)i\right) & j \equiv 1 \pmod 2, \\ \frac{\sqrt{2}}{384}\left(\left(\frac{2}{j/2}\right) - \left(\frac{-2}{j/2}\right)i\right) & j \equiv 2 \pmod 4, \\ \frac{1}{192}\left(-2 - \left(\frac{-1}{j/4}\right)i\right) & j \equiv 4 \pmod 8, \\ \frac{1}{256}\left(-4 + \left(\frac{2}{j/8}\right)\sqrt{2} - \left(\frac{-2}{j/8}\right)i\sqrt{2}\right) & j \equiv 8 \pmod{16}, \\ \frac{1}{384}\left(-2 - 3\left(\frac{-1}{j/16}\right)i\right) & j \equiv 16 \pmod{32}, \\ \frac{1}{384}\left(-2 + 3(-1)^{j/32}\right) & j \equiv 0 \pmod{32}. \end{cases}$$

(xiv) *Let* $f := 3x^2 + 3y^2 + 4z^2 + 3xy$ *(form B1). Then with* $g := x^2 + y^2 + 4z^2 + xy$ *we have for all* $w \in \mathcal{H}$

$$\theta(f; w) = \sum_{j=1}^{3} a_j \theta\left(g; w + \frac{j}{3}\right) - \frac{\eta^2(3w)\eta^2(12w)}{\eta(6w)} - 2\frac{\eta^5(6w)}{\eta^2(3w)},$$

*where for* $j \in \{1, 2, 3\}$

$$a_j = \begin{cases} \frac{1}{12}\left(3 - \left(\frac{-3}{j}\right)i\sqrt{3}\right) & j \not\equiv 0 \pmod 3, \\ \frac{1}{2} & j \equiv 0 \pmod 3. \end{cases}$$

(xv) *Let* $f := 3x^2 + 4y^2 + 4z^2 + 4yz + 3zx + 3xy$ *(form B2). Then with* $g := x^2 + 2y^2 + 2z^2 + yz + zx + xy$ *we have for all* $w \in \mathcal{H}$

$$\theta(f; w) = \sum_{j=1}^{3} a_j \theta\left(g; w + \frac{j}{3}\right) - \frac{\eta^2(3w)\eta^2(12w)}{\eta(6w)},$$

*where for* $j \in \{1,2,3\}$

$$a_j = \begin{cases} \frac{1}{12}\left(3-\left(\frac{-3}{j}\right)i\sqrt{3}\right) & j \not\equiv 0 \pmod 3, \\ \frac{1}{2} & j \equiv 0 \pmod 3. \end{cases}$$

(xvi) *Let* $f := x^2+7y^2+12z^2+xy$ *(form B3). Then with* $g := x^2+y^2+12z^2+xy$ *we have for all* $w \in \mathcal{H}$

$$\theta(f;w) = \sum_{j=1}^{9} a_j \theta\left(g; w+\frac{j}{9}\right) + \frac{\eta^2(9w)\eta^2(36w)}{\eta(18w)} - 4\frac{\eta^5(72w)}{\eta^2(144w)},$$

*where for* $j \in \{1,2,\ldots,9\}$

$$a_j = \begin{cases} \frac{1}{36}(3-i\left(\frac{-3}{j}\right)\sqrt{3}) & j \equiv 1,2 \pmod 3, \\ \frac{1}{18}(2-i\left(\frac{-3}{j/3}\right)\sqrt{3}) & j \equiv 3,6 \pmod 9, \\ \frac{5}{18} & j \equiv 0 \pmod 9. \end{cases}$$

(xvii) *Let* $f := 4x^2+4y^2+9z^2+4xy$ *(form B5). Then with* $g := x^2+3y^2+9z^2$ *we have for all* $w \in \mathcal{H}$

$$\theta(f;w) = \sum_{j=1}^{8} a_j \theta\left(g; w+\frac{j}{8}\right) + \frac{1}{2}\theta(g;4w) - \frac{\eta^2(3w)\eta^2(12w)}{\eta(6w)},$$

*where for* $j \in \{1,2,\ldots,8\}$

$$a_j = \begin{cases} \frac{\sqrt{2}}{64}\left(-\left(\frac{2}{j}\right)+\left(\frac{-2}{j}\right)i\right) & j \equiv 1 \pmod 2, \\ \frac{1}{32}\left(4-5\left(\frac{-1}{j/2}\right)i\right) & j \equiv 2 \pmod 4, \\ \frac{1}{32}\left(4+5(-1)^{j/4}\right) & j \equiv 0 \pmod 4. \end{cases}$$

(xviii) *Let* $f := 3x^2+4y^2+9z^2$ *(form B6). Then with* $g := x^2+3y^2+9z^2$ *we have for all* $w \in \mathcal{H}$

$$\theta(f;w) = \sum_{j=1}^{8} a_j \theta\left(g; w+\frac{j}{8}\right) + \frac{1}{2}\theta(g;4w) - 2\frac{\eta^5(24w)}{\eta^2(48w)} + \frac{\eta^2(3w)\eta^2(12w)}{\eta(6w)},$$

*where for* $j \in \{1,2,\ldots,8\}$

$$a_j = \begin{cases} \frac{\sqrt{2}}{64}\left(\left(\frac{2}{j}\right)-\left(\frac{-2}{j}\right)i\right) & j \equiv 1 \pmod 2, \\ \frac{1}{32}\left(4+5\left(\frac{-1}{j/2}\right)i\right) & j \equiv 2 \pmod 4, \\ \frac{1}{32}(4+11(-1)^{j/4}) & j \equiv 0 \pmod 4. \end{cases}$$

(xix) *Let* $f := 4x^2 + 9y^2 + 12z^2$ *(form B7). Then with* $g := x^2 + 3y^2 + 9z^2$ *we have for all* $w \in \mathcal{H}$

$$\theta(f; w) = \sum_{j=1}^{8} a_j \theta\left(g; w + \frac{j}{8}\right) + \theta(g; 4w) - \frac{\eta^5(24w)}{\eta^2(48w)},$$

*where for* $j \in \{1, 2, \dots, 8\}$

$$a_j = \begin{cases} \frac{\sqrt{2}}{64}\left(\left(\frac{2}{j}\right) - \left(\frac{-2}{j}\right) i\right) & j \equiv 1 \pmod{2}, \\ -\frac{3}{32}\left(\frac{-1}{j/2}\right) i & j \equiv 2 \pmod{4}, \\ \frac{3}{32}(-1)^{j/4} & j \equiv 0 \pmod{4}. \end{cases}$$

(xx) *Let* $f := 4x^2 + 9y^2 + 28z^2 + 4zx$ *(form B8). Then with* $g := x^2 + 4y^2 + 4z^2 + 4yz$ *we have for all* $w \in \mathcal{H}$

$$\theta(f; w) = \sum_{j=1}^{3} a_j \theta\left(g; w + \frac{j}{3}\right) + \theta(g; 9w) - \frac{1}{3}\frac{\eta^2(3w)\eta^2(12w)}{\eta(6w)},$$

*where for* $j \in \{1, 2, 3\}$

$$a_j = \begin{cases} \frac{1}{36}\left(-1 - \left(\frac{-3}{j}\right) i\sqrt{3}\right) & j \not\equiv 0 \pmod{3}, \\ \frac{1}{18} & j \equiv 0 \pmod{3}. \end{cases}$$

(xxi) *Let* $f := 9x^2 + 16y^2 + 16z^2 + 16yz$ *(form B9). Then with* $g := x^2 + 3y^2 + 9z^2$ *we have for all* $w \in \mathcal{H}$

$$\theta(f; w) = \sum_{j=1}^{8} a_j \theta\left(g; w + \frac{j}{8}\right) + \frac{5}{4}\theta(g; 4w) - \frac{3}{2}\frac{\eta^5(24w)}{\eta^2(48w)} + \frac{1}{2}\frac{\eta^2(3w)\eta^2(12w)}{\eta(6w)},$$

*where for* $j \in \{1, 2, \dots, 8\}$

$$a_j = \begin{cases} \frac{\sqrt{2}}{32}\left(\left(\frac{2}{j}\right) - \left(\frac{-2}{j}\right) i\right) & j \equiv 1 \pmod{2}, \\ \frac{1}{16}\left(-1 - \left(\frac{-1}{j/2}\right) i\right) & j \equiv 2 \pmod{4}, \\ \frac{1}{16}(-1 + (-1)^{j/4}) & j \equiv 0 \pmod{4}. \end{cases}$$

(xxii) *Let* $f := 13x^2 + 13y^2 + 16z^2 - 8yz + 8zx + 10xy$ *(form B10). Then with* $g := x^2 + 3y^2 + 9z^2$ *we have for all* $w \in \mathcal{H}$

$$\theta(f; w) = \sum_{j=1}^{8} a_j \theta\left(g; w + \frac{j}{8}\right) + \frac{5}{4}\theta(g; 4w) - \frac{1}{2}\frac{\eta^5(24w)}{\eta^2(48w)} + \frac{1}{2}\frac{\eta^2(3w)\eta^2(12w)}{\eta(6w)},$$

*where for* $j \in \{1, 2, \ldots, 8\}$

$$a_j = \begin{cases} \frac{\sqrt{2}}{64}\left(-\left(\frac{2}{j}\right) + \left(\frac{-2}{j}\right) i\right) & j \equiv 1 \pmod 2, \\ \frac{1}{32}\left(-2 - \left(\frac{-1}{j/2}\right) i\right) & j \equiv 2 \pmod 4, \\ \frac{1}{32}\left(-2 + (-1)^{j/4}\right) & j \equiv 0 \pmod 4. \end{cases}$$

(xxiii) *Let* $f := 9x^2 + 16y^2 + 112z^2 + 16yz$ *(form B12). Then with* $g := x^2 + 16y^2 + 16z^2 + 16yz$ *we have for all* $w \in \mathcal{H}$

$$\theta(f; w) = \sum_{j=1}^{3} a_j \theta\left(g; w + \frac{j}{3}\right) + \theta(g; 9w) - \frac{1}{2}\frac{\eta^5(24w)}{\eta^2(48w)} + \frac{1}{6}\frac{\eta^2(3w)\eta^2(12w)}{\eta(6w)},$$

*where for* $j \in \{1, 2, 3\}$

$$a_j = \begin{cases} \frac{1}{36}\left(-1 - \left(\frac{-3}{j}\right) i\sqrt{3}\right) & j \not\equiv 0 \pmod 3, \\ \frac{1}{18} & j \equiv 0 \pmod 3. \end{cases}$$

(xxiv) *Let* $f := 2x^2+7y^2+8z^2+7yz+zx$ *(form C1). Then with* $g := x^2+y^2+2z^2+zx$ *we have for all* $w \in \mathcal{H}$

$$\theta(f; w) = \sum_{j=1}^{7} a_j \theta\left(g; w + \frac{j}{7}\right) - \sum_{n=1}^{\infty}\left(\frac{-7}{n}\right) n e^{2\pi i n^2 w},$$

*where for* $j \in \{1, 2, 3, 4, 5, 6, 7\}$

$$a_j = \begin{cases} \frac{1}{56}\left(7 - \left(\frac{-7}{j}\right) i\sqrt{7}\right) & j \not\equiv 0 \pmod 7, \\ \frac{1}{4} & j \equiv 0 \pmod 7. \end{cases}$$

(xxv) *Let* $f := 7x^2 + 8y^2 + 9z^2 + 6yz + 7zx$ *(form C2). Then with* $g := x^2 + 3y^2 + 3z^2 + 2yz + zx + xy$*, we have for all* $w \in \mathcal{H}$

$$\theta(f; w) = \sum_{j=1}^{7} a_j \theta\left(g; w + \frac{j}{7}\right) - \frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{-7}{n}\right) n e^{2\pi i n^2 w},$$

*where for* $j \in \{1, 2, 3, 4, 5, 6, 7\}$

$$a_j = \begin{cases} \frac{1}{56}\left(7 - \left(\frac{-7}{j}\right) i\sqrt{7}\right) & j \not\equiv 0 \pmod 7, \\ \frac{1}{4} & j \equiv 0 \pmod 7. \end{cases}$$

(xxvi) *Let* $f := 8x^2 + 9y^2 + 25z^2 + 2yz + 4zx + 8xy$ *(form C3). Then with* $g := x^2 + 4y^2 + 8z^2 + 4yz$*, we have for all* $w \in \mathcal{H}$

$$\theta(f; w) = \sum_{j=1}^{7} a_j \theta\left(g; w + \frac{j}{7}\right) - \frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{-7}{n}\right) n e^{2\pi i n^2 w},$$

*where for* $j \in \{1,2,3,4,5,6,7\}$

$$a_j = \begin{cases} \frac{1}{56}\left(7 - \left(\frac{-7}{j}\right) i\sqrt{7}\right) & j \not\equiv 0 \pmod 7, \\ \frac{1}{4} & j \equiv 0 \pmod 7. \end{cases}$$

(xxvii) *Let* $f := 29x^2 + 32y^2 + 36z^2 + 32yz + 12zx + 24xy$ *(form C4). Then with* $g := 4x^2 + 5y^2 + 29z^2 + 2yz + 4zx + 4xy$ *we have for all* $w \in \mathcal{H}$

$$\theta(f;w) = \sum_{j=1}^{7} a_j \theta\left(g; w + \frac{j}{7}\right) - \frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{-7}{n}\right) n e^{8\pi i n^2 w},$$

*where for* $j \in \{1,2,3,4,5,6,7\}$

$$a_j = \begin{cases} \frac{1}{56}\left(7 - \left(\frac{-7}{j}\right) i\sqrt{7}\right) & j \not\equiv 0 \pmod 7, \\ \frac{1}{4} & j \equiv 0 \pmod 7. \end{cases}$$

We now indicate how we found the identities given in Theorem 2.1. We illustrate the approach taken by giving the details in the case of the spinor regular ternary form A1, namely $f = f(x,y,z) := 2x^2 + 2y^2 + 5z^2 + 2yz + 2zx$. It follows from the work of Bateman [2] and Lomadze [18] that the number of representations of a positive integer $n$ by the simplest ternary $x^2 + y^2 + z^2$ is given as follows:

$$r(1,1,1,0,0,0;n) = k(n)l(n)h(\mathbb{Q}(\sqrt{-n^*})), \tag{2.3}$$

where $n^*$ and $l(n)$ are defined in (1.5) and (1.8), respectively, and

$$k(n) = \begin{cases} 6 & \text{if } \alpha = 0 \pmod 2, g = 1, \\ 8 & \text{if } \alpha = 0 \pmod 2, g = 3, \\ 12 & \text{if } \alpha = 0 \pmod 2, g \equiv 1 \pmod 4, g \neq 1, \\ 24 & \text{if } \alpha = 0 \pmod 2, g \equiv 3 \pmod 8, g \neq 3, \\ 0 & \text{if } \alpha = 0 \pmod 2, g \equiv 7 \pmod 8, \\ 12 & \text{if } \alpha = 1 \pmod 2, \end{cases}$$

where $\alpha = \nu_2(n)$ and $g$ is defined in (1.6). The formula (2.3) is obtained from the formulas of Bateman [2] and Lomadze [18] by replacing the $L$-series in their formulas by the class number using Dirichlet's class number formula. Indeed from the work of Lomadze [18] we know that a similar formula holds for $r(a,b,c,0,0,0;n)$ for any regular ternary $ax^2+by^2+cz^2$ which is alone in its genus. This is probably also true for non-diagonal ternaries alone in their genus. We computed $r(2,2,5,2,2,0;n)$ for all $n$ up to $3 \cdot 10^6$ and found that a similar formula to (2.3) held for all $n \in \mathbb{N}$ except for those satisfying $(\alpha, g) = (0,1)$, that is, $n = h^2$ ($h$ odd). In that case, the numerical evidence suggested a formula of the type

$$r(f;n) = r(2,2,5,2,2,0;n) = 2l(n) - 2\left(\frac{-1}{h}\right) h.$$

If this is the case, the contribution of these $n$ to

$$\theta(f;w) = \theta(2,2,5,2,2,0;w) = \sum_{n=0}^{\infty} r(2,2,5,2,2,0;n)e^{2\pi iwn}$$

would involve the term

$$\begin{aligned} -2\sum_{\substack{n=1\\ n=h^2\\ h \text{ odd}}}^{\infty} \left(\frac{-1}{h}\right) he^{2\pi iwn} &= -2\sum_{\substack{h=1\\ h \text{ odd}}}^{\infty} \left(\frac{-1}{h}\right) he^{2\pi iwh^2} \\ &= -2\sum_{m=1}^{\infty} \left(\frac{-4}{m}\right) me^{2\pi iwm^2} = -2\eta^3(8w). \end{aligned}$$

The next step was to specify a set of regular ternaries $g$ (each alone in its genus) to test if $f$ could be derived (in the sense of Definition 2.1 with the last term in (2.1) equal to $-2\eta^3(8w)$) from any of the $g$. We chose the test set to consist of those regular ternaries alone in their genus having

$$\frac{\text{disc}(f)}{\text{disc}(g)} = \text{perfect square} > 1.$$

Since $\text{disc}(f) = 2^6$, this meant that $\text{disc}(g) = 4$ or $16$ (as no ternary has discriminant equal to 1) and the test set comprised

$$\{x^2+y^2+z^2 \text{ (disc} = 4), \quad x^2+2y^2+2z^2 \text{ (disc} = 16), \quad x^2+y^2+4z^2 \text{ (disc} = 16)\}.$$

For each $g$ in the test set we calculated $r(g;n)$ for $n$ up to $3\cdot 10^6$ and then searched for a divisor (denoted by $R$) of $\text{disc}(f)$ such that there are complex numbers $c_1,\ldots,c_R$ and $b_d$ $(1 < d \mid R)$ such that the coefficient of $q^n$ $(q = e^{2\pi iw})$ in

$$\sum_{j=1}^{R} c_j \sum_{m=0}^{\infty} r(g;Rm+j)q^{Rm+j} + \sum_{1<d|R} b_d \sum_{m=0}^{\infty} r(g;m)q^{d^2m} - 2\sum_{m=1}^{\infty} \left(\frac{-4}{m}\right) mq^{m^2}$$

is equal to $r(f;n)$ for $n = 1,2,\ldots,3\cdot 10^6$. We found with $g = x^2+y^2+4z^2$ and $R = 8$ the values

$$\begin{aligned} &c_1 = \frac{1}{2},\ c_2 = 1,\ c_3 = 0,\ c_4 = 0,\ c_5 = \frac{1}{2},\ c_6 = 0,\ c_7 = 0,\ c_8 = 0, \\ &b_2 = 1,\ b_4 = 0,\ b_8 = 0. \end{aligned}$$

With these values

$$\sum_{j=1}^{8} c_j \sum_{m=0}^{\infty} r(g;8m+j)q^{8m+j}$$

$$= \sum_{n=1}^{\infty} \left\{ \begin{smallmatrix} 1/2 \text{ if } n\equiv 1 \pmod 4 \\ 1 \text{ if } n\equiv 2 \pmod 8 \\ 0 \text{ otherwise} \end{smallmatrix} \right\} r(1,1,4,0,0,0;n)q^n$$
$$= \sum_{n=1}^{\infty} \left( -\frac{i}{8}\omega_8^n - \frac{1+i}{8}\omega_8^{2n} + \frac{i}{8}\omega_8^{3n} - \frac{i}{8}\omega_8^{5n} - \frac{1-i}{8}\omega_8^{6n} + \frac{i}{8}\omega_8^{7n} + \frac{1}{4} \right) r(g;n)q^n$$
$$= \sum_{j=1}^{8} a_j \theta(g; w + \frac{j}{8}),$$

where

$$a_1 = a_5 = -\frac{i}{8},\ a_3 = a_7 = \frac{i}{8},\ a_2 = \frac{-1-i}{8},\ a_6 = \frac{-1+i}{8},\ a_4 = 0,\ a_8 = \frac{1}{4},$$

and

$$\sum_{1<d|R} b_d \sum_{m=0}^{\infty} r(g;m)q^{d^2 m} = b_2 \sum_{m=0}^{\infty} r(g;m)q^{4m} = \theta(g;4w),$$

we are led to the identity of Theorem 2.1(i). We now prove a result that we need for the proof of Theorem 2.1.

**Proposition 2.3.** *Let $R, N \in \mathbb{N}$ and let $N^* = \mathrm{lcm}(R^2, N)$. Let $\chi$ be a Dirichlet character with conductor dividing $N^*/R$. Suppose that $e_i \in \mathbb{C}$ $(i = 1, \dots, R)$ satisfy the following property:*

$$\begin{aligned} &\textit{For all integers } \delta \textit{ satisfying } \gcd(\delta, N^*) = 1 \textit{ we have } e_i = e_j, \\ &\textit{where } j \equiv \delta^2 i \pmod R \textit{ and } j \in \{1, \dots, R\}. \end{aligned} \tag{2.4}$$

*If $f(w) \in M_k(\Gamma_0(N), \chi)$, where $k \in \mathbb{N}$, then*

$$\sum_{i=1}^{R} e_i f\left(w + \frac{i}{R}\right) \in M_k\left(\Gamma_0(N^*), \chi\right).$$

*Proof.* Let $f(w) \in M_k(\Gamma_0(N), \chi)$. Then clearly $\sum_{i=1}^{R} e_i f(w+i/R)$ is holomorphic in $\mathcal{H}$ and at all the cusps. Now we prove that $\sum_{i=1}^{R} e_i f(w+i/R)$ satisfies the requisite transformation property for all matrices in $\Gamma_0(N^*)$. Let $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma_0(N^*)$. Then we have

$$\alpha\delta - \beta\gamma = 1, \quad N^* \mid \gamma,$$

and

$$f\left(\frac{\alpha w + \beta}{\gamma w + \delta}\right) = \chi(\delta)(\gamma w + \delta)^k f(w). \tag{2.5}$$

Since $N^* = \mathrm{lcm}(R^2, N)$, we have

$$R^2 \mid N^*, \quad N \mid N^*.$$

Thus, $R^2 \mid \gamma$ and for $i = 1, 2, \ldots, R$ we have

$$\begin{aligned} \alpha + \frac{i\gamma}{R} &\in \mathbb{Z}, \\ \beta - i\delta\left(\frac{\beta\gamma}{R} + \frac{i\delta\gamma}{R^2}\right) &\in \mathbb{Z}, \\ \delta - \frac{\gamma\delta^2 i}{R} &\in \mathbb{Z}, \\ \left(\alpha + \frac{i\gamma}{R}\right)\left(\delta - \frac{\gamma\delta^2 i}{R}\right) - \left(\beta - i\delta\left(\frac{\beta\gamma}{R} + \frac{i\delta\gamma}{R^2}\right)\right)\gamma &= 1, \\ \gamma &\equiv 0 \pmod{N^*}, \end{aligned}$$

so

$$\begin{pmatrix} \alpha + \frac{i\gamma}{R} & \beta - i\delta\left(\frac{\beta\gamma}{R} + \frac{i\delta\gamma}{R^2}\right) \\ \gamma & \delta - \frac{\gamma\delta^2 i}{R} \end{pmatrix} \in \Gamma_0(N^*).$$

Next we determine

$$\begin{aligned} f\left(\frac{\alpha w + \beta}{\gamma w + \delta} + \frac{i}{R}\right) &= f\left(\frac{(R\alpha + i\gamma)w + R\beta + i\delta}{R\gamma w + R\delta}\right) \\ &= f\left(\frac{(R\alpha + i\gamma)(w + \delta^2 i/R) + R\beta - i\delta(\beta\gamma + i\delta\gamma/R)}{R\gamma(w + \delta^2 i/R) + R\delta - \gamma\delta^2 i}\right) \\ &= f\left(\frac{(\alpha + i\gamma/R)(w + \delta^2 i/R) + \beta - i\delta(\beta\gamma/R + i\delta\gamma/R^2)}{\gamma(w + \delta^2 i/R) + \delta - \gamma\delta^2 i/R}\right) \\ &= \chi\left(\delta - \frac{\gamma\delta^2 i}{R}\right)(\gamma(w + \delta^2 i/R) + \delta - \gamma\delta^2 i/R)^k f\left(w + \frac{\delta^2 i}{R}\right) \\ &= \chi(\delta)(\gamma w + \delta)^k f\left(w + \frac{\delta^2 i}{R}\right). \end{aligned}$$

In the second step we used $\alpha\delta - \beta\gamma = 1$, in the fourth step we used (2.5), and in the final step we used the fact that $\chi$ is a character with conductor dividing $N^*/R$, which divides $\gamma/R$. Thus, we have

$$\begin{aligned} \sum_{i=1}^{R} e_i f\left(\frac{\alpha w + \beta}{\gamma w + \delta} + \frac{i}{R}\right) &= \sum_{i=1}^{R} e_i \chi(\delta)(\gamma w + \delta)^k f\left(w + \frac{\delta^2 i}{R}\right) \\ &= \chi(\delta)(\gamma w + \delta)^k \sum_{i=1}^{R} e_i f\left(w + \frac{\delta^2 i}{R}\right) \end{aligned}$$

$$= \chi(\delta)(\gamma w+\delta)^k \sum_{j=1}^{R} \sum_{\substack{i=1 \\ j\equiv \delta^2 i \pmod{R}}}^{R} e_i f\left(w+\frac{\delta^2 i}{R}\right)$$

$$= \chi(\delta)(\gamma w+\delta)^k \sum_{j=1}^{R} e_j f\left(w+\frac{j}{R}\right)$$

as $\gcd(\delta, N^*) = 1$ (so (2.4) holds) and $f(w+m) = f(w)$ for any integer $m$. Hence,

$$\sum_{i=1}^{R} e_i f\left(w+\frac{i}{R}\right) \in M_k\left(\Gamma_0(N^*), \chi\right),$$

as asserted. □

Results similar to Proposition 2.3 can be found in [20], where the modularity of the sum of the terms of the Fourier expansion of a modular form with index congruent to $r$ modulo $t$ is studied. Further, one can also use [5, Prop. 10.3.18] to give a statement that is similar to ours.

*Proof of Theorem 2.1.* (i) The reach of the identity we wish to prove is $R = 8$. The ternary quadratic forms involved in the identity are $f = 2x^2+2y^2+5z^2+2yz+2zx$ and $g = x^2+y^2+4z^2$. We define quaternary quadratic forms $F$ and $G_{d^2}$ (where $d^2 \mid R$) in terms of $f$ and $g$, respectively, by introducing a fourth variable $u$ as follows:

$$F := f + Ru^2 = 2x^2+2y^2+5z^2+8u^2+2yz+2zx$$

and

$$G_{d^2} := g + (R/d^2)u^2 = x^2+y^2+4z^2+(8/d^2)u^2.$$

The matrix of the quaternary quadratic form $F$ is

$$M(F) = \begin{pmatrix} 4 & 0 & 2 & 0 \\ 0 & 4 & 2 & 0 \\ 2 & 2 & 10 & 0 \\ 0 & 0 & 0 & 16 \end{pmatrix}.$$

The determinant of $M(F)$ is $2048 = 2^{11}$ and the inverse of $M(F)$ is

$$M(F)^{-1} = \begin{pmatrix} \frac{9}{32} & \frac{1}{32} & -\frac{1}{16} & 0 \\ \frac{1}{32} & \frac{9}{32} & -\frac{1}{16} & 0 \\ -\frac{1}{16} & -\frac{1}{16} & \frac{1}{8} & 0 \\ 0 & 0 & 0 & \frac{1}{16} \end{pmatrix}.$$

The level of $F$ (that is the minimal integer $N$ such that $NM(F)^{-1}$ is an integral matrix with even diagonal entries) is 64 and the character associated to $F$ is

$$\left(\frac{\det M(F)}{*}\right) = \left(\frac{2^{11}}{*}\right) = \left(\frac{8}{*}\right) = \chi_8.$$

Hence, by [23, Theorem 10.1] we have

$$\theta(F;w) \in M_2\left(\Gamma_0(64),\chi_8\right). \tag{2.6}$$

Similarly, we have

$$\theta(G_1;w) \in M_2\left(\Gamma_0(32),\chi_8\right) \tag{2.7}$$

and

$$\theta(G_4;4w) \in M_2\left(\Gamma_0(64),\chi_8\right). \tag{2.8}$$

By [5, Proposition 5.9.2] we have

$$\frac{\eta(8w)\eta^5(16w)}{\eta^2(32w)} \in M_2\left(\Gamma_0(64),\chi_8\right). \tag{2.9}$$

Furthermore, since $\theta(G_1;w) \in M_2\left(\Gamma_0(32),\chi_8\right)$ it follows from Proposition 2.3 that

$$\begin{aligned}
&-\frac{i}{8}\theta\left(G_1;w+\frac{1}{8}\right)-\left(\frac{1}{8}+\frac{i}{8}\right)\theta\left(G_1;w+\frac{2}{8}\right)+\frac{i}{8}\theta\left(G_1;w+\frac{3}{8}\right)\\
&\quad-\frac{i}{8}\theta\left(G_1;w+\frac{5}{8}\right)+\left(-\frac{1}{8}+\frac{i}{8}\right)\theta\left(G_1;w+\frac{6}{8}\right)+\frac{i}{8}\theta\left(G_1;w+\frac{7}{8}\right)\\
&\quad+\frac{1}{4}\theta\left(G_1;w+\frac{8}{8}\right)\in M_2\left(\Gamma_0(64),\chi_8\right).
\end{aligned}$$

Thus, by (2.6)–(2.9) we have

$$\begin{aligned}
H(w) := &-\frac{i}{8}\theta\left(G_1;w+\frac{1}{8}\right)-\left(\frac{1}{8}+\frac{i}{8}\right)\theta\left(G_1;w+\frac{2}{8}\right)+\frac{i}{8}\theta\left(G_1;w+\frac{3}{8}\right)\\
&-\frac{i}{8}\theta\left(G_1;w+\frac{5}{8}\right)+\left(-\frac{1}{8}+\frac{i}{8}\right)\theta\left(G_1;w+\frac{6}{8}\right)+\frac{i}{8}\theta\left(G_1;w+\frac{7}{8}\right)\\
&+\frac{1}{4}\theta\left(G_1;w+\frac{8}{8}\right)+\theta(G_4;4w)-2\frac{\eta(8w)\eta^5(16w)}{\eta^2(32w)}-\theta(F;w)\\
&\in M_2\left(\Gamma_0(64),\chi_8\right).
\end{aligned}$$

We compute the first $\frac{64}{6}\prod_{p\mid 64}\frac{p+1}{p}+1=17$ coefficients of the $q$-series expansion of $H(w)$ and obtain

$$H(w) = 0 + O(q^{18}).$$

Thus, by the Sturm theorem [5, Corollary 5.6.14], we deduce that $H(w)=0$, and this gives the identity

$$\theta(F;w) = -\frac{i}{8}\theta\left(G_1,w+\frac{1}{8}\right)-\left(\frac{1}{8}+\frac{i}{8}\right)\theta\left(G_1,w+\frac{2}{8}\right)+\frac{i}{8}\theta\left(G_1,w+\frac{3}{8}\right)$$

$$-\frac{i}{8}\theta\left(G_1, w+\frac{5}{8}\right)+\left(-\frac{1}{8}+\frac{i}{8}\right)\theta\left(G_1, w+\frac{6}{8}\right)+\frac{i}{8}\theta\left(G_1, w+\frac{7}{8}\right)$$
$$+\frac{1}{4}\theta\left(G_1, w+\frac{8}{8}\right)+\theta(G_4, 4w)-2\frac{\eta(8w)\eta^5(16w)}{\eta^2(32w)}. \tag{2.10}$$

Next, recalling the definition of $\theta(w)$ from (1.2) and Jacobi's theorem from (1.3), we have

$$\theta(F; w)=\theta(f; w)\theta(8w),$$
$$\theta\left(G_1; w+\frac{r}{8}\right)=\theta\left(g; w+\frac{r}{8}\right)\theta(8w), \quad r=1,2,\ldots,8,$$
$$\theta(G_4; 4w)=\theta(g; 4w)\theta(8w),$$
$$\frac{\eta(8w)\eta^5(16w)}{\eta^2(32w)}=\eta^3(8w)\frac{\eta^5(16w)}{\eta^2(8w)\eta^2(32w)}=\eta^3(8w)\theta(8w),$$

so dividing both sides of (2.10) by $\theta(8w)$, we obtain the identity of Theorem 2.1(i).

The remaining parts of Theorem 2.1 can all be proved in a similar manner. For the convenience of the reader we summarize the quantities needed in Table 2.1.

The only significant difference in the proofs occurs for the ternaries C1–C4. The identities in parts (xxiv)–(xxvii) contain the series $\sum_{n=1}^{\infty}\left(\frac{-7}{n}\right)ne^{2\pi iwn^2}$ rather than an eta quotient for these ternaries. In these four parts we use our final proposition of this section to determine the modular space to which the function

$$\theta(7w)\sum_{n=1}^{\infty}\left(\frac{-7}{n}\right)ne^{2\pi iwn^2}$$

belongs. □

**Proposition 2.4.**

$$\theta(7w)\sum_{n=1}^{\infty}\left(\frac{-7}{n}\right)ne^{2\pi iwn^2}\in M_2(\Gamma_0(196), \chi_1),$$

*where* $\chi_1$ *denotes the trivial character.*

*Proof.* Let

$$f(w):=\theta(7w)\sum_{n=1}^{\infty}\left(\frac{-7}{n}\right)ne^{2\pi iwn^2}, \quad w\in\mathcal{H}.$$

$f(w)$ is clearly holomorphic on $\mathcal{H}$ and at the cusps, see [5, Remark 2.3.22]. Now we prove the transformation property. Let $\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\in\Gamma_0(196)$, then by [5, Corollary 2.3.21] we have

$$f\left(\frac{\alpha w+\beta}{\gamma w+\delta}\right)=\left(\frac{-7}{\delta}\right)\left(\frac{-4}{\delta}\right)^{-\frac{1}{2}}\left(\frac{\gamma}{\delta}\right)\left(\frac{-4}{\delta}\right)^{-\frac{1}{2}}\left(\frac{\gamma/7}{\delta}\right)(\gamma w+\delta)^2 f(w).$$

| Form Number | Level | Sturm Bound | Added Term | $R$ |
|---|---|---|---|---|
| A1 | 64 | 17 | $(8/d^2)u^2$ | 8 |
| A2 | 128 | 33 | $(4/d^2)u^2$ | 4 |
| A3 | 1024 | 257 | $(32/d^2)u^2$ | 32 |
| A4 | 64 | 17 | $(4/d^2)u^2$ | 2 |
| A5 | 64 | 17 | $(8/d^2)u^2$ | 8 |
| A6 | 256 | 65 | $(2/d^2)u^2$ | 2 |
| A7 | 1024 | 257 | $(32/d^2)u^2$ | 32 |
| A8 | 256 | 65 | $(4/d^2)u^2$ | 4 |
| A9 | 256 | 65 | $(16/d^2)u^2$ | 16 |
| A10 | 128 | 33 | $(4/d^2)u^2$ | 2 |
| A11 | 256 | 65 | $(8/d^2)u^2$ | 8 |
| A12 | 16384 | 4097 | $(128/d^2)u^2$ | 128 |
| A13 | 4096 | 1025 | $(64/d^2)u^2$ | 64 |
| Form Number | Level | Sturm Bound | Added Term | $R$ |
| B1 | 144 | 49 | $(3/d^2)u^2$ | 3 |
| B2 | 144 | 49 | $(3/d^2)u^2$ | 3 |
| B3 | 1296 | 433 | $(9/d^2)u^2$ | 9 |
| B5 | 576 | 193 | $(8/d^2)u^2$ | 8 |
| B6 | 1152 | 385 | $(8/d^2)u^2$ | 8 |
| B7 | 576 | 193 | $(8/d^2)u^2$ | 8 |
| B8 | 108 | 37 | $(9/d^2)u^2$ | 3 |
| B9 | 288 | 97 | $(8/d^2)u^2$ | 8 |
| B10 | 288 | 97 | $(8/d^2)u^2$ | 8 |
| B12 | 432 | 145 | $(9/d^2)u^2$ | 3 |
| Form Number | Level | Sturm Bound | Added Term | $R$ |
| C1 | 196 | 57 | $(7/d^2)u^2$ | 7 |
| C2 | 784 | 225 | $(7/d^2)u^2$ | 7 |
| C3 | 196 | 57 | $(7/d^2)u^2$ | 7 |
| C4 | 784 | 225 | $(7/d^2)u^2$ | 7 |

Table 2.1: Level, Sturm bound and reach $R$ for the 27 spinor regular ternaries alone in their spinor genus

We now show that

$$\left(\frac{-7}{\delta}\right)\left(\frac{-4}{\delta}\right)^{-\frac{1}{2}}\left(\frac{\gamma}{\delta}\right)\left(\frac{-4}{\delta}\right)^{-\frac{1}{2}}\left(\frac{\gamma/7}{\delta}\right)=1. \tag{2.11}$$

As $\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\in\Gamma_0(196)$ we have

$$\alpha\delta-\beta\gamma=1, \quad 196 \mid \gamma.$$

Then, as $196 = 2^2 7^2$, we deduce that

$$4 \mid \gamma,\ 49 \mid \gamma,\ 2 \nmid \delta,\ 7 \nmid \delta, \quad (\gamma, \delta) = 1.$$

Thus

$$\left(\frac{-7}{\delta}\right)\left(\frac{\gamma}{\delta}\right)\left(\frac{\gamma/7}{\delta}\right) = \left(\frac{-\gamma^2}{\delta}\right) = \left(\frac{-1}{\delta}\right)$$

and

$$\left(\frac{-4}{\delta}\right)^{-\frac{1}{2}} = \left.\begin{cases} 1 & \text{if } \delta \equiv 1 \pmod 4 \\ (-1)^{-\frac{1}{2}} & \text{if } \delta \equiv 3 \pmod 4 \end{cases}\right\} = \begin{cases} 1 & \text{if } \delta \equiv 1 \pmod 4 \\ -i & \text{if } \delta \equiv 3 \pmod 4 \end{cases}$$

so

$$\left(\frac{-4}{\delta}\right)^{-\frac{1}{2}}\left(\frac{-4}{\delta}\right)^{-\frac{1}{2}} = \left.\begin{cases} 1 & \text{if } \delta \equiv 1 \pmod 4 \\ -1 & \text{if } \delta \equiv 3 \pmod 4 \end{cases}\right\} = \left(\frac{-1}{\delta}\right).$$

This proves (2.11). Hence, we have

$$f\left(\frac{\alpha w + \beta}{\gamma w + \delta}\right) = (\gamma w + \delta)^2 f(w),$$

for all $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma_0(196)$, proving $f(w) \in M_2(\Gamma_0(196), \chi_1)$. □

## 3. Two Lemmas

In this section we prove two elementary arithmetic lemmas that will be useful in deducing exactly which integers are represented by a particular spinor regular ternary quadratic form from its representation number.

**Lemma 3.1.** (i) *If $n = 2^\alpha h^2$, where $\alpha \in \mathbb{N}_0, h \in \mathbb{N}$ and $(h, 2) = 1$, then*

$$\prod_{p \mid h}\left(\sigma\left(p^{\nu_p(h)}\right) - \left(\frac{-1}{p}\right)\sigma\left(p^{\nu_p(h)-1}\right)\right) = \left(\frac{-1}{h}\right) h$$

*if and only if $n \in 2^\alpha M_4^2$.*

(ii) *If $n = 2^\alpha h^2$, where $\alpha \in \mathbb{N}_0, h \in \mathbb{N}$ and $(h, 2) = 1$, then*

$$\prod_{p \mid h}\left(\sigma\left(p^{\nu_p(h)}\right) - \left(\frac{-2}{p}\right)\sigma\left(p^{\nu_p(h)-1}\right)\right) = \left(\frac{-2}{h}\right) h$$

*if and only if $n \in 2^\alpha M_8^2$.*

(iii) *If* $n = 2^\alpha h^2$, *where* $\alpha \in \mathbb{N}_0, h \in \mathbb{N}$ *and* $(h, 6) = 1$, *then*

$$\prod_{p|h} \left( \sigma\left(p^{\nu_p(h)}\right) - \left(\frac{-3}{p}\right) \sigma\left(p^{\nu_p(h)-1}\right) \right) = \left(\frac{-3}{h}\right) h$$

*if and only if* $n \in 2^\alpha M_3^2$.

(iv) *If* $n = 2^\alpha h^2$, *where* $\alpha \in \mathbb{N}_0, h \in \mathbb{N}$ *and* $(h, 14) = 1$, *then*

$$\prod_{p|h} \left( \sigma\left(p^{\nu_p(h)}\right) - \left(\frac{-7}{p}\right) \sigma\left(p^{\nu_p(h)-1}\right) \right) = \left(\frac{-7}{h}\right) h$$

*if and only if* $n \in 2^\alpha M_7^2$.

*Proof.* (i) As $h$ is odd, we can define positive odd integers $h_{1,4}$ and $h_{3,4}$ by

$$h_{1,4} := \prod_{\substack{p|h \\ p \equiv 1 \pmod 4}} p^{\nu_p(h)}, \qquad h_{3,4} := \prod_{\substack{p|h \\ p \equiv 3 \pmod 4}} p^{\nu_p(h)},$$

so that

$$h = h_{1,4} h_{3,4}, \quad n = 2^\alpha h_{1,4}^2 h_{3,4}^2, \quad (h_{1,4}, h_{3,4}) = 1,$$
$$h_{1,4} \equiv 1 \pmod 4, \quad h_{3,4} \equiv (-1)^x \pmod 4,$$

where

$$x = x(h_{3,4}) := \sum_{p|h_{3,4}} \nu_p(h_{3,4}).$$

We have

$$\prod_{p|h} \left( \sigma\left(p^{\nu_p(h)}\right) - \left(\frac{-1}{p}\right) \sigma\left(p^{\nu_p(h)-1}\right) \right) = h_{1,4} \prod_{p|h_{3,4}} \frac{p^{\nu_p(h_{3,4})+1} + p^{\nu_p(h_{3,4})} - 2}{p-1}$$

and

$$\left(\frac{-1}{h}\right) h = \left(\frac{-1}{h_{3,4}}\right) h_{1,4} h_{3,4} = (-1)^x h_{1,4} h_{3,4}.$$

Thus

$$\prod_{p|h} \left( \sigma\left(p^{\nu_p(h)}\right) - \left(\frac{-1}{p}\right) \sigma\left(p^{\nu_p(h)-1}\right) \right) = \left(\frac{-1}{h}\right) h$$

$$\text{if and only if } \prod_{p|h_{3,4}} \frac{p^{\nu_p(h_{3,4})+1} + p^{\nu_p(h_{3,4})} - 2}{p-1} = (-1)^x h_{3,4}$$

$$\text{if and only if } \prod_{p|h_{3,4}} \frac{p + 1 - \frac{2}{p^{\nu_p(h_{3,4})}}}{p-1} = (-1)^x.$$

Now

$$\frac{p+1-\frac{2}{p^{\nu_p(h_{3,4})}}}{p-1}\begin{cases}=1 & \text{if } v_p(h_{3,4})=0,\\ >1 & \text{if } v_p(h_{3,4})>0,\end{cases}$$

so

$$\begin{aligned}\prod_{p\mid h_{3,4}}\frac{p+1-\frac{2}{p^{\nu_p(h_{3,4})}}}{p-1}=(-1)^x &\text{ if and only if } \nu_p(h_{3,4})=0 \text{ for all } p\mid h_{3,4}\\ &\text{ if and only if } h_{3,4}=1\\ &\text{ if and only if } n=2^\alpha h_{1,4}^2\\ &\text{ if and only if } n\in 2^\alpha M_4^2.\end{aligned}$$

(ii) As $h$ is odd, we can define positive odd integers $h_{1,8}$, $h_{3,8}$ $h_{5,8}$ and $h_{7,8}$ by

$$h_{1,8}:=\prod_{\substack{p\mid h\\ p\equiv 1 \pmod 8}}p^{\nu_p(h)},\qquad h_{3,8}:=\prod_{\substack{p\mid h\\ p\equiv 3 \pmod 8}}p^{\nu_p(h)},$$

$$h_{5,8}:=\prod_{\substack{p\mid h\\ p\equiv 5 \pmod 8}}p^{\nu_p(h)},\qquad h_{7,8}:=\prod_{\substack{p\mid h\\ p\equiv 7 \pmod 8}}p^{\nu_p(h)},$$

and $x_j=x(h_{j,8}):=\sum_{p\mid h_{j,8}}\nu_p(h_{j,8}), j=1,3,5,7$, so that

$$h=h_{1,8}h_{3,8}h_{5,8}h_{7,8},\quad n=2^\alpha h_{1,8}^2h_{3,8}^2h_{5,8}^2h_{7,8}^2,\quad (h_{i,8},h_{j,8})=1 \text{ for all } i\neq j,$$

$$h_{1,8}\equiv 1 \pmod 8,\quad h_{7,8}\equiv(-1)^{x_7} \pmod 8,$$

$$h_{3,8}\equiv\begin{cases}1 \pmod 8 & \text{if } x_3 \text{ is even},\\ 3 \pmod 8 & \text{if } x_3 \text{ is odd},\end{cases}$$

and

$$h_{5,8}\equiv\begin{cases}1 \pmod 8 & \text{if } x_5 \text{ is even},\\ 5 \pmod 8 & \text{if } x_5 \text{ is odd}.\end{cases}$$

We have

$$\begin{aligned}&\prod_{p\mid h}\left(\sigma\left(p^{\nu_p(h)}\right)-\left(\frac{-2}{p}\right)\sigma\left(p^{\nu_p(h)-1}\right)\right)\\ &=h_{1,8}h_{3,8}\prod_{p\mid h_{5,8}}\frac{p^{\nu_p(h_{5,8})+1}+p^{\nu_p(h_{5,8})}-2}{p-1}\prod_{p\mid h_{7,8}}\frac{p^{\nu_p(h_{7,8})+1}+p^{\nu_p(h_{7,8})}-2}{p-1}\\ &=h_{1,8}h_{3,8}h_{5,8}h_{7,8}\prod_{p\mid h_{5,8}}\frac{p+1-\frac{2}{p^{\nu_p(h_{5,8})}}}{p-1}\prod_{p\mid h_{7,8}}\frac{p+1-\frac{2}{p^{\nu_p(h_{7,8})}}}{p-1}\end{aligned}$$

and

$$\left(\frac{-2}{h}\right) h = \left(\frac{-2}{h_{5,8}h_{7,8}}\right) h_{1,8}h_{3,8}h_{5,8}h_{7,8} = (-1)^{x_5}(-1)^{x_7} h_{1,8}h_{3,8}h_{5,8}h_{7,8}.$$

Thus

$$\prod_{p|h}\left(\sigma\left(p^{\nu_p(h)}\right) - \left(\frac{-2}{p}\right)\sigma\left(p^{\nu_p(h)-1}\right)\right) = \left(\frac{-2}{h}\right) h$$

$$\text{if and only if } \prod_{p|h_{5,8}} \frac{p+1-\frac{2}{p^{\nu_p(h_{5,8})}}}{p-1} \prod_{p|h_{7,8}} \frac{p+1-\frac{2}{p^{\nu_p(h_{7,8})}}}{p-1} = (-1)^{x_5+x_7}$$

$$\text{if and only if } \prod_{p|h_{5,8}h_{7,8}} \frac{p+1-\frac{2}{p^{\nu_p(h_{5,8}h_{7,8})}}}{p-1} = (-1)^{\sum_{p|h_{5,8}h_{7,8}} \nu_p(h_{5,8}h_{7,8})}.$$

Similar to the proof of (i), we have

$$\prod_{p|h_{5,8}h_{7,8}} \frac{p+1-\frac{2}{p^{\nu_p(h_{5,8}h_{7,8})}}}{p-1} = (-1)^{\sum_{p|h_{5,8}h_{7,8}} \nu_p(h_{5,8}h_{7,8})}$$

$$\begin{aligned}&\text{if and only if } \nu_p(h_{5,8}h_{7,8}) = 0 \text{ for all } p \mid h_{5,8}h_{7,8}\\ &\text{if and only if } h_{5,8}h_{7,8} = 1\\ &\text{if and only if } n = 2^\alpha h_{1,8}^2 h_{3,8}^2\\ &\text{if and only if } n \in 2^\alpha M_8^2.\end{aligned}$$

(iii) The proof is similar to that of (i), except that

$$h_{1,3} := \prod_{\substack{p|h \\ p\equiv 1 \pmod 3}} p^{\nu_p(h)}, \text{ and } \quad h_{2,3} := \prod_{\substack{p|h \\ p\equiv 2 \pmod 3}} p^{\nu_p(h)},$$

are used in place of $h_{1,4}$ and $h_{3,4}$.

(iv) The proof is similar to the previous parts using

$$h_{r,7} := \prod_{\substack{p|h \\ p\equiv r \pmod 7}} p^{\nu_p(h)}, \quad r = 1,2,3,4,5,6.$$

□

**Lemma 3.2.** *Let $h \in \mathbb{N}$ satisfy $(h,6) = 1$. Let $A, B, C \in \mathbb{R}$ satisfy $A > C > 0$ and $B > 0$. Let $m \in \mathbb{N}_0$ satisfy*

$$2^m > \frac{B}{A-C}.$$

*Then*

$$(A2^m - B)\prod_{p\mid h}\left(\sigma\left(p^{\nu_p(h)}\right) - \left(\frac{-3}{p}\right)\sigma\left(p^{\nu_p(h)-1}\right)\right) > C2^m h.$$

*Proof.* As $(h,6) = 1$, we can define positive integers $h_{1,3}$ and $h_{2,3}$ as in the proof of Lemma 3.1 (iii) so that $h = h_{1,3}h_{2,3}$. We have

$$\prod_{p\mid h}\left(\sigma\left(p^{\nu_p(h)}\right) - \left(\frac{-3}{p}\right)\sigma\left(p^{\nu_p(h)-1}\right)\right) = h_{1,3}\prod_{p\mid h_{2,3}}\frac{p^{\nu_p(h_{2,3})+1} + p^{\nu_p(h_{2,3})} - 2}{p-1}.$$

Hence, we obtain

$$\frac{1}{h}\prod_{p\mid h}\left(\sigma\left(p^{\nu_p(h)}\right) - \left(\frac{-3}{p}\right)\sigma\left(p^{\nu_p(h)-1}\right)\right) = \prod_{p\mid h_{2,3}}\frac{p+1-\frac{2}{p^{\nu_p(h_{2,3})}}}{p-1} \geq 1.$$

As $2^m > \frac{B}{A-C}$ we have $A2^m - B > C2^m$. Thus

$$\frac{A2^m - B}{h}\prod_{p\mid h}\left(\sigma\left(p^{\nu_p(h)}\right) - \left(\frac{-3}{p}\right)\sigma\left(p^{\nu_p(h)-1}\right)\right) > C2^m$$

from which the asserted inequality follows. □

## 4. Spinor Regular Ternaries with Discriminant $2^r$

The thirteen spinor regular positive-definite ternary quadratic forms $f = f(x,y,z)$ which are not regular and have discriminant $\Delta = 2^r$ for some $r \in \mathbb{N}$ are those with identification numbers A1–A13 in Table 1.1. We determine their representation numbers in this section.

When considering the representation of $n \in \mathbb{N}$ by $f(x,y,z)$, we use the integers $\alpha = \nu_2(n)$, and $g$, $h$ and $n^*$, which are defined uniquely in terms of $n$ by (1.6), (1.7) and (1.5), respectively. We have

$$n = 2^\alpha g h^2, \tag{4.1}$$

where

$$\alpha \in \mathbb{N}_0, \quad g, h \in \mathbb{N}, \quad g \text{ squarefree}, \quad (gh, 2) = 1, \tag{4.2}$$

and

$$n^* = \begin{cases} 2^{\alpha - 2[\alpha/2]}g & \text{if } r \text{ is even,} \\ 2^{\alpha+1-2[(\alpha+1)/2]}g & \text{if } r \text{ is odd.} \end{cases} \tag{4.3}$$

We now state and prove formulas for the representation numbers of the forms A1–A13. All of these formulas involve the quantities defined in (4.1)–(4.3), as well as $l(n)$, which is defined in (1.8). We begin with A1.

**Theorem 4.1.** *Let $f$ denote the form A1, that is, $f = 2x^2 + 2y^2 + 5z^2 + 2yz + 2zx$. If $(\alpha, g) \neq (0, 1)$, we have*

$$r(f;n) = k_f(n)l(n)h(\mathbb{Q}(\sqrt{-n^*})),$$

*where the values of $k_f(n)$ are given in Table 4.1.*

| $\alpha$ | $g$ | $k_f(n)$ |
|---|---|---|
| 0 | $g \equiv 1 \pmod 4, g \neq 1$ | 4 |
| | $g \equiv 3 \pmod 4$ | 0 |
| 1 | $g \equiv 1 \pmod 4$ | 4 |
| | $g \equiv 3 \pmod 4$ | 0 |
| 2 | $g = 1$ | 4 |
| | $g \equiv 1 \pmod 4, g \neq 1$ | 8 |
| | $g \equiv 3 \pmod 4$ | 0 |
| 3 | | 4 |
| *even* $\geq 4$ | $g = 1$ | 6 |
| | $g \equiv 1 \pmod 4, g \neq 1$ | 12 |
| | $g = 3$ | 8 |
| | $g \equiv 3 \pmod 8, g \neq 3$ | 24 |
| | $g \equiv 7 \pmod 8$ | 0 |
| *odd* $\geq 5$ | | 12 |

Table 4.1: Values of $k_f(n)$

*If $(\alpha, g) = (0, 1)$, we have*

$$r(f;n) = 2l(n) - 2\left(\frac{-1}{h}\right)h.$$

*Proof.* Recall for $k \in \mathbb{N}$ we set $\omega_k = e^{2\pi i/k}$, so that $\omega_8 = \frac{1}{2}(1+i)\sqrt{2}$. By Theorem 2.1 (i) and Proposition 2.1, we obtain for all $n \in \mathbb{N}$

$$r(f;n) = s(n)r(1,1,4,0,0,0;n) + r\left(1,1,4,0,0,0;\frac{n}{4}\right) + t(n),$$

where

$$\begin{aligned} s(n) &= -\frac{i}{8}\omega_8^n - \frac{1+i}{8}\omega_8^{2n} + \frac{i}{8}\omega_8^{3n} - \frac{i}{8}\omega_8^{5n} + \frac{-1+i}{8}\omega_8^{6n} + \frac{i}{8}\omega_8^{7n} + \frac{1}{4} \\ &= \begin{cases} 0 & \text{if } n \equiv 0,3 \pmod 4 \text{ or } n \equiv 6 \pmod 8, \\ \frac{1}{2} & \text{if } n \equiv 1 \pmod 4, \\ 1 & \text{if } n \equiv 2 \pmod 8, \end{cases} \end{aligned}$$

$$= \begin{cases} \frac{1}{2} & \text{if } \alpha = 0, g \equiv 1 \pmod 4, \\ 0 & \text{if } \alpha = 0, g \equiv 3 \pmod 4, \\ 1 & \text{if } \alpha = 1, g \equiv 1 \pmod 4, \\ 0 & \text{if } \alpha = 1, g \equiv 3 \pmod 4, \\ 0 & \text{if } \alpha \geq 2, \end{cases}$$

and

$$t(n) = \left. \begin{cases} -2\left(\frac{-4}{\sqrt{n}}\right)\sqrt{n} & \text{if } n = \text{odd square} \\ 0 & \text{otherwise} \end{cases} \right\} = \begin{cases} -2\left(\frac{-1}{h}\right)h & \text{if } \alpha = 0, g = 1, \\ 0 & \text{otherwise.} \end{cases}$$

By Table 1.1 we have $\Delta = 2^6$ so that $r = 6$ is even and thus by (4.3) $n^* = 2^{\alpha - 2[\alpha/2]}g$. If $4 \mid n$, we have $\left(\frac{n}{4}\right)^* = n^*$ and $l(n/4) = l(n)$ by (1.9) and (1.10).

By Proposition A.1 we have

$$r(1,1,4,0,0,0;n) = k(n)l(n)h(\mathbb{Q}(\sqrt{-n^*})),$$

where

$$k(n) = \begin{cases} 4 & \text{if } \alpha = 0, g = 1, \\ 8 & \text{if } \alpha = 0, g \equiv 1 \pmod 4, g \neq 1, \\ 0 & \text{if } \alpha = 0, g \equiv 3 \pmod 4, \\ 4 & \text{if } \alpha = 1, \\ 6 & \text{if } \alpha \text{ (even)} \geq 2, g = 1, \\ 12 & \text{if } \alpha \text{ (even)} \geq 2, g \equiv 1 \pmod 4, g \neq 1, \\ 8 & \text{if } \alpha \text{ (even)} \geq 2, g = 3, \\ 24 & \text{if } \alpha \text{ (even)} \geq 2, g \equiv 3 \pmod 8, g \neq 3, \\ 0 & \text{if } \alpha \text{ (even)} \geq 2, g \equiv 7 \pmod 8, \\ 12 & \text{if } \alpha \text{ (odd)} \geq 3. \end{cases}$$

Also by Proposition A.1 we have

$$\begin{aligned} r(1,1,4,0,0,0;n/4) &= k(n/4)l(n/4)h\left(\mathbb{Q}\left(\sqrt{-\left(\frac{n}{4}\right)^*}\right)\right) \\ &= k(n/4)l(n)h(\mathbb{Q}(\sqrt{-n^*})), \end{aligned}$$

where

$$k(n/4) = \begin{cases} 0 & \text{if } \alpha = 0, 1, \\ 4 & \text{if } \alpha = 2, g = 1, \\ 8 & \text{if } \alpha = 2, g \equiv 1 \pmod 4, g \neq 1, \\ 0 & \text{if } \alpha = 2, g \equiv 3 \pmod 4, \\ 4 & \text{if } \alpha = 3, \\ 6 & \text{if } \alpha \text{ (even)} \geq 4, g = 1, \\ 12 & \text{if } \alpha \text{ (even)} \geq 4, g \equiv 1 \pmod 4, g \neq 1, \\ 8 & \text{if } \alpha \text{ (even)} \geq 4, g = 3, \\ 24 & \text{if } \alpha \text{ (even)} \geq 4, g \equiv 3 \pmod 8, g \neq 3, \\ 0 & \text{if } \alpha \text{ (even)} \geq 4, g \equiv 7 \pmod 8, \\ 12 & \text{if } \alpha \text{ (odd)} \geq 5. \end{cases}$$

Thus, we have

$$s(n)k(n) + k(n/4) = \begin{cases} \frac{1}{2} \cdot 4 + 0 = 2 & \text{if } \alpha = 0, g = 1, \\ \frac{1}{2} \cdot 8 + 0 = 4 & \text{if } \alpha = 0, g \equiv 1 \pmod 4, g \neq 1, \\ 0 \cdot 0 + 0 = 0 & \text{if } \alpha = 0, g \equiv 3 \pmod 4, \\ 1 \cdot 4 + 0 = 4 & \text{if } \alpha = 1, g \equiv 1 \pmod 4, \\ 0 \cdot 4 + 0 = 0 & \text{if } \alpha = 1, g \equiv 3 \pmod 4, \\ 0 \cdot 6 + 4 = 4 & \text{if } \alpha = 2, g = 1, \\ 0 \cdot 12 + 8 = 8 & \text{if } \alpha = 2, g \equiv 1 \pmod 4, g \neq 1, \\ 0 \cdot (8, 24, 0) + 0 = 0 & \text{if } \alpha = 2, g \equiv 3 \pmod 4, \\ 0 \cdot 12 + 4 = 4 & \text{if } \alpha = 3, \\ 0 \cdot 6 + 6 = 6 & \text{if } \alpha \text{ (even)} \geq 4, g = 1, \\ 0 \cdot 12 + 12 = 12 & \text{if } \alpha \text{ (even)} \geq 4, g \equiv 1 \pmod 4, g \neq 1, \\ 0 \cdot 8 + 8 = 8 & \text{if } \alpha \text{ (even)} \geq 4, g = 3, \\ 0 \cdot 24 + 24 = 24 & \text{if } \alpha \text{ (even)} \geq 4, g \equiv 3 \pmod 8, g \neq 3, \\ 0 \cdot 0 + 0 = 0 & \text{if } \alpha \text{ (even)} \geq 4, g \equiv 7 \pmod 8, \\ 0 \cdot 12 + 12 = 12 & \text{if } \alpha \text{ (odd)} \geq 5, \end{cases}$$

$$= \begin{cases} 2 & \text{if } (\alpha, g) = (0, 1), \\ k_f(n) \text{ (of Table 4.1)} & \text{if } (\alpha, g) \neq (0, 1). \end{cases}$$

Hence, we obtain

$$\begin{aligned} r(f; n) &= (s(n)k(n) + k(n/4))\, l(n) h(\mathbb{Q}(\sqrt{-n^*})) + t(n) \\ &= \begin{cases} 2l(n)h(\mathbb{Q}(\sqrt{-n^*})) - 2\left(\frac{-1}{h}\right) h & \text{if } (\alpha, g) = (0, 1), \\ k_f(n) l(n) h(\mathbb{Q}(\sqrt{-n^*})) & \text{if } (\alpha, g) \neq (0, 1). \end{cases} \end{aligned}$$

The proof is completed by noting that when $(\alpha, g) = (0, 1)$ we have $n^* = 1$ so $h(\mathbb{Q}(\sqrt{-n^*})) = h(\mathbb{Q}(\sqrt{-1})) = 1$. □

Theorem 4.1 enables us to determine the positive integers $n$ that are not represented by $f$ by determining those $n$ such that $r(f;n) = 0$. For those $n$ with $(\alpha, g) \neq (0, 1)$ these integers follow from Table 4.1. For those with $(\alpha, g) = (0, 1)$ these $n$ are given by $l(n) - \left(\frac{-1}{h}\right) h = 0$ and can be determined by means of Lemma 3.1. We carry out the details and the positive integers not represented by $f = (2, 2, 5, 2, 2, 0)$ are given in Table A.17. These integers were first determined by Lomadze [19, Corollary 2, p. 141].

If $(\alpha, g) \neq (0, 1)$, we deduce from Theorem 4.1 that $r(f;n) = 0$ occurs precisely when $\alpha$ and $g$ satisfy

$$\begin{aligned}
&\alpha = 0,\ g \equiv 3 \pmod 8, \text{ or}\\
&\alpha = 1,\ g \equiv 3 \pmod 4, \text{ or}\\
&\alpha = 2,\ g \equiv 3 \pmod 8, \text{ or}\\
&\alpha \text{ (even)} \geq 0,\ g \equiv 7 \pmod 8,
\end{aligned}$$

that is, when $n = 8l+3, 8l+6, 32l+12, 4^k(8l+7)$ for some $k, l \in \mathbb{N}_0$. If $(\alpha, g) = (0, 1)$ by Theorem 4.1 and Lemma 3.1 (i) we have

$$r(f;n) = 0 \text{ if and only if } l(n) = \left(\frac{-1}{h}\right) h \text{ if and only if } n \in M_4^2$$

Hence, $n$ is represented by $f$ if and only if $n$ does not belong to any of the progressions $8l + 3, 8l + 6, 32l + 12, 4^k(8l + 7)$ $(k, l \in \mathbb{N}_0), M_4^2$.

Since the method of finding the positive integers not represented by each of the 27 spinor regular forms alone in their spinor genus is in principle the same, we only give the details for a few of the forms and leave the remainder to the reader. For seven of these forms Lemma 3.2 is used. The non-represented positive integers are listed in Table A.17. Next, we give the representation number $r(f;n)$ when $f$ is the form A2.

**Theorem 4.2.** *Let $f$ denote the form A2, that is, $f = x^2 + 4y^2 + 9z^2 + 4yz$. If $(\alpha, g) \neq (1, 1)$, we have*

$$r(f;n) = k_f(n) l(n) h(\mathbb{Q}(\sqrt{-n^*})),$$

*where the values of $k_f(n)$ are given in Table 4.2.*
*If $(\alpha, g) = (1, 1)$, we have*

$$r(f;n) = 2l(n) - 2\left(\frac{-1}{h}\right) h.$$

*Proof.* By Theorem 2.1 (ii) and Proposition 2.1, we obtain for all $n \in \mathbb{N}$

$$r(f;n) = s(n) r(1, 1, 8, 0, 0, 0; n) + r\left(1, 1, 8, 0, 0, 0; \frac{n}{4}\right) + t(n),$$

| $\alpha$ | $g$ | $k_f(n)$ |
| --- | --- | --- |
| 0 | $g \equiv 1 \pmod 4$ | 2 |
| | $g \equiv 3 \pmod 4$ | 0 |
| 1 | $g \equiv 1 \pmod 4, g \neq 1$ | 4 |
| | $g \equiv 3 \pmod 4$ | 0 |
| 2 | $g \equiv 1 \pmod 4$ | 4 |
| | $g \equiv 3 \pmod 4$ | 0 |
| 3 | $g = 1$ | 4 |
| | $g \equiv 1 \pmod 4, g \neq 1$ | 8 |
| | $g \equiv 3 \pmod 4$ | 0 |
| 4 | | 4 |
| odd $\geq 5$ | $g = 1$ | 6 |
| | $g \equiv 1 \pmod 4, g \neq 1$ | 12 |
| | $g = 3$ | 8 |
| | $g \equiv 3 \pmod 8, g \neq 3$ | 24 |
| | $g \equiv 7 \pmod 8$ | 0 |
| even $\geq 6$ | | 12 |

Table 4.2: Values of $k_f(n)$

where

$$s(n) = \left(-\frac{1}{8} - \frac{i}{8}\right) i^n + \left(-\frac{1}{8} + \frac{i}{8}\right) (-i)^n + \frac{1}{4} = \begin{cases} \frac{1}{2} & \text{if } n \equiv 1, 2 \pmod 4, \\ 0 & \text{if } n \equiv 0, 3 \pmod 4, \end{cases}$$
$$= \begin{cases} \frac{1}{2} & \text{if } \alpha = 0, g \equiv 1 \pmod 4 \text{ or } \alpha = 1, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$t(n) = \begin{cases} -2\left(\frac{-1}{\sqrt{n/2}}\right)\sqrt{n/2} & \text{if } n = 2 \times \text{odd square}, \\ 0 & \text{otherwise,} \end{cases}$$
$$= \begin{cases} -2\left(\frac{-1}{h}\right) h & \text{if } \alpha = 1, g = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Appealing to Proposition A.2 for the formula for $r(1, 1, 8, 0, 0, 0; n)$, we obtain Theorem 4.2. □

From Theorem 4.2 using Lemma 3.1 we determine exactly which positive integers are not represented by the form A2. These are given in Table A.17. Next, we determine the representation number $r(f; n)$ when $f$ is the form A3.

**Theorem 4.3.** *Let $f$ denote the form A3, that is, $f = 2x^2+5y^2+8z^2+4yz+2xy$. If $(\alpha, g) \neq (0,1),(2,1)$, we have*

$$r(f;n) = k_f(n)l(n)h(\mathbb{Q}(\sqrt{-n^*})),$$

*where the values of $k_f(n)$ are given in Table 4.3.*

| $\alpha$ | $g$ | $k_f(n)$ |
|---|---|---|
| 0 | $g \equiv 1 \pmod 4, g \neq 1$ | 2 |
| | $g \equiv 3 \pmod 4$ | 0 |
| 1 | $g \equiv 1 \pmod 4$ | 2 |
| | $g \equiv 3 \pmod 4$ | 0 |
| 2 | $g \equiv 1 \pmod 4, g \neq 1$ | 4 |
| | $g \equiv 3 \pmod 4$ | 0 |
| 3 | $g \equiv 1 \pmod 4$ | 4 |
| | $g \equiv 3 \pmod 4$ | 0 |
| 4 | $g = 1$ | 4 |
| | $g \equiv 1 \pmod 4, g \neq 1$ | 8 |
| | $g \equiv 3 \pmod 4$ | 0 |
| 5 | | 4 |
| *even* $\geq 6$ | $g = 1$ | 6 |
| | $g \equiv 1 \pmod 4, g \neq 1$ | 12 |
| | $g = 3$ | 8 |
| | $g \equiv 3 \pmod 8, g \neq 3$ | 24 |
| | $g \equiv 7 \pmod 8$ | 0 |
| *odd* $\geq 7$ | | 12 |

Table 4.3: Values of $k_f(n)$

*If $(\alpha, g) = (0,1)$, we have*

$$r(f;n) = l(n) - \left(\frac{-1}{h}\right) h.$$

*If $(\alpha, g) = (2,1)$, we have*

$$r(f;n) = 2l(n) - 2\left(\frac{-1}{h}\right) h.$$

*Proof.* By Theorem 2.1 (iii) and Proposition 2.1, we obtain for all $n \in \mathbb{N}$

$$r(f;n) = s(n)r(1,1,4,0,0,0;n) + r\left(1,1,4,0,0,0;\frac{n}{16}\right) + t(n),$$

where

$$s(n) = \sum_{j=1}^{32} a_j \omega_{32}^{jn}$$

and the $a_j$ are given by

$$a_j = \begin{cases} -\left(\frac{-1}{j}\right)\frac{i}{96} & j = 1,3,5,\ldots,31, \\ -\frac{1}{96} - \frac{1}{48}\left(\frac{-1}{j/2}\right)i & j = 2,6,10,\ldots,30, \\ -\frac{1}{96} - \frac{1}{16}\left(\frac{-1}{j/4}\right)i & j = 4,12,20,28, \\ -\frac{1}{32} - \frac{1}{16}\left(\frac{-1}{j/8}\right)i & j = 8,24, \\ \frac{1}{32} & j = 16, \\ \frac{5}{32} & j = 32, \end{cases}$$

and

$$t(n) = \begin{cases} -\left(\frac{-1}{\sqrt{n}}\right)\sqrt{n} & \text{if } n = \text{odd square}, \\ -2\left(\frac{-1}{\sqrt{n/4}}\right)\sqrt{n/4} & \text{if } n = 4 \times \text{odd square}, \\ 0 & \text{otherwise}. \end{cases}$$

A short calculation using

$$\omega_{32} = \frac{1}{2}\sqrt{2+\sqrt{2+\sqrt{2}}} + \frac{i}{2}\sqrt{2-\sqrt{2+\sqrt{2}}}$$

yields

$$\begin{aligned} s(n) &= \begin{cases} \frac{1}{4} & \text{if } n \equiv 1 \pmod 4, \\ \frac{1}{2} & \text{if } n \equiv 2 \pmod 8, \\ \frac{1}{3} & \text{if } n \equiv 4 \pmod{16} \text{ or } n \equiv 8 \pmod{32}, \\ 0 & \text{otherwise}, \end{cases} \\ &= \begin{cases} \frac{1}{4} & \text{if } \alpha = 0, g \equiv 1 \pmod 4, \\ \frac{1}{2} & \text{if } \alpha = 1, g \equiv 1 \pmod 4, \\ \frac{1}{3} & \text{if } \alpha = 2, g \equiv 1 \pmod 4 \text{ or } \alpha = 3, g \equiv 1 \pmod 4, \\ 0 & \text{otherwise}. \end{cases} \end{aligned}$$

Also

$$t(n) = \begin{cases} -\left(\frac{-1}{h}\right)h & \text{if } (\alpha, g) = (0,1), \\ -2\left(\frac{-1}{h}\right)h & \text{if } (\alpha, g) = (2,1), \\ 0 & \text{otherwise}. \end{cases}$$

Appealing to Proposition A.1 for the formula for $r(1,1,4,0,0,0;n)$, we obtain Theorem 4.3. □

The positive integers not represented by the form A3 are given in Table A.17. Next, we determine the representation number $r(f;n)$ when $f$ is the form A4.

**Theorem 4.4.** *Let $f$ denote the form A4, that is, $f = 4x^2 + 4y^2 + 5z^2 + 4zx$. If $(\alpha, g) \neq (0, 1)$, we have*

$$r(f;n) = k_f(n)l(n)h(\mathbb{Q}(\sqrt{-n^*})),$$

*where the values of $k_f(n)$ are given in Table 4.4.*

| $\alpha$ | $g$ | $k_f(n)$ |
|---|---|---|
| 0 | $g \equiv 1 \pmod 4, g \neq 1$ | 2 |
| | $g \equiv 3 \pmod 4$ | 0 |
| 1 | | 0 |
| 2 | $g = 1$ | 4 |
| | $g \equiv 1 \pmod 4, g \neq 1$ | 8 |
| | $g \equiv 3 \pmod 4$ | 0 |
| 3 | | 4 |
| *even* $\geq 4$ | $g = 1$ | 6 |
| | $g \equiv 1 \pmod 4, g \neq 1$ | 12 |
| | $g = 3$ | 8 |
| | $g \equiv 3 \pmod 8, g \neq 3$ | 24 |
| | $g \equiv 7 \pmod 8$ | 0 |
| *odd* $\geq 5$ | | 12 |

Table 4.4: Values of $k_f(n)$

*If $(\alpha, g) = (0, 1)$, we have*

$$r(f;n) = l(n) - \left(\frac{-1}{h}\right) h.$$

*Proof.* By Theorem 2.1 (iv) and Proposition 2.1, we obtain for all $n \in \mathbb{N}$

$$r(f;n) = s(n)r(1,1,4,0,0,0;n) + r\left(1,1,4,0,0,0;\frac{n}{4}\right) + t(n),$$

where

$$s(n) = -\frac{1}{8}(-1)^n + \frac{1}{8} = \left\{\begin{matrix} 0 & \text{if } n \text{ even} \\ \frac{1}{4} & \text{if } n \text{ odd} \end{matrix}\right\} = \begin{cases} 0 & \text{if } \alpha \geq 1, \\ \frac{1}{4} & \text{if } \alpha = 0, \end{cases}$$

and

$$t(n) = \left\{\begin{matrix} -\left(\frac{-1}{\sqrt{n}}\right)\sqrt{n} & \text{if } n = \text{odd square} \\ 0 & \text{otherwise} \end{matrix}\right\} = \begin{cases} -\left(\frac{-1}{h}\right)h & \text{if } \alpha = 0, g = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Appealing to Proposition A.1 for the formula for $r(1,1,4,0,0,0;n)$, we obtain Theorem 4.4. □

The positive integers which are not represented by the form A4 are given in Table A.17. They were first determined by Lomadze [19, Corollary 2, p. 144]. Next, we determine the representation number $r(f;n)$ when $f$ is the form A5.

**Theorem 4.5.** *Let $f$ denote the form A5, that is, $f = 4x^2 + 9y^2 + 9z^2 + 2yz + 4zx + 4xy$. If $(\alpha, g) \neq (0,1)$, we have*

$$r(f;n) = k_f(n)l(n)h(\mathbb{Q}(\sqrt{-n^*})),$$

*where the values of $k_f(n)$ are given in Table 4.5.*

| $\alpha$ | $g$ | $k_f(n)$ |
|---|---|---|
| 0 | $g \equiv 1 \pmod 8, g \neq 1$ | 2 |
| | $g \equiv 3,5,7 \pmod 8$ | 0 |
| 1 | | 0 |
| 2 | $g = 1$ | 2 |
| | $g \equiv 1 \pmod 4, g \neq 1$ | 4 |
| | $g \equiv 3 \pmod 4$ | 0 |
| 3 | | 0 |
| *even* $\geq 4$ | $g = 1$ | 6 |
| | $g \equiv 1 \pmod 4, g \neq 1$ | 12 |
| | $g = 3$ | 8 |
| | $g \equiv 3 \pmod 8, g \neq 3$ | 24 |
| | $g \equiv 7 \pmod 8$ | 0 |
| *odd* $\geq 5$ | | 12 |

Table 4.5: Values of $k_f(n)$

*If $(\alpha, g) = (0,1)$, we have*

$$r(f;n) = l(n) - \left(\frac{-1}{h}\right) h.$$

*Proof.* By Theorem 2.1 (v) and Proposition 2.1, we obtain for all $n \in \mathbb{N}$

$$r(f;n) = s(n)r(1,4,4,0,0,0;n) + r\left(1,4,4,0,0,0;\frac{n}{4}\right) + t(n),$$

where

$$s(n) = \begin{cases} \frac{1}{2} & \text{if } n \equiv 1 \pmod 8 \\ 0 & \text{otherwise} \end{cases} = \begin{cases} \frac{1}{2} & \text{if } \alpha = 0, g \equiv 1 \pmod 8, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$t(n) = \begin{cases} -\left(\frac{-1}{\sqrt{n}}\right)\sqrt{n} & \text{if } n = \text{odd square} \\ 0 & \text{otherwise} \end{cases} = \begin{cases} -\left(\frac{-1}{h}\right)h & \text{if } \alpha = 0, g = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Appealing to Proposition A.3 for the formula for $r(1, 4, 4, 0, 0, 0; n)$, we obtain Theorem 4.5. □

The positive integers not represented by the ternary quadratic form A5 are given in Table A.17. They were first determined by Lomadze [19, Corollary 2, p. 146]. Next, we determine the representation number $r(f; n)$ when $f$ is the form A6.

**Theorem 4.6.** *Let $f$ denote the form A6, that is, $f = 4x^2 + 5y^2 + 13z^2 + 2yz$. If $(\alpha, g) \neq (0, 1)$, we have*

$$r(f; n) = k_f(n)l(n)h(\mathbb{Q}(\sqrt{-n^*})),$$

*where the values of $k_f(n)$ are given in Table* 4.6.

| $\alpha$ | $g$ | $k_f(n)$ |
|---|---|---|
| 0 | $g \equiv 1 \pmod 4, g \neq 1$ | 1 |
| | $g \equiv 3 \pmod 4$ | 0 |
| 1 | | 0 |
| 2 | $g = 1$ | 2 |
| | $g \equiv 1 \pmod 4, g \neq 1$ | 4 |
| | $g \equiv 3 \pmod 4$ | 0 |
| 3 | $g \equiv 1 \pmod 4$ | 0 |
| | $g \equiv 3 \pmod 4$ | 4 |
| 4 | $g = 1$ | 4 |
| | $g \equiv 1 \pmod 4, g \neq 1$ | 8 |
| | $g \equiv 3 \pmod 4$ | 0 |
| 5 | | 4 |
| *even* $\geq 6$ | $g = 1$ | 6 |
| | $g \equiv 1 \pmod 4, g \neq 1$ | 12 |
| | $g = 3$ | 8 |
| | $g \equiv 3 \pmod 8, g \neq 3$ | 24 |
| | $g \equiv 7 \pmod 8$ | 0 |
| *odd* $\geq 7$ | | 12 |

Table 4.6: Values of $k_f(n)$

*If* $(\alpha, g) = (0,1)$, *we have*

$$r(f;n) = \frac{1}{2}l(n) - \frac{1}{2}\left(\frac{-1}{h}\right)h.$$

*Proof.* By Theorem 2.1 (vi) and Proposition 2.1, we obtain for all $n \in \mathbb{N}$

$$r(f;n) = s(n)r(1,4,5,4,0,0;n) + r\left(1,4,5,4,0,0;\frac{n}{4}\right) + t(n),$$

where

$$s(n) = -\frac{1}{8}(-1)^n + \frac{1}{8} = \left.\begin{cases} 0 & \text{if } n \text{ even} \\ \frac{1}{4} & \text{if } n \text{ odd} \end{cases}\right\} = \begin{cases} 0 & \text{if } \alpha \geq 1, \\ \frac{1}{4} & \text{if } \alpha = 0, \end{cases}$$

and

$$t(n) = \left.\begin{cases} -\frac{1}{2}\left(\frac{-1}{\sqrt{n}}\right)\sqrt{n} & \text{if } n = \text{odd square} \\ 0 & \text{otherwise} \end{cases}\right\} = \begin{cases} -\frac{1}{2}\left(\frac{-1}{h}\right)h & \text{if } \alpha = 0, g = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Appealing to Proposition A.4 for the formula for $r(1,4,5,4,0,0;n)$, we obtain Theorem 4.6. □

The positive integers not represented by the form A6 are listed in Table A.17. Next, we determine the representation number $r(f;n)$ when $f$ is the form A7.

**Theorem 4.7.** *Let* $f$ *denote the form A7, that is,* $f = 5x^2 + 8y^2 + 8z^2 + 4zx + 4xy$. *If* $(\alpha, g) \neq (0,1), (2,1)$, *we have*

$$r(f;n) = k_f(n)l(n)h(\mathbb{Q}(\sqrt{-n^*})),$$

*where the values of* $k_f(n)$ *are given in Table 4.7.*

| $\alpha$ | $g$ | $k_f(n)$ |
|---|---|---|
| 0 | $g \equiv 1 \pmod 4, g \neq 1$ | 1 |
| | $g \equiv 3 \pmod 4$ | 0 |
| 1 | | 0 |
| 2 | $g \equiv 1 \pmod 4, g \neq 1$ | 4 |
| | $g \equiv 3 \pmod 4$ | 0 |
| 3 | $g \equiv 1 \pmod 4$ | 4 |
| | $g \equiv 3 \pmod 4$ | 0 |
| 4 | $g = 1$ | 4 |
| | $g \equiv 1 \pmod 4, g \neq 1$ | 8 |
| | $g \equiv 3 \pmod 4$ | 0 |
| 5 | | 4 |
| | | *Continued on next page* |

| $\alpha$ | $g$ | $k_f(n)$ |
|---|---|---|
| *even* $\geq 6$ | $g = 1$ | 6 |
| | $g \equiv 1 \pmod 4, g \neq 1$ | 12 |
| | $g = 3$ | 8 |
| | $g \equiv 3 \pmod 8, g \neq 3$ | 24 |
| | $g \equiv 7 \pmod 8$ | 0 |
| *odd* $\geq 7$ | | 12 |

Table 4.7: Values of $k_f(n)$

*If* $(\alpha, g) = (0, 1)$, *we have*

$$r(f; n) = \frac{1}{2}l(n) - \frac{1}{2}\left(\frac{-1}{h}\right)h.$$

*If* $(\alpha, g) = (2, 1)$, *we have*

$$r(f; n) = 2l(n) - 2\left(\frac{-1}{h}\right)h.$$

*Proof.* By Theorem 2.1 (vii) and Proposition 2.1, we obtain for all $n \in \mathbb{N}$

$$r(f; n) = s(n)r(1, 1, 4, 0, 0, 0; n) + r\left(1, 1, 4, 0, 0, 0; \frac{n}{16}\right) + t(n),$$

where

$$s(n) = \begin{cases} \frac{1}{8} & \text{if } n \equiv 1 \pmod 4, \\ \frac{1}{3} & \text{if } n \equiv 4 \pmod{16} \text{ or } n \equiv 8 \pmod{32}, \\ 0 & \text{otherwise}, \end{cases}$$

$$= \begin{cases} \frac{1}{8} & \text{if } \alpha = 0, g \equiv 1 \pmod 4, \\ \frac{1}{3} & \text{if } \alpha = 2, g \equiv 1 \pmod 4 \text{ or } \alpha = 3, g \equiv 1 \pmod 4, \\ 0 & \text{otherwise}, \end{cases}$$

and

$$t(n) = \begin{cases} -\frac{1}{2}\left(\frac{-1}{\sqrt{n}}\right)\sqrt{n} & \text{if } n = \text{odd square}, \\ -2\left(\frac{-1}{\sqrt{n/4}}\right)\sqrt{n/4} & \text{if } n = 4 \times \text{odd square}, \\ 0 & \text{otherwise}, \end{cases}$$

$$= \begin{cases} -\frac{1}{2}\left(\frac{-1}{h}\right)h & \text{if } \alpha = 0, g = 1, \\ -2\left(\frac{-1}{h}\right)h & \text{if } \alpha = 2, g = 1, \\ 0 & \text{otherwise}. \end{cases}$$

Appealing to Proposition A.1 for the formula for $r(1, 1, 4, 0, 0, 0; n)$, we obtain Theorem 4.7. □

The positive integers not represented by the form A7 are given in Table A.17. Next, we determine the representation number $r(f;n)$ when $f$ is the form A8.

**Theorem 4.8.** *Let $f$ denote the form A8, that is, $f = 4x^2 + 8y^2 + 17z^2 + 4zx$. If $(\alpha, g) \neq (0,1)$, we have*

$$r(f;n) = k_f(n)l(n)h(\mathbb{Q}(\sqrt{-n^*})),$$

*where the values of $k_f(n)$ are given in Table 4.8.*

| $\alpha$ | $g$ | $k_f(n)$ |
|---|---|---|
| 0 | $g \equiv 1 \pmod 8, g \neq 1$ | 1 |
| | $g \equiv 3,5,7 \pmod 8$ | 0 |
| 1 | | 0 |
| 2 | $g \equiv 1,3 \pmod 8$ | 2 |
| | $g \equiv 5,7 \pmod 8$ | 0 |
| 3 | $g = 1$ | 2 |
| | $g \equiv 1 \pmod 8, g \neq 1$ | 4 |
| | $g = 3$ | 4 |
| | $g \equiv 3 \pmod 8, g \neq 3$ | 12 |
| | $g \equiv 5,7 \pmod 8$ | 0 |
| 4 | | 2 |
| 5 | $g = 1$ | 2 |
| | $g \equiv 1 \pmod 4, g \neq 1$ | 4 |
| | $g = 3$ | 8 |
| | $g \equiv 3 \pmod 8, g \neq 3$ | 24 |
| | $g \equiv 7 \pmod 8$ | 0 |
| 6 | | 4 |
| *odd* $\geq 7$ | $g = 1$ | 6 |
| | $g \equiv 1 \pmod 4, g \neq 1$ | 12 |
| | $g = 3$ | 8 |
| | $g \equiv 3 \pmod 8, g \neq 3$ | 24 |
| | $g \equiv 7 \pmod 8$ | 0 |
| *even* $\geq 8$ | | 12 |

Table 4.8: Values of $k_f(n)$

*If $(\alpha, g) = (0,1)$, we have*

$$r(f;n) = l(n) - \left(\frac{-2}{h}\right)h.$$

*Proof.* By Theorem 2.1 (viii) and Proposition 2.1, we obtain for all $n \in \mathbb{N}$

$$r(f;n) = s(n)r(1,2,16,0,0,0;n) + r\left(1,2,16,0,0,0;\frac{n}{4}\right) + t(n),$$

where

$$s(n) = -\frac{i}{8}i^n - \frac{1}{8}(-1)^n + \frac{i}{8}(-i)^n + \frac{1}{8} = \begin{cases} \frac{1}{2} & \text{if } n \equiv 1 \pmod 4, \\ 0 & \text{otherwise,} \end{cases}$$
$$= \begin{cases} \frac{1}{2} & \text{if } \alpha = 0, g \equiv 1 \pmod 4, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$t(n) = \left\{ \begin{matrix} -\left(\frac{-2}{\sqrt{n}}\right)\sqrt{n} & \text{if } n = \text{odd square} \\ 0 & \text{otherwise} \end{matrix} \right\} = \begin{cases} -\left(\frac{-2}{h}\right)h & \text{if } \alpha = 0, g = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Appealing to Proposition A.5 for the formula for $r(1,2,16,0,0,0;n)$, we obtain Theorem 4.8. □

The positive integers not represented by the form A8 are given in Table A.17. They were first determined by Lomadze [19, Corollary 2, p. 150]. Next, we determine the representation number $r(f;n)$ when $f$ is the form A9.

**Theorem 4.9.** *Let $f$ denote the form A9, that is, $f = 9x^2 + 9y^2 + 16z^2 + 8yz + 8zx + 2xy$. If $(\alpha, g) \neq (0,1), (2,1)$, we have*

$$r(f;n) = k_f(n)l(n)h(\mathbb{Q}(\sqrt{-n^*})),$$

*where the values of $k_f(n)$ are given in Table 4.9.*

| $\alpha$ | $g$ | $k_f(n)$ |
|---|---|---|
| 0 | $g \equiv 1 \pmod 8, g \neq 1$ | 1 |
| | $g \equiv 3,5,7 \pmod 8$ | 0 |
| 1 | | 0 |
| 2 | $g \equiv 1 \pmod 4, g \neq 1$ | 2 |
| | $g \equiv 3 \pmod 4$ | 0 |
| 3 | | 0 |
| 4 | $g = 1$ | 4 |
| | $g \equiv 1 \pmod 4, g \neq 1$ | 8 |
| | $g \equiv 3 \pmod 4$ | 0 |
| 5 | | 4 |
| *even* $\geq 6$ | $g = 1$ | 6 |
| | $g \equiv 1 \pmod 4, g \neq 1$ | 12 |
| | $g = 3$ | 8 |
| | $g \equiv 3 \pmod 8, g \neq 3$ | 24 |
| | $g \equiv 7 \pmod 8$ | 0 |
| *odd* $\geq 7$ | | 12 |

Table 4.9: Values of $k_f(n)$

*If* $(\alpha, g) = (0, 1)$*, we have*

$$r(f;n) = \frac{1}{2}l(n) - \frac{1}{2}\left(\frac{-1}{h}\right)h.$$

*If* $(\alpha, g) = (2, 1)$*, we have*

$$r(f;n) = l(n) - \left(\frac{-1}{h}\right)h.$$

*Proof.* We recall that $\omega_{16} = e^{2\pi i/16} = \frac{1}{2}\left(\sqrt{2+\sqrt{2}} + i\sqrt{2-\sqrt{2}}\right)$. By Theorem 2.1 (ix) and Proposition 2.1, we obtain for all $n \in \mathbb{N}$

$$r(f;n) = s(n)r(1,1,4,0,0,0;n) + r\left(1,1,4,0,0,0;\frac{n}{16}\right) + t(n),$$

where

$$\begin{aligned}
s(n) &= \frac{-i}{96}\left(\omega_{16}^{n} + \omega_{16}^{5n} + \omega_{16}^{9n} + \omega_{16}^{13n}\right) + \frac{i}{96}\left(\omega_{16}^{3n} + \omega_{16}^{7n} + \omega_{16}^{11n} + \omega_{16}^{15n}\right) \\
&\quad + \frac{1}{384}\left(-4 + 3\sqrt{2} - 3i\sqrt{2}\right)\omega_{16}^{2n} + \frac{1}{384}\left(-4 - 3\sqrt{2} - 3i\sqrt{2}\right)\omega_{16}^{6n} \\
&\quad + \frac{1}{384}\left(-4 - 3\sqrt{2} + 3i\sqrt{2}\right)\omega_{16}^{10n} + \frac{1}{384}\left(-4 + 3\sqrt{2} + 3i\sqrt{2}\right)\omega_{16}^{14n} \\
&\quad + \frac{1}{192}(2-3i)\omega_{16}^{4n} + \frac{1}{192}(2+3i)\omega_{16}^{12n} - \frac{1}{192}\omega_{16}^{8n} + \frac{5}{192} \\
&= \left.\begin{cases} \frac{1}{8} & \text{if } n \equiv 1 \pmod 8 \\ \frac{1}{6} & \text{if } n \equiv 4 \pmod{16} \\ 0 & \text{otherwise} \end{cases}\right\} = \begin{cases} \frac{1}{8} & \text{if } \alpha = 0, g \equiv 1 \pmod 8, \\ \frac{1}{6} & \text{if } \alpha = 2, g \equiv 1 \pmod 4, \\ 0 & \text{otherwise,} \end{cases}
\end{aligned}$$

and

$$\begin{aligned}
t(n) &= \begin{cases} -\frac{1}{2}\left(\frac{-1}{\sqrt{n}}\right)\sqrt{n} & \text{if } n = \text{odd square}, \\ -\left(\frac{-1}{\sqrt{n/4}}\right)\sqrt{n/4} & \text{if } n = 4 \times \text{odd square}, \\ 0 & \text{otherwise,} \end{cases} \\
&= \begin{cases} -\frac{1}{2}\left(\frac{-1}{h}\right)h & \text{if } \alpha = 0, g = 1, \\ -\left(\frac{-1}{h}\right)h & \text{if } \alpha = 2, g = 1, \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

Appealing to Proposition A.1 for the formula for $r(1,1,4,0,0,0;n)$, we obtain Theorem 4.9. □

The positive integers not represented by the spinor regular ternary form A9 are given in Table A.17. Next, we determine the representation number $r(f;n)$ when $f$ is the form A10.

**Theorem 4.10.** *Let $f$ denote the form A10, that is, $f = 4x^2 + 9y^2 + 32z^2 + 4xy$. If $(\alpha, g) \neq (0, 1)$, we have*

$$r(f; n) = k_f(n)l(n)h(\mathbb{Q}(\sqrt{-n^*})),$$

*where the values of $k_f(n)$ are given in Table* 4.10.

| $\alpha$ | $g$ | $k_f(n)$ |
|---|---|---|
| 0 | $g \equiv 1 \pmod 8, g \neq 1$ | 1 |
| | $g \equiv 3, 5, 7 \pmod 8$ | 0 |
| 1 | | 0 |
| 2 | $g = 1$ | 2 |
| | $g \equiv 1 \pmod 8, g \neq 1$ | 4 |
| | $g \equiv 3, 5, 7 \pmod 8$ | 0 |
| 3 | | 0 |
| 4 | $g = 1$ | 2 |
| | $g \equiv 1 \pmod 4, g \neq 1$ | 4 |
| | $g = 3$ | 8 |
| | $g \equiv 3 \pmod 8, g \neq 3$ | 24 |
| | $g \equiv 7 \pmod 8$ | 0 |
| 5 | | 4 |
| *even* $\geq 6$ | $g = 1$ | 6 |
| | $g \equiv 1 \pmod 4, g \neq 1$ | 12 |
| | $g = 3$ | 8 |
| | $g \equiv 3 \pmod 8, g \neq 3$ | 24 |
| | $g \equiv 7 \pmod 8$ | 0 |
| *odd* $\geq 7$ | | 12 |

Table 4.10: Values of $k_f(n)$

*If $(\alpha, g) = (0, 1)$, we have*

$$r(f; n) = \frac{1}{2}l(n) - \frac{1}{2}\left(\frac{-1}{h}\right)h.$$

*Proof.* By Theorem 2.1 (x) and Proposition 2.1, we obtain for all $n \in \mathbb{N}$

$$r(f; n) = s(n)r(1, 8, 8, 0, 0, 0; n) + r\left(1, 8, 8, 0, 0, 0; \frac{n}{4}\right) + t(n),$$

where

$$s(n) = -\frac{1}{8}(-1)^n + \frac{1}{8} = \left.\begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{1}{4} & \text{if } n \text{ is odd} \end{cases}\right\} = \begin{cases} 0 & \text{if } \alpha \geq 1, \\ \frac{1}{4} & \text{if } \alpha = 0, \end{cases}$$

and

$$t(n) = \begin{Bmatrix} \begin{cases} -\frac{1}{2}\left(\frac{-1}{\sqrt{n}}\right)\sqrt{n} & \text{if } n = \text{odd square} \\ 0 & \text{if } n \neq \text{odd square} \end{cases} \end{Bmatrix} = \begin{cases} -\frac{1}{2}\left(\frac{-1}{h}\right)h & \text{if } \alpha = 0, g = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Appealing to Proposition A.6 for the formula for $r(1,8,8,0,0,0;n)$, we obtain Theorem 4.10. □

The positive integers not represented by the spinor regular ternary form A10 are given in Table A.17. Next, we determine the representation number $r(f;n)$ when $f$ is the form A11.

**Theorem 4.11.** *Let $f$ denote the form A11, that is, $f = 5x^2 + 13y^2 + 16z^2 + 2xy$. If $(\alpha, g) \neq (2,1)$, we have*

$$r(f;n) = k_f(n)l(n)h(\mathbb{Q}(\sqrt{-n^*})),$$

*where the values of $k_f(n)$ are given in Table 4.11.*

| $\alpha$ | $g$ | $k_f(n)$ |
|---|---|---|
| 0 | $g \equiv 1,3,7 \pmod 8$ | 0 |
| | $g \equiv 5 \pmod 8$ | 1 |
| 1 | | 0 |
| 2 | $g \equiv 1 \pmod 4, g \neq 1$ | 2 |
| | $g \equiv 3 \pmod 4$ | 0 |
| 3 | | 0 |
| 4 | $g = 1$ | 4 |
| | $g \equiv 1 \pmod 4, g \neq 1$ | 8 |
| | $g \equiv 3 \pmod 4$ | 0 |
| 5 | | 4 |
| *even* $\geq 6$ | $g = 1$ | 6 |
| | $g \equiv 1 \pmod 4, g \neq 1$ | 12 |
| | $g = 3$ | 8 |
| | $g \equiv 3 \pmod 8, g \neq 3$ | 24 |
| | $g \equiv 7 \pmod 8$ | 0 |
| *odd* $\geq 7$ | | 12 |

Table 4.11: Values of $k_f(n)$

*If $(\alpha, g) = (2,1)$, we have*

$$r(f;n) = l(n) - \left(\frac{-1}{h}\right)h.$$

*Proof.* By Theorem 2.1 (xi) and Proposition 2.1, we obtain for all $n \in \mathbb{N}$

$$r(f;n) = s(n)r(1,1,4,0,0,0;n) + r\left(1,1,4,0,0,0;\frac{n}{16}\right) + t(n),$$

where

$$\begin{aligned} s(n) = &\frac{-i}{96}\left(\omega_{16}^{n} + \omega_{16}^{5n} + \omega_{16}^{9n} + \omega_{16}^{13n}\right) + \frac{i}{96}\left(\omega_{16}^{3n} + \omega_{16}^{7n} + \omega_{16}^{11n} + \omega_{16}^{15n}\right) \\ &+ \frac{1}{384}\left(-4 - 3\sqrt{2} + 3i\sqrt{2}\right)\omega_{16}^{2n} + \frac{1}{384}\left(-4 + 3\sqrt{2} + 3i\sqrt{2}\right)\omega_{16}^{6n} \\ &+ \frac{1}{384}\left(-4 + 3\sqrt{2} - 3i\sqrt{2}\right)\omega_{16}^{10n} + \frac{1}{384}\left(-4 - 3\sqrt{2} - 3i\sqrt{2}\right)\omega_{16}^{14n} \\ &+ \frac{1}{192}(2-3i)\omega_{16}^{4n} + \frac{1}{192}(2+3i)\omega_{16}^{12n} - \frac{1}{192}\omega_{16}^{8n} + \frac{5}{192} \end{aligned}$$

and

$$t(n) = \begin{cases} -\left(\frac{-1}{\sqrt{n/2}}\right)\frac{\sqrt{n}}{2} & \text{if } n = 4 \times \text{odd square}, \\ 0 & \text{otherwise.} \end{cases}$$

Using

$$\omega_{16}^{n} + \omega_{16}^{5n} + \omega_{16}^{9n} + \omega_{16}^{13n} = \begin{cases} (-1)^{n/8}4 & \text{if } n \equiv 0 \pmod 8, \\ (-1)^{(n-4)/8}4i & \text{if } n \equiv 4 \pmod 8, \\ 0 & \text{if } n \not\equiv 0 \pmod 4, \end{cases}$$

$$\omega_{16}^{3n} + \omega_{16}^{7n} + \omega_{16}^{11n} + \omega_{16}^{15n} = \begin{cases} (-1)^{n/8}4 & \text{if } n \equiv 0 \pmod 8, \\ (-1)^{(n+4)/8}4i & \text{if } n \equiv 4 \pmod 8, \\ 0 & \text{if } n \not\equiv 0 \pmod 4, \end{cases}$$

$$\omega_{16}^{2n} + \omega_{16}^{6n} + \omega_{16}^{10n} + \omega_{16}^{14n} = \begin{cases} (-1)^{n/4}4 & \text{if } n \equiv 0 \pmod 4, \\ 0 & \text{if } n \not\equiv 0 \pmod 4, \end{cases}$$

$$\omega_{16}^{2n} - \omega_{16}^{6n} - \omega_{16}^{10n} + \omega_{16}^{14n} = \begin{cases} 0 & \text{if } n \equiv 0 \pmod 2, \\ \omega_8^n(2 - (-1)^{(n-1)/2}2i) & \text{if } n \equiv 1 \pmod 2, \end{cases}$$

$$\omega_{16}^{2n} + \omega_{16}^{6n} - \omega_{16}^{10n} - \omega_{16}^{14n} = \begin{cases} 0 & \text{if } n \equiv 0 \pmod 2, \\ \omega_8^n(2 + (-1)^{(n-1)/2}2i) & \text{if } n \equiv 1 \pmod 2, \end{cases}$$

$$\omega_{16}^{4n} + \omega_{16}^{12n} = \begin{cases} 2(-1)^{n/2} & \text{if } n \equiv 0 \pmod 2, \\ 0 & \text{if } n \equiv 1 \pmod 2, \end{cases}$$

$$\omega_{16}^{4n} - \omega_{16}^{12n} = \begin{cases} 0 & \text{if } n \equiv 0 \pmod 2, \\ 2i^n & \text{if } n \equiv 1 \pmod 2, \end{cases}$$

$$\omega_{16}^{8n} = (-1)^n,$$

we obtain

$$s(n) = \begin{cases} 0 & \text{if } n \equiv 0,1,2,3,6,7 \pmod 8, \\ \frac{1}{12}\left(1+(-1)^{(n-4)/8}\right) & \text{if } n \equiv 4 \pmod 8, \\ \frac{1}{8} & \text{if } n \equiv 5 \pmod 8. \end{cases}$$

Also

$$t(n) = \begin{cases} -\left(\frac{-1}{h}\right)h & \text{if } (\alpha, g) = (2,1), \\ 0 & \text{otherwise.} \end{cases}$$

Appealing to Proposition A.1 for the formula for $r(1,1,4,0,0,0;n)$, we obtain Theorem 4.11. □

The positive integers not represented by the spinor regular ternary form A11 are given in Table A.17. Next, we determine the representation number $r(f;n)$ when $f$ is the form A12.

**Theorem 4.12.** *Let $f$ denote the form A12, that is, $f = 9x^2+17y^2+32z^2-8yz+8zx+6xy$. If $(\alpha, g) \neq (0,1),(2,1),(4,1)$, we have*

$$r(f;n) = k_f(n)l(n)h(\mathbb{Q}(\sqrt{-n^*})),$$

*where the values of $k_f(n)$ are given in Table* 4.12.

| $\alpha$ | $g$ | $k_f(n)$ |
|---|---|---|
| 0 | $g \equiv 1 \pmod 8, g \neq 1$ | $\frac{1}{2}$ |
| | $g \equiv 3,5,7 \pmod 8$ | 0 |
| 1 | | 0 |
| 2 | $g \equiv 1 \pmod 4, g \neq 1$ | 1 |
| | $g \equiv 3 \pmod 4$ | 0 |
| 3 | | 0 |
| 4 | $g \equiv 1 \pmod 4, g \neq 1$ | 4 |
| | $g \equiv 3 \pmod 4$ | 0 |
| 5 | $g \equiv 1 \pmod 4$ | 4 |
| | $g \equiv 3 \pmod 4$ | 0 |
| 6 | $g = 1$ | 4 |
| | $g \equiv 1 \pmod 4, g \neq 1$ | 8 |
| | $g \equiv 3 \pmod 4$ | 0 |
| 7 | | 4 |
| *even* $\geq 8$ | $g = 1$ | 6 |
| | $g \equiv 1 \pmod 4, g \neq 1$ | 12 |
| | $g = 3$ | 8 |
| | $g \equiv 3 \pmod 8, g \neq 3$ | 24 |
| | $g \equiv 7 \pmod 8$ | 0 |
| *odd* $\geq 9$ | | 12 |

Table 4.12: Values of $k_f(n)$

*If* $(\alpha, g) = (0,1)$*, we have*

$$r(f;n) = \frac{1}{4}l(n) - \frac{1}{4}\left(\frac{-1}{h}\right)h.$$

*If* $(\alpha, g) = (2,1)$*, we have*

$$r(f;n) = \frac{1}{2}l(n) - \frac{1}{2}\left(\frac{-1}{h}\right)h.$$

*If* $(\alpha, g) = (4,1)$*, we have*

$$r(f;n) = 2l(n) - 2\left(\frac{-1}{h}\right)h.$$

*Proof.* By Theorem 2.1 (xii) and Proposition 2.1, we obtain for all $n \in \mathbb{N}$

$$r(f;n) = s(n)r(1,1,4,0,0,0;n) + r\left(1,1,4,0,0,0;\frac{n}{64}\right) + t(n),$$

where

$$\begin{aligned} s(n) = &-\frac{i}{384}\sum_{\substack{j=1\\ j\equiv 1 \pmod 2}}^{127}\left(\frac{-1}{j}\right)\omega_{128}^{jn} - \frac{1}{384}\sum_{\substack{j=2\\ j\equiv 2 \pmod 4}}^{126}\omega_{128}^{jn} \\ &- \frac{i}{192}\sum_{\substack{j=2\\ j\equiv 2 \pmod 4}}^{126}\left(\frac{-1}{j/2}\right)\omega_{128}^{jn} - \frac{1}{384}\sum_{\substack{j=4\\ j\equiv 4 \pmod 8}}^{124}\omega_{128}^{jn} \\ &+ \frac{1}{128}\sum_{\substack{j=8\\ j\equiv 8 \pmod{16}}}^{120}\omega_{128}^{jn} - \frac{i}{192}\sum_{\substack{j=8\\ j\equiv 8 \pmod{16}}}^{120}\left(\frac{-1}{j/8}\right)\omega_{128}^{jn} \\ &+ \frac{1}{384}\sum_{\substack{j=16\\ j\equiv 16 \pmod{32}}}^{112}\omega_{128}^{jn} + \frac{\sqrt{2}}{256}\sum_{\substack{j=16\\ j\equiv 16 \pmod{32}}}^{112}\left(\frac{2}{j/16}\right)\omega_{128}^{jn} \\ &- \frac{i\sqrt{2}}{256}\sum_{\substack{j=16\\ j\equiv 16 \pmod{32}}}^{112}\left(\frac{-2}{j/16}\right)\omega_{128}^{jn} + \frac{5}{384}\left(\omega_{128}^{32n} + \omega_{128}^{96n}\right) \\ &- \frac{i}{128}\left(\omega_{128}^{32n} - \omega_{128}^{96n}\right) + \frac{1}{192}(-1)^n + \frac{1}{48} \end{aligned}$$

and

$$t(n)=\begin{cases}-\frac{1}{4}\left(\frac{-1}{\sqrt{n}}\right)\sqrt{n} & \text{if } n=\text{odd square},\\ -\frac{1}{2}\left(\frac{-1}{\sqrt{n/4}}\right)\sqrt{n/4} & \text{if } n=4\times\text{odd square},\\ -2\left(\frac{-1}{\sqrt{n/16}}\right)\sqrt{n/16} & \text{if } n=16\times\text{odd square},\\ 0 & \text{otherwise},\end{cases}$$
$$=\begin{cases}-\frac{1}{4}\left(\frac{-1}{h}\right)h & \text{if } \alpha=0, g=1,\\ -\frac{1}{2}\left(\frac{-1}{h}\right)h & \text{if } \alpha=2, g=1,\\ -2\left(\frac{-1}{h}\right)h & \text{if } \alpha=4, g=1,\\ 0 & \text{otherwise}.\end{cases}$$

Making use of the results

$$\sum_{\substack{j=1\\ j\equiv 1 \ (\mathrm{mod}\ 2)}}^{127}\left(\frac{-1}{j}\right)\omega_{128}^{jn}=\begin{cases}(-1)^{(n-32)/64}64i & \text{if } n\equiv 32 \pmod{64},\\ 0 & \text{if } n\not\equiv 32 \pmod{64},\end{cases}$$
$$\sum_{\substack{j=2\\ j\equiv 2 \ (\mathrm{mod}\ 4)}}^{126}\omega_{128}^{jn}=\begin{cases}(-1)^{n/32}32 & \text{if } n\equiv 0 \pmod{32},\\ 0 & \text{if } n\not\equiv 0 \pmod{32},\end{cases}$$
$$\sum_{\substack{j=2\\ j\equiv 2 \ (\mathrm{mod}\ 4)}}^{126}\left(\frac{-1}{j/2}\right)\omega_{128}^{jn}=\begin{cases}(-1)^{(n-16)/32}32i & \text{if } n\equiv 16 \pmod{32},\\ 0 & \text{if } n\not\equiv 16 \pmod{32},\end{cases}$$
$$\sum_{\substack{j=4\\ j\equiv 4 \ (\mathrm{mod}\ 8)}}^{124}\omega_{128}^{jn}=\begin{cases}(-1)^{n/16}16 & \text{if } n\equiv 0 \pmod{16},\\ 0 & \text{if } n\not\equiv 0 \pmod{16},\end{cases}$$
$$\sum_{\substack{j=8\\ j\equiv 8 \ (\mathrm{mod}\ 16)}}^{120}\omega_{128}^{jn}=\begin{cases}(-1)^{n/8}8 & \text{if } n\equiv 0 \pmod{8},\\ 0 & \text{if } n\not\equiv 0 \pmod{8},\end{cases}$$
$$\sum_{\substack{j=8\\ j\equiv 8 \ (\mathrm{mod}\ 16)}}^{120}\left(\frac{-1}{j/8}\right)\omega_{128}^{jn}=\begin{cases}(-1)^{(n-4)/8}8i & \text{if } n\equiv 4 \pmod{8},\\ 0 & \text{if } n\not\equiv 4 \pmod{8},\end{cases}$$
$$\sum_{\substack{j=16\\ j\equiv 16 \ (\mathrm{mod}\ 32)}}^{112}\omega_{128}^{jn}=\begin{cases}(-1)^{n/4}4 & \text{if } n\equiv 0 \pmod{4},\\ 0 & \text{if } n\not\equiv 0 \pmod{4},\end{cases}$$
$$\sum_{\substack{j=16\\ j\equiv 16 \ (\mathrm{mod}\ 32)}}^{112}\left(\frac{2}{j/16}\right)\omega_{128}^{jn}=\begin{cases}(-1)^{(n-1)/2}2\sqrt{2} & \text{if } n\equiv 1 \pmod{2},\\ 0 & \text{if } n\equiv 0 \pmod{2},\end{cases}$$

$$\sum_{\substack{j=16 \\ j\equiv 16 \pmod{32}}}^{112} \left(\frac{-2}{j/16}\right)\omega_{128}^{jn} = \begin{cases} 2\sqrt{2}i & \text{if } n \equiv 1,3 \pmod 8, \\ -2\sqrt{2}i & \text{if } n \equiv 5,7 \pmod 8, \\ 0 & \text{if } n \not\equiv 1 \pmod 2, \end{cases}$$

$$\omega_{128}^{32n} + \omega_{128}^{96n} = i^n + i^{3n} = \begin{cases} (-1)^{n/2}2 & \text{if } n \equiv 0 \pmod 2, \\ 0 & \text{if } n \equiv 1 \pmod 2, \end{cases}$$

$$i\left(\omega_{128}^{32n} - \omega_{128}^{96n}\right) = i(i^n - i^{3n}) = \begin{cases} 0 & \text{if } n \equiv 0 \pmod 2, \\ (-1)^{(n+1)/2}2 & \text{if } n \equiv 1 \pmod 2, \end{cases}$$

we obtain

$$s(n) = \begin{cases} \frac{1}{16} & \text{if } n \equiv 1 \pmod 8, \\ \frac{1}{12} & \text{if } n \equiv 4 \pmod{16}, \\ \frac{1}{3} & \text{if } n \equiv 16, 32, 80 \pmod{128}, \\ 0 & \text{otherwise}, \end{cases}$$

$$= \begin{cases} \frac{1}{16} & \text{if } \alpha = 0, g \equiv 1 \pmod 8, \\ \frac{1}{12} & \text{if } \alpha = 2, g \equiv 1 \pmod 4, \\ \frac{1}{3} & \text{if } \alpha = 4, 5, g \equiv 1 \pmod 4, \\ 0 & \text{otherwise}. \end{cases}$$

Appealing to Proposition A.1 for the formula for $r(1,1,4,0,0,0;n)$, we obtain Theorem 4.12. □

The positive integers not represented by the spinor regular form A12 follow from Theorem 4.12 and are listed in Table A.17. They were obtained by Berkovich [3, Theorem 4.7] in a different way. Next, we determine the representation number $r(f;n)$ when $f$ is the form A13.

**Theorem 4.13.** *Let $f$ denote the form A13, that is, $f = 9x^2+16y^2+36z^2+16yz+4zx+8xy$. If $(\alpha, g) \neq (0,1), (2,1)$, we have*

$$r(f;n) = k_f(n)l(n)h(\mathbb{Q}(\sqrt{-n^*})),$$

*where the values of $k_f(n)$ are given in Table 4.13.*
*If $(\alpha, g) = (0,1)$, we have*

$$r(f;n) = \frac{1}{4}l(n) - \frac{1}{4}\left(\frac{-1}{h}\right)h.$$

*If $(\alpha, g) = (2,1)$, we have*

$$r(f;n) = l(n) - \left(\frac{-1}{h}\right)h.$$

| $\alpha$ | $g$ | $k_f(n)$ |
|---|---|---|
| 0 | $g \equiv 1 \pmod 8, g \neq 1$ | $\frac{1}{2}$ |
| | $g \equiv 3, 5, 7 \pmod 8$ | 0 |
| 1 | | 0 |
| 2 | $g \equiv 1 \pmod 8, g \neq 1$ | 2 |
| | $g \equiv 3, 5, 7 \pmod 8$ | 0 |
| 3 | | 0 |
| 4 | $g = 1$ | 2 |
| | $g \equiv 1 \pmod 4, g \neq 1$ | 4 |
| | $g \equiv 3 \pmod 4$ | 0 |
| 5 | | 0 |
| even $\geq 6$ | $g = 1$ | 6 |
| | $g \equiv 1 \pmod 4, g \neq 1$ | 12 |
| | $g = 3$ | 8 |
| | $g \equiv 3 \pmod 8, g \neq 3$ | 24 |
| | $g \equiv 7 \pmod 8$ | 0 |
| odd $\geq 7$ | | 12 |

Table 4.13: Values of $k_f(n)$

*Proof.* By Theorem 2.1 (xiii) and Proposition 2.1, we obtain for all $n \in \mathbb{N}$

$$r(f;n) = s(n)r(1,1,4,0,0,0;n) + r\left(1,1,4,0,0,0;\frac{n}{16}\right) + t(n),$$

where

$$\begin{aligned}
s(n) = \frac{1}{192} &\sum_{\substack{j=1 \\ j\equiv 1 \pmod 2}}^{63} \omega_{64}^{jn} + \frac{1}{192} \sum_{\substack{j=1 \\ j\equiv 1 \pmod 2}}^{63} \left(\frac{-1}{j}\right) \omega_{64}^{jn} \\
&+ \frac{\sqrt{2}}{384} \sum_{\substack{j=2 \\ j\equiv 2 \pmod 4}}^{62} \left(\frac{2}{j/2}\right) \omega_{64}^{jn} - \frac{i\sqrt{2}}{384} \sum_{\substack{j=2 \\ j\equiv 2 \pmod 4}}^{62} \left(\frac{-2}{j/2}\right) \omega_{64}^{jn} \\
&- \frac{1}{96} \sum_{\substack{j=4 \\ j\equiv 4 \pmod 8}}^{60} \omega_{64}^{jn} - \frac{i}{192} \sum_{\substack{j=4 \\ j\equiv 4 \pmod 8}}^{60} \left(\frac{-1}{j/4}\right) \omega_{64}^{jn} \\
&- \frac{1}{64} \sum_{\substack{j=8 \\ j\equiv 8 \pmod{16}}}^{56} \omega_{64}^{jn} + \frac{\sqrt{2}}{256} \sum_{\substack{j=8 \\ j\equiv 8 \pmod{16}}}^{56} \left(\frac{2}{j/8}\right) \omega_{64}^{jn} \\
&- \frac{i\sqrt{2}}{256} \sum_{\substack{j=8 \\ j\equiv 8 \pmod{16}}}^{56} \left(\frac{-2}{j/8}\right) \omega_{64}^{jn} - \frac{1}{192} \left(\omega_{64}^{16n} + \omega_{64}^{48n}\right)
\end{aligned}$$

$$- \frac{i}{128}\left(\omega_{64}^{16n} - \omega_{64}^{48n}\right) - \frac{5}{384}\omega_{64}^{32n} + \frac{1}{384}$$

and

$$t(n) = \begin{cases} -\frac{1}{4}\left(\frac{-1}{\sqrt{n}}\right)\sqrt{n} & \text{if } n = \text{odd square}, \\ -\left(\frac{-1}{\sqrt{n/4}}\right)\sqrt{n/4} & \text{if } n = 4 \times \text{odd square}, \\ 0 & \text{otherwise}, \end{cases}$$
$$= \begin{cases} -\frac{1}{4}\left(\frac{-1}{h}\right)h & \text{if } \alpha = 0, g = 1, \\ -\left(\frac{-1}{h}\right)h & \text{if } \alpha = 2, g = 1, \\ 0 & \text{otherwise}. \end{cases}$$

Making use of the results

$$\sum_{\substack{j=1 \\ j\equiv 1 \pmod 2}}^{63} \omega_{64}^{jn} = \begin{cases} (-1)^{n/32}32 & \text{if } n \equiv 0 \pmod{32}, \\ 0 & \text{if } n \not\equiv 0 \pmod{32}, \end{cases}$$

$$\sum_{\substack{j=1 \\ j\equiv 1 \pmod 2}}^{63} \left(\frac{-1}{j}\right)\omega_{64}^{jn} = \begin{cases} (-1)^{(n-16)/32}32i & \text{if } n \equiv 16 \pmod{32}, \\ 0 & \text{if } n \not\equiv 16 \pmod{32}, \end{cases}$$

$$\sum_{\substack{j=2 \\ j\equiv 2 \pmod 4}}^{62} \left(\frac{2}{j/2}\right)\omega_{64}^{jn} = \begin{cases} 8\sqrt{2} & \text{if } n \equiv 4, 28 \pmod{32}, \\ -8\sqrt{2} & \text{if } n \equiv 12, 20 \pmod{32}, \\ 0 & \text{if } n \not\equiv 4 \pmod 8, \end{cases}$$

$$\sum_{\substack{j=2 \\ j\equiv 2 \pmod 4}}^{62} \left(\frac{-2}{j/2}\right)\omega_{64}^{jn} = \begin{cases} 8\sqrt{2}i & \text{if } n \equiv 4, 12 \pmod{32}, \\ -8\sqrt{2}i & \text{if } n \equiv 20, 28 \pmod{32}, \\ 0 & \text{if } n \not\equiv 4 \pmod 8, \end{cases}$$

$$\sum_{\substack{j=4 \\ j\equiv 4 \pmod 8}}^{60} \omega_{64}^{jn} = \begin{cases} (-1)^{n/8}8 & \text{if } n \equiv 0 \pmod 8, \\ 0 & \text{if } n \not\equiv 0 \pmod 8, \end{cases}$$

$$\sum_{\substack{j=4 \\ j\equiv 4 \pmod 8}}^{60} \left(\frac{-1}{j/4}\right)\omega_{64}^{jn} = \begin{cases} (-1)^{(n-4)/8}8i & \text{if } n \equiv 4 \pmod 8, \\ 0 & \text{if } n \not\equiv 4 \pmod 8, \end{cases}$$

$$\sum_{\substack{j=8 \\ j\equiv 8 \pmod{16}}}^{56} \omega_{64}^{jn} = \begin{cases} (-1)^{n/4}4 & \text{if } n \equiv 0 \pmod 4, \\ 0 & \text{if } n \not\equiv 0 \pmod 4, \end{cases}$$

$$\sum_{\substack{j=8 \\ j\equiv 8 \pmod{16}}}^{56} \left(\frac{2}{j/8}\right)\omega_{64}^{jn} = \begin{cases} 2\sqrt{2} & \text{if } n \equiv 1,7 \pmod 8, \\ -2\sqrt{2} & \text{if } n \equiv 3,5 \pmod 8, \\ 0 & \text{if } n \equiv 0 \pmod 2, \end{cases}$$

$$\sum_{\substack{j=8 \\ j\equiv 8 \pmod{16}}}^{56} \left(\frac{-2}{j/8}\right)\omega_{64}^{jn} = \begin{cases} 2\sqrt{2}i & \text{if } n \equiv 1,3 \pmod 8, \\ -2\sqrt{2}i & \text{if } n \equiv 5,7 \pmod 8, \\ 0 & \text{if } n \equiv 0 \pmod 2, \end{cases}$$

$$\omega_{64}^{16n} + \omega_{64}^{48n} = i^n + i^{3n} = \begin{cases} (-1)^{n/2}2 & \text{if } n \equiv 0 \pmod 2, \\ 0 & \text{if } n \equiv 1 \pmod 2, \end{cases}$$

$$i\left(\omega_{64}^{16n} - \omega_{64}^{48n}\right) = i(i^n - i^{3n}) = \begin{cases} 0 & \text{if } n \equiv 0 \pmod 2, \\ (-1)^{(n+1)/2} & \text{if } n \equiv 1 \pmod 2, \end{cases}$$

we obtain

$$s(n) = \begin{cases} \frac{1}{16} & \text{if } n \equiv 1 \pmod 8, \\ \frac{1}{6} & \text{if } n \equiv 4 \pmod{32}, \\ -\frac{1}{3} & \text{if } n \equiv 16, 32 \pmod{64}, \\ 0 & \text{otherwise}, \end{cases}$$

$$= \begin{cases} \frac{1}{16} & \text{if } \alpha = 0, g \equiv 1 \pmod 8, \\ \frac{1}{6} & \text{if } \alpha = 2, g \equiv 1 \pmod 8, \\ -\frac{1}{3} & \text{if } \alpha = 4, g \equiv 1 \pmod 4 \text{ or } \alpha = 5, \\ 0 & \text{otherwise}. \end{cases}$$

Appealing to Proposition A.1 for the formula for $r(1, 1, 4, 0, 0, 0; n)$, we obtain Theorem 4.13. □

The positive integers not represented by the spinor regular form A13 follow from Theorem 4.13 and are given in Table A.17. They were obtained by Berkovich [3, Theorem 4.5] in a different way.

## 5. Spinor Regular Ternaries with Discriminant $2^r \cdot 3^s$

The twelve spinor regular positive-definite ternary quadratic forms $f = f(x, y, z)$ which are not regular and have discriminant $\Delta = 2^r \cdot 3^s$ for some $r, s \in \mathbb{N}$ (with $r$ even) are those with identification numbers B1–B12 in Table 1.1. For those forms which are alone in their spinor genus, that is, all except B4 and B11, we determine their representation numbers for all $n \in \mathbb{N}$. For B4 and B11 we determine their representation numbers for all even $n \in \mathbb{N}$.

When considering the representation of $n \in \mathbb{N}$ by $f(x, y, z)$, we use the integers $\alpha = \nu_2(n)$, $\beta = \nu_3(n)$, as well as $g$, $h$ and $n^*$, which are defined uniquely in terms of $n$ by (1.6), (1.7) and (1.5), respectively. We have

$$n = 2^\alpha 3^\beta g h^2, \tag{5.1}$$

where

$$\alpha, \beta \in \mathbb{N}_0, \quad g, h \in \mathbb{N}, \quad g \text{ squarefree}, \quad (gh, 6) = 1, \tag{5.2}$$

and

$$n^* = \begin{cases} 2^{\alpha - 2[\alpha/2]} 3^{\beta - 2[\beta/2]} g & \text{if } s \text{ is even}, \\ 2^{\alpha - 2[\alpha/2]} 3^{\beta + 1 - 2[(\beta+1)/2]} g & \text{if } s \text{ is odd}. \end{cases} \tag{5.3}$$

We now state and prove formulas for the representation numbers of the forms B1–B12. All of these formulas involve the quantities defined in (5.1)–(5.3) as well as $l(n)$, which is defined in (1.8). We begin with B1.

**Theorem 5.1.** *Let $f$ denote the form B1, that is, $f = 3x^2 + 3y^2 + 4z^2 + 3xy$. If $(\alpha, \beta, g) \neq (2k, 0, 1)$ for all $k \in \mathbb{N}_0$, we have*

$$r(f; n) = k_f(n) l(n) h(\mathbb{Q}(\sqrt{-n^*})),$$

*where the values of $k_f(n)$ are given in Table* 5.1.

| $\alpha, \beta$ | $g$ | $k_f(n)$ |
|---|---|---|
| *$\alpha$(even) $\beta$(even)* | | |
| $\alpha = 0, \beta = 0$ | $g \equiv 1 \pmod{24}, g \neq 1$ | 9 |
| | $g \equiv 5, 11, 17, 23 \pmod{24}$ | 0 |
| | $g \equiv 7, 13, 19 \pmod{24}$ | 3 |
| $\alpha = 0, \beta \geq 2$ | $g = 1$ | 6 |
| | $g \equiv 1, 17 \pmod{24}, g \neq 1$ | 18 |
| | $g \equiv 5, 7, 11, 13, 19, 23 \pmod{24}$ | 6 |
| $\alpha \geq 2, \beta = 0$ | $g \equiv 1 \pmod{24}, g \neq 1$ | $9 \cdot 2^{\alpha/2} - 6$ |
| | $g \equiv 5, 11, 17, 23 \pmod{24}$ | 0 |
| | $g \equiv 7, 19 \pmod{24}$ | $3 \cdot 2^{\alpha/2} - 3$ |
| | $g \equiv 13 \pmod{24}$ | $3 \cdot 2^{\alpha/2}$ |
| $\alpha \geq 2, \beta \geq 2$ | $g = 1$ | $3 \cdot 2^{\alpha/2+1} - 4$ |
| | $g \equiv 1, 17 \pmod{24}, g \neq 1$ | $9 \cdot 2^{\alpha/2+1} - 12$ |
| | $g \equiv 5, 13 \pmod{24}$ | $3 \cdot 2^{\alpha/2+1}$ |
| | $g \equiv 7, 11, 19, 23 \pmod{24}$ | $3 \cdot 2^{\alpha/2+1} - 6$ |
| *Continued on next page* | | |

| $\alpha, \beta$ | $g$ | $k_f(n)$ |
|---|---|---|
| | $\alpha$*(even)* $\beta$*(odd)* | |
| $\alpha = 0, \beta \geq 1$ | $g = 1$ | 6 |
| | $g \equiv 1, 7, 13 \pmod{24}, g \neq 1$ | 12 |
| | $g \equiv 5, 11, 17, 23 \pmod{24}$ | 0 |
| | $g \equiv 19 \pmod{24}$ | 36 |
| $\alpha \geq 2, \beta \geq 1$ | $g = 1$ | $3 \cdot 2^{\alpha/2+1} - 6$ |
| | $g \equiv 1, 13 \pmod{24}, g \neq 1$ | $3 \cdot 2^{\alpha/2+2} - 12$ |
| | $g \equiv 5, 11, 17, 23 \pmod{24}$ | 0 |
| | $g \equiv 7 \pmod{24}$ | $3 \cdot 2^{\alpha/2+2}$ |
| | $g \equiv 19 \pmod{24}$ | $9 \cdot 2^{\alpha/2+2} - 24$ |
| | $\alpha$*(odd)* $\beta$*(even)* | |
| $\alpha = 1, \beta \geq 0$ | | 0 |
| $\alpha \geq 3, \beta = 0$ | $g \equiv 1 \pmod{6}$ | 0 |
| | $g \equiv 5 \pmod{6}$ | $3 \cdot 2^{(\alpha-1)/2} - 3$ |
| $\alpha \geq 3, \beta \geq 2$ | | $3 \cdot 2^{(\alpha+1)/2} - 6$ |
| | $\alpha$*(odd)* $\beta$*(odd)* | |
| $\alpha = 1, \beta \geq 1$ | | 0 |
| $\alpha \geq 3, \beta \geq 1$ | $g \equiv 1 \pmod{6}$ | 0 |
| | $g \equiv 5 \pmod{6}$ | $3 \cdot 2^{(\alpha+3)/2} - 12$ |

Table 5.1: Values of $k_f(n)$

*If* $(\alpha, \beta, g) = (0, 0, 1)$*, we have*

$$r(f; n) = 3l(n) - 3\left(\frac{-3}{h}\right)h.$$

*If* $(\alpha, \beta, g) = (2k, 0, 1)$*, where* $k \in \mathbb{N}$*, we have*

$$r(f; n) = (3 \cdot 2^{\alpha/2} - 2)l(n) + 2^{\alpha/2}\left(\frac{-3}{2^{\alpha/2}h}\right)h.$$

*Proof.* By Theorem 2.1 (xiv) and Proposition 2.1, we obtain for all $n \in \mathbb{N}$

$$r(f; n) = s(n)r(1, 1, 4, 0, 0, 1; n) + t(n),$$

where

$$\begin{aligned} s(n) &= \frac{1}{12}\left(3 - i\sqrt{3}\right)\omega_3^n + \frac{1}{12}\left(3 + i\sqrt{3}\right)\omega_3^{2n} + \frac{1}{2} \\ &= \begin{cases} \frac{1}{2} & \text{if } n \equiv 1 \pmod{3}, \\ 0 & \text{if } n \equiv 2 \pmod{3}, \\ 1 & \text{if } n \equiv 0 \pmod{3}, \end{cases} \end{aligned}$$

$$= \begin{cases} \frac{1}{2} & \text{if } \alpha \equiv 0 \pmod 2, \beta = 0, g \equiv 1 \pmod 3 \\ & \text{or } \alpha \equiv 1 \pmod 2, \beta = 0, g \equiv 2 \pmod 3, \\ 0 & \text{if } \alpha \equiv 0 \pmod 2, \beta = 0, g \equiv 2 \pmod 3 \\ & \text{or } \alpha \equiv 1 \pmod 2, \beta = 0, g \equiv 1 \pmod 3, \\ 1 & \text{if } \beta \geq 1, \end{cases}$$

and

$$t(n) = \begin{cases} -3\left(\frac{-3}{\sqrt{n}}\right)\sqrt{n} & \text{if } n = \text{odd square}, \\ \left(\frac{-3}{\sqrt{n}}\right)\sqrt{n} & \text{if } n = \text{even square}, \\ 0 & \text{otherwise}, \end{cases}$$
$$= \begin{cases} -3\left(\frac{-3}{h}\right)h & \text{if } (\alpha, \beta, g) = (0, 0, 1), \\ 2^k\left(\frac{-3}{2^k h}\right)h & \text{if } (\alpha, \beta, g) = (2k, 0, 1) \ (k \in \mathbb{N}), \\ 0 & \text{otherwise}. \end{cases}$$

Appealing to Proposition A.7 for the formula for $r(1, 1, 4, 0, 0, 1; n)$, we obtain Theorem 5.1. □

The positive integers not represented by the spinor regular form B1 follow from Theorem 5.1 using Lemmas 3.1 and 3.2. We give the details of the proof as we are using Lemma 3.2 for the first time.

We define the conditions $V, W, X, Y, Z$ by

$$\begin{aligned} V: &\quad \alpha = 1, \\ W: &\quad \alpha \equiv 0 \pmod 2, \beta = 0, g \equiv 5 \pmod 6, \\ X: &\quad \alpha \equiv 1 \pmod 2, \alpha \geq 3, \beta = 0, g \equiv 1 \pmod 6, \\ Y: &\quad \alpha \equiv 0 \pmod 2, \beta \equiv 1 \pmod 2, g \equiv 5 \pmod 6, \\ Z: &\quad \alpha \equiv 1 \pmod 2, \alpha \geq 3, \beta \equiv 1 \pmod 2, g \equiv 1 \pmod 6. \end{aligned}$$

Then we have

$$\begin{aligned} n = 4l + 2 &\text{ if and only if } V \text{ holds}, \\ n = 3l + 2, n \neq 4l + 2 &\text{ if and only if } W \text{ or } X \text{ holds}, \\ n = 9^k(9l + 6), n \neq 4l + 2 &\text{ if and only if } Y \text{ or } Z \text{ holds}. \end{aligned}$$

If $(\alpha, \beta, g) \neq (2k, 0, 1)$ for all $k \in \mathbb{N}_0$, by Theorem 5.1, we have

$$\begin{aligned} r(f; n) = 0 &\text{ if and only if } k_f(n) = 0 \\ &\text{ if and only if } V, W, X, Y \text{ or } Z \text{ holds} \end{aligned}$$

$$\text{if and only if } n = 3l+2, 4l+2, 9^k(9l+6).$$

If $(\alpha, \beta, g) = (0, 0, 1)$, by Theorem 5.1 and Lemma 3.1 (ii), we have

$$r(f;n) = 0 \text{ if and only if } l(n) = \left(\frac{-3}{h}\right) h \text{ if and only if } n \in M_3^2.$$

If $(\alpha, \beta, g) = (2k, 0, 1)$ for some $k \in \mathbb{N}$, by Theorem 5.1 and Lemma 3.2 (with $A = 3, B = 2, C = 1$ and $m = \frac{\alpha}{2} \geq 1$), we have

$$r(f;n) = (3 \cdot 2^{\alpha/2} - 2)l(n) + 2^{\alpha/2} \left(\frac{-3}{2^{\alpha/2}h}\right) h > 2^{\alpha/2}h - 2^{\alpha/2}h = 0.$$

This completes the proof that $n$ is not represented by B1 if and only if $n$ belongs to at least one of the progressions $3l+2, 4l+2, 9^k(9l+6)$ $(k, l \in \mathbb{N}_0), M_3^2$, as stated in Table A.17. Next, we determine the representation number $r(f;n)$ when $f$ is the form B2.

**Theorem 5.2.** *Let $f$ denote the form B2, that is, $f = 3x^2 + 4y^2 + 4z^2 + 4yz + 3zx + 3xy$. If $(\alpha, \beta, g) \neq (2k, 0, 1)$ for all $k \in \mathbb{N}_0$, we have*

$$r(f;n) = k_f(n)l(n)h(\mathbb{Q}(\sqrt{-n^*})),$$

*where the values of $k_f(n)$ are given in Table 5.2.*

| $\alpha, \beta$ | $g$ | $k_f(n)$ |
|---|---|---|
| | $\alpha$(even) $\beta$(even) | |
| $\alpha = 0, \beta = 0$ | $g \equiv 1 \pmod{24}, g \neq 1$ | $3$ |
| | $g \equiv 5, 11, 17, 23 \pmod{24}$ | $0$ |
| | $g \equiv 7, 13, 19 \pmod{24}$ | $1$ |
| $\alpha = 0, \beta \geq 2$ | $g = 1$ | $2$ |
| | $g \equiv 1, 17 \pmod{24}, g \neq 1$ | $6$ |
| | $g \equiv 5, 7, 11, 13, 19, 23 \pmod{24}$ | $2$ |
| $\alpha \geq 2, \beta = 0$ | $g \equiv 1 \pmod{24}, g \neq 1$ | $15 \cdot 2^{\alpha/2} - 6$ |
| | $g \equiv 5, 11, 17, 23 \pmod{24}$ | $0$ |
| | $g \equiv 7, 19 \pmod{24}$ | $5 \cdot 2^{\alpha/2} - 3$ |
| | $g \equiv 13 \pmod{24}$ | $5 \cdot 2^{\alpha/2}$ |
| $\alpha \geq 2, \beta \geq 2$ | $g = 1$ | $5 \cdot 2^{\alpha/2+1} - 4$ |
| | $g \equiv 1, 17 \pmod{24}, g \neq 1$ | $15 \cdot 2^{\alpha/2+1} - 12$ |
| | $g \equiv 5, 13 \pmod{24}$ | $5 \cdot 2^{\alpha/2+1}$ |
| | $g \equiv 7, 11, 19, 23 \pmod{24}$ | $5 \cdot 2^{\alpha/2+1} - 6$ |
| | | *Continued on next page* |

| $\alpha, \beta$ | $g$ | $k_f(n)$ |
|---|---|---|
| | $\alpha$*(even)* $\beta$*(odd)* | |
| $\alpha = 0, \beta \geq 1$ | $g = 1$ | $2$ |
| | $g \equiv 1, 7, 13 \pmod{24}, g \neq 1$ | $4$ |
| | $g \equiv 5, 11, 17, 23 \pmod{24}$ | $0$ |
| | $g \equiv 19 \pmod{24}$ | $12$ |
| $\alpha \geq 2, \beta \geq 1$ | $g = 1$ | $5 \cdot 2^{\alpha/2+1} - 6$ |
| | $g \equiv 1, 13 \pmod{24}, g \neq 1$ | $5 \cdot 2^{\alpha/2+2} - 12$ |
| | $g \equiv 5, 11, 17, 23 \pmod{24}$ | $0$ |
| | $g \equiv 7 \pmod{24}$ | $5 \cdot 2^{\alpha/2+2}$ |
| | $g \equiv 19 \pmod{24}$ | $15 \cdot 2^{\alpha/2+2} - 24$ |
| | $\alpha$*(odd)* $\beta$*(even)* | |
| $\alpha \geq 1, \beta = 0$ | $g \equiv 1 \pmod 6$ | $0$ |
| | $g \equiv 5 \pmod 6$ | $5 \cdot 2^{(\alpha-1)/2} - 3$ |
| $\alpha \geq 1, \beta \geq 2$ | | $5 \cdot 2^{(\alpha+1)/2} - 6$ |
| | $\alpha$*(odd)* $\beta$*(odd)* | |
| $\alpha \geq 1, \beta \geq 1$ | $g \equiv 1 \pmod 6$ | $0$ |
| | $g \equiv 5 \pmod 6$ | $5 \cdot 2^{(\alpha+3)/2} - 12$ |

Table 5.2: Values of $k_f(n)$

*If* $(\alpha, \beta, g) = (0, 0, 1)$*, we have*

$$r(f; n) = l(n) - \left(\frac{-3}{h}\right) h.$$

*If* $(\alpha, \beta, g) = (2k, 0, 1)$*, where* $k \in \mathbb{N}$*, we have*

$$r(f; n) = (5 \cdot 2^{\alpha/2} - 2) l(n) - \left(\frac{-3}{2^{\alpha/2} h}\right) 2^{\alpha/2} h.$$

*Proof.* By Theorem 2.1 (xv) and Proposition 2.1, we obtain for all $n \in \mathbb{N}$

$$r(f; n) = s(n) r(1, 2, 2, 1, 1, 1; n) + t(n),$$

where

$$\begin{aligned} s(n) &= \frac{1}{12}\left(3 - i\sqrt{3}\right)\omega_3^n + \frac{1}{12}\left(3 + i\sqrt{3}\right)\omega_3^{2n} + \frac{1}{2} \\ &= \begin{cases} \frac{1}{2} & \text{if } n \equiv 1 \pmod 3, \\ 0 & \text{if } n \equiv 2 \pmod 3, \\ 1 & \text{if } n \equiv 0 \pmod 3, \end{cases} \end{aligned}$$

$$= \begin{cases} \frac{1}{2} & \text{if } \alpha \equiv 0 \pmod 2, \beta = 0, g \equiv 1 \pmod 3 \\ & \text{or } \alpha \equiv 1 \pmod 2, \beta = 0, g \equiv 2 \pmod 3, \\ 0 & \text{if } \alpha \equiv 0 \pmod 2, \beta = 0, g \equiv 2 \pmod 3 \\ & \text{or } \alpha \equiv 1 \pmod 2, \beta = 0, g \equiv 1 \pmod 3, \\ 1 & \text{if } \beta \geq 1, \end{cases}$$

and

$$t(n) = \begin{cases} -\left(\frac{-3}{\sqrt{n}}\right)\sqrt{n} & \text{if } n = \text{square}, \\ 0 & \text{otherwise}, \end{cases}$$

$$= \begin{cases} -\left(\frac{-3}{2^k h}\right) 2^k h & \text{if } (\alpha, \beta, g) = (2k, 0, 1) \ (k \in \mathbb{N}_0), \\ 0 & \text{otherwise}. \end{cases}$$

Appealing to Proposition A.8 for the formula for $r(1, 2, 2, 1, 1, 1; n)$, we obtain Theorem 5.2. □

The positive integers not represented by the spinor regular form B2 are given in Table A.17. They follow from Theorem 5.2 using Lemmas 3.1 and 3.2 similarly to the proof for the form B1. Next, we determine the representation number $r(f; n)$ when $f$ is the form B3.

**Theorem 5.3.** *Let $f$ denote the form B3, that is, $f = x^2 + 7y^2 + 12z^2 + xy$. If $(\alpha, \beta, g) \neq (2k, 1, 1)$ for all $k \in \mathbb{N}_0$, we have*

$$r(f; n) = k_f(n) l(n) h(\mathbb{Q}(\sqrt{-n^*})),$$

*where the values of $k_f(n)$ are given in Table 5.3.*

| $\alpha, \beta$ | $g$ | $k_f(n)$ |
|---|---|---|
| | *$\alpha$(even) $\beta$(even)* | |
| $\alpha = 0, \beta = 0$ | $g = 1$ | 2 |
| | $g \equiv 1, 7, 13 \pmod{24}, g \neq 1$ | 4 |
| | $g \equiv 5, 11, 17, 23 \pmod{24}$ | 0 |
| | $g \equiv 19 \pmod{24}$ | 12 |
| $\alpha = 0, \beta \geq 2$ | $g = 1$ | 6 |
| | $g \equiv 1, 7, 13 \pmod{24}, g \neq 1$ | 12 |
| | $g \equiv 5, 11, 17, 23 \pmod{24}$ | 0 |
| | $g \equiv 19 \pmod{24}$ | 36 |
| $\alpha \geq 2, \beta = 0$ | $g = 1$ | $2^{\alpha/2+1} - 2$ |
| | $g \equiv 1, 13 \pmod{24}, g \neq 1$ | $2^{\alpha/2+2} - 4$ |
| | $g \equiv 5, 11, 17, 23 \pmod{24}$ | 0 |
| | $g \equiv 7 \pmod{24}$ | $2^{\alpha/2+2}$ |
| | $g \equiv 19 \pmod{24}$ | $3 \cdot 2^{\alpha/2+2} - 8$ |
| | | *Continued on next page* |

| $\alpha, \beta$ | $g$ | $k_f(n)$ |
|---|---|---|
| $\alpha \geq 2, \beta \geq 2$ | $g = 1$ | $3 \cdot 2^{\alpha/2+1} - 6$ |
| | $g \equiv 1, 13 \pmod{24}, g \neq 1$ | $3 \cdot 2^{\alpha/2+2} - 12$ |
| | $g \equiv 5, 11, 17, 23 \pmod{24}$ | $0$ |
| | $g \equiv 7 \pmod{24}$ | $3 \cdot 2^{\alpha/2+2}$ |
| | $g \equiv 19 \pmod{24}$ | $9 \cdot 2^{\alpha/2+2} - 24$ |
| | *$\alpha$(even) $\beta$(odd)* | |
| $\alpha = 0, \beta = 1$ | $g \equiv 1 \pmod{24}, g \neq 1$ | $9$ |
| | $g \equiv 5, 11, 17, 23 \pmod{24}$ | $0$ |
| | $g \equiv 7, 13, 19 \pmod{24}$ | $3$ |
| $\alpha = 0, \beta \geq 3$ | $g = 1$ | $6$ |
| | $g \equiv 1, 17 \pmod{24}, g \neq 1$ | $18$ |
| | $g \equiv 5, 7, 11, 13, 19, 23 \pmod{24}$ | $6$ |
| $\alpha \geq 2, \beta = 1$ | $g \equiv 1 \pmod{24}, g \neq 1$ | $9 \cdot 2^{\alpha/2} - 6$ |
| | $g \equiv 5, 11, 17, 23 \pmod{24}$ | $0$ |
| | $g \equiv 7, 19 \pmod{24}$ | $3 \cdot 2^{\alpha/2} - 3$ |
| | $g \equiv 13 \pmod{24}$ | $3 \cdot 2^{\alpha/2}$ |
| $\alpha \geq 2, \beta \geq 3$ | $g = 1$ | $3 \cdot 2^{\alpha/2+1} - 4$ |
| | $g \equiv 1, 17 \pmod{24}, g \neq 1$ | $9 \cdot 2^{\alpha/2+1} - 12$ |
| | $g \equiv 5, 13 \pmod{24}$ | $3 \cdot 2^{\alpha/2+1}$ |
| | $g \equiv 7, 11, 19, 23 \pmod{24}$ | $3 \cdot 2^{\alpha/2+1} - 6$ |
| | *$\alpha$(odd) $\beta$(even)* | |
| $\alpha = 1, \beta \geq 0$ | | $0$ |
| $\alpha \geq 3, \beta = 0$ | $g \equiv 1 \pmod 6$ | $0$ |
| | $g \equiv 5 \pmod 6$ | $2^{(\alpha+3)/2} - 4$ |
| $\alpha \geq 3, \beta \geq 2$ | $g \equiv 1 \pmod 6$ | $0$ |
| | $g \equiv 5 \pmod 6$ | $3 \cdot 2^{(\alpha+3)/2} - 12$ |
| | *$\alpha$(odd) $\beta$(odd)* | |
| $\alpha = 1, \beta \geq 1$ | | $0$ |
| $\alpha \geq 3, \beta = 1$ | $g \equiv 1 \pmod 6$ | $0$ |
| | $g \equiv 5 \pmod 6$ | $3 \cdot 2^{(\alpha-1)/2} - 3$ |
| $\alpha \geq 3, \beta \geq 3$ | | $3 \cdot 2^{(\alpha+1)/2} - 6$ |

Table 5.3: Values of $k_f(n)$

*If* $(\alpha, \beta, g) = (0, 1, 1)$*, we have*

$$r(f; n) = 3l(n) - 3\left(\frac{-3}{h}\right)h.$$

*If* $(\alpha, \beta, g) = (2k, 1, 1)$*, where* $k \in \mathbb{N}$*, we have*

$$r(f; n) = (3 \cdot 2^{\alpha/2} - 2)l(n) + 2^{\alpha/2}\left(\frac{-3}{2^{\alpha/2}h}\right)h.$$

*Proof.* By Theorem 2.1 (xvi) and Proposition 2.1, we obtain for all $n \in \mathbb{N}$

$$r(f;n) = s(n)r(1,1,12,0,0,1;n) + t(n),$$

where

$$s(n) = \frac{1}{36}\left(3 - i\sqrt{3}\right)\left(\omega_9^n + \omega_9^{4n} + \omega_9^{7n}\right) + \frac{1}{36}\left(3 + i\sqrt{3}\right)\left(\omega_9^{2n} + \omega_9^{5n} + \omega_9^{8n}\right) + \frac{1}{18}\left(2 - i\sqrt{3}\right)\omega_9^{3n} + \frac{1}{18}\left(2 + i\sqrt{3}\right)\omega_9^{6n} + \frac{5}{18}$$

and

$$t(n) = \begin{cases} -3\left(\frac{-3}{\sqrt{n/3}}\right)\sqrt{n/3} & \text{if } n = 3 \times \text{odd square}, \\ \left(\frac{-3}{\sqrt{n/3}}\right)\sqrt{n/3} & \text{if } n = 3 \times \text{even square}, \\ 0 & \text{if } n \neq 3 \times \text{square}. \end{cases}$$

As

$$\omega_9^n + \omega_9^{4n} + \omega_9^{7n} = \begin{cases} 3\omega_3^{n/3} & \text{if } n \equiv 0 \pmod 3, \\ 0 & \text{if } n \not\equiv 0 \pmod 3, \end{cases}$$

and

$$\omega_9^{2n} + \omega_9^{5n} + \omega_9^{8n} = \begin{cases} 3\omega_3^{2n/3} & \text{if } n \equiv 0 \pmod 3, \\ 0 & \text{if } n \not\equiv 0 \pmod 3, \end{cases}$$

we deduce

$$s(n) = \begin{cases} \frac{1}{3} & \text{if } n \equiv 1,4,7 \pmod 9, \\ \frac{1}{2} & \text{if } n \equiv 3 \pmod 9, \\ 1 & \text{if } n \equiv 0 \pmod 9, \\ 0 & \text{if } n \equiv 2,5,6,8 \pmod 9, \end{cases}$$

$$= \begin{cases} \frac{1}{3} & \text{if } \alpha \equiv 0 \pmod 2, \beta = 0, g \equiv 1 \pmod 3 \\ & \text{or } \alpha \equiv 1 \pmod 2, \beta = 0, g \equiv 2 \pmod 3, \\ \frac{1}{2} & \text{if } \alpha \equiv 0 \pmod 2, \beta = 1, g \equiv 1 \pmod 3 \\ & \text{or } \alpha \equiv 1 \pmod 2, \beta = 1, g \equiv 2 \pmod 3, \\ 1 & \text{if } \beta \geq 2, \\ 0 & \text{otherwise}, \end{cases}$$

and

$$t(n) = \begin{cases} -3\left(\frac{-3}{h}\right)h & \text{if } (\alpha,\beta,g) = (0,1,1), \\ \left(\frac{-3}{2^k h}\right)2^k h & \text{if } (\alpha,\beta,g) = (2k,1,1) \ (k \in \mathbb{N}), \\ 0 & \text{otherwise}. \end{cases}$$

Appealing to Proposition A.9 for the formula for $r(1,1,12,0,0,1;n)$, we obtain Theorem 5.3. $\square$

The positive integers not represented by the spinor regular form B3 are given in Table A.17. They follow from Theorem 5.3 using Lemmas 3.1 and 3.2. Next, we consider the representation number $r(f;n)$ when $f$ is the form B4. This spinor regular form is not alone in its spinor genus. We are only able to evaluate $r(f;n)$ for even values of $n \in \mathbb{N}$.

**Theorem 5.4.** *Let $f$ denote the form B4, that is, $f = 3x^2 + 7y^2 + 7z^2 + 5yz + 3zx + 3xy$. Suppose that $n \in \mathbb{N}$ is even so that $\alpha \geq 1$. If $(\alpha, \beta, g) \neq (2k, 0, 1)$ for all $k \in \mathbb{N}$ then*

$$r(f;n) = k_f(n)l(n)h(\mathbb{Q}(\sqrt{-n^*})),$$

*where the values of $k_f(n)$ are given in Table 5.4.*
*If $(\alpha, \beta, g) = (2, 0, 1)$, then*

$$r(f;n) = 3l(n) - 3\left(\frac{-3}{h}\right)h.$$

*If $(\alpha, \beta, g) = (2k, 0, 1)$, where $k \geq 2$, then*

$$r(f;n) = (3 \cdot 2^{\alpha/2-1} - 2)l(n) - 2^{\alpha/2-1}\left(\frac{-3}{2^{\alpha/2}h}\right)h.$$

*Proof.* We have

$$3x^2 + 7y^2 + 7z^2 + 5yz + 3zx + 3xy \equiv \begin{cases} 0 \pmod 4 & \text{if } x \equiv y \equiv z \pmod 2, \\ 1 \pmod 2 & \text{otherwise,} \end{cases}$$

so

$$r(f;n) = 0 \text{ if } n \equiv 2 \pmod 4.$$

Now suppose that $n \equiv 0 \pmod 4$ so that any solution $(x,y,z) \in \mathbb{Z}^3$ of $n = 3x^2 + 7y^2 + 7z^2 + 5yz + 3zx + 3xy$ satisfies $x \equiv y \equiv z \pmod 2$, so that $\frac{1}{2}(x+y)$, $\frac{1}{2}(x+z)$, and $\frac{1}{2}(y+z)$ are all integers. Let

$$A := \left\{(x,y,z) \in \mathbb{Z}^3 \mid n = 3x^2 + 7y^2 + 7z^2 + 5yz + 3zx + 3xy\right\}$$

and

$$B := \left\{(x,y,z) \in \mathbb{Z}^3 \mid \frac{n}{4} = 3x^2 + 3y^2 + 4z^2 + 3xy\right\}.$$

The mapping $\lambda : A \to B$ defined by

$$\lambda(x,y,z) = \left(\frac{1}{2}(x+y), -\frac{1}{2}(x+z), -\frac{1}{2}(y+z)\right)$$

| $\alpha, \beta$ | $g$ | $k_f(n)$ |
|---|---|---|
| | $\alpha$(even) $\beta$(even) | |
| $\alpha = 2, \beta = 0$ | $g \equiv 1 \pmod{24}, g \neq 1$ | 9 |
| | $g \equiv 5, 11, 17, 23 \pmod{24}$ | 0 |
| | $g \equiv 7, 13, 19 \pmod{24}$ | 3 |
| $\alpha = 2, \beta \geq 2$ | $g = 1$ | 6 |
| | $g \equiv 1, 17 \pmod{24}, g \neq 1$ | 18 |
| | $g \equiv 5, 7, 11, 13, 19, 23 \pmod{24}$ | 6 |
| $\alpha \geq 4, \beta = 0$ | $g \equiv 1 \pmod{24}, g \neq 1$ | $9 \cdot 2^{\alpha/2-1} - 6$ |
| | $g \equiv 5, 11, 17, 23 \pmod{24}$ | 0 |
| | $g \equiv 7, 19 \pmod{24}$ | $3 \cdot 2^{\alpha/2-1} - 3$ |
| | $g \equiv 13 \pmod{24}$ | $3 \cdot 2^{\alpha/2-1}$ |
| $\alpha \geq 4, \beta \geq 2$ | $g = 1$ | $3 \cdot 2^{\alpha/2} - 4$ |
| | $g \equiv 1, 17 \pmod{24}, g \neq 1$ | $9 \cdot 2^{\alpha/2} - 12$ |
| | $g \equiv 5, 13 \pmod{24}$ | $3 \cdot 2^{\alpha/2}$ |
| | $g \equiv 7, 11, 19, 23 \pmod{24}$ | $3 \cdot 2^{\alpha/2} - 6$ |
| | $\alpha$(even) $\beta$(odd) | |
| $\alpha = 2, \beta \geq 1$ | $g = 1$ | 6 |
| | $g \equiv 1, 7, 13 \pmod{24}, g \neq 1$ | 12 |
| | $g \equiv 5, 11, 17, 23 \pmod{24}$ | 0 |
| | $g \equiv 19 \pmod{24}$ | 36 |
| $\alpha \geq 4, \beta \geq 1$ | $g = 1$ | $3 \cdot 2^{\alpha/2} - 6$ |
| | $g \equiv 1, 13 \pmod{24}, g \neq 1$ | $3 \cdot 2^{\alpha/2+1} - 12$ |
| | $g \equiv 5, 11, 17, 23 \pmod{24}$ | 0 |
| | $g \equiv 7 \pmod{24}$ | $3 \cdot 2^{\alpha/2+1}$ |
| | $g \equiv 19 \pmod{24}$ | $9 \cdot 2^{\alpha/2+1} - 24$ |
| | $\alpha$(odd) $\beta$(even) | |
| $\alpha = 1, 3, \beta \geq 0$ | | 0 |
| $\alpha \geq 5, \beta = 0$ | $g \equiv 1 \pmod{6}$ | 0 |
| | $g \equiv 5 \pmod{6}$ | $3 \cdot 2^{(\alpha-3)/2} - 3$ |
| $\alpha \geq 5, \beta \geq 2$ | | $3 \cdot 2^{(\alpha-1)/2} - 6$ |
| | $\alpha$(odd) $\beta$(odd) | |
| $\alpha = 1, 3, \beta \geq 1$ | | 0 |
| $\alpha \geq 5, \beta \geq 1$ | $g \equiv 1 \pmod{6}$ | 0 |
| | $g \equiv 5 \pmod{6}$ | $3 \cdot 2^{(\alpha+1)/2} - 12$ |

Table 5.4: Values of $k_f(n)$

is a bijection. Thus

$$r(f; n) = \text{card } A = \text{card } B = r\left(3, 3, 4, 0, 0, 3; \frac{n}{4}\right)$$

and the theorem follows from Theorem 5.1. □

We determine the even positive integers which are not represented by $f$. As $n \in \mathbb{N}$ is even we have $n \equiv 0 \pmod 4$ or $n \equiv 2 \pmod 4$. If $n \equiv 0 \pmod 4$, appealing to Table A.17 for the positive integers not represented by the form $3x^2+3y^2+4z^2+3xy$, we have

$$\begin{aligned} r(f;n) = 0 \text{ if and only if } & r(3,3,4,0,0,3;n/4) = 0 \\ \text{if and only if } & n/4 = 3l+2, 4l+2, 9^k(9l+6) \text{ for some } k,l \in \mathbb{N}_0 \text{ or} \\ & n/4 \in M_3^2. \end{aligned}$$

If $n \equiv 2 \pmod 4$ we have $r(f;n) = 0$. Thus $n$ is represented by $3x^2 + 7y^2 + 7z^2 + 5yz + 3zx + 3xy$ if and only if $n$ does not belong to any of the progressions $4l+2, 12l+8, 16l+8, 4 \cdot 9^k(9l+6), 4M_3^2$. Next, we determine the representation number $r(f;n)$ when $f$ is the form B5.

**Theorem 5.5.** *Let $f$ denote the form B5, that is, $f = 4x^2+4y^2+9z^2+4xy$. If $(\alpha, \beta, g) \neq (2k, 0, 1)$ for all $k \in \mathbb{N}_0$, we have*

$$r(f;n) = k_f(n) l(n) h(\mathbb{Q}(\sqrt{-n^*})),$$

*where the values of $k_f(n)$ are given in Table 5.5.*

| $\alpha, \beta$ | $g$ | $k_f(n)$ |
|---|---|---|
| | *$\alpha$(even) $\beta$(even)* | |
| $\alpha = 0, \beta = 0$ | $g \equiv 1 \pmod{12}, g \neq 1$ | 3 |
| | $g \equiv 5, 7, 11 \pmod{12}$ | 0 |
| $\alpha = 0, \beta \geq 2$ | $g = 1$ | 2 |
| | $g \equiv 1, 5 \pmod{12}, g \neq 1$ | 6 |
| | $g \equiv 7, 11 \pmod{12}$ | 0 |
| $\alpha \geq 2, \beta = 0$ | $g \equiv 1 \pmod{24}, g \neq 1$ | $9 \cdot 2^{\alpha/2} - 6$ |
| | $g \equiv 5, 11, 17, 23 \pmod{24}$ | 0 |
| | $g \equiv 7, 19 \pmod{24}$ | $3 \cdot 2^{\alpha/2} - 3$ |
| | $g \equiv 13 \pmod{24}$ | $3 \cdot 2^{\alpha/2}$ |
| $\alpha \geq 2, \beta \geq 2$ | $g = 1$ | $3 \cdot 2^{\alpha/2+1} - 4$ |
| | $g \equiv 1, 17 \pmod{24}, g \neq 1$ | $9 \cdot 2^{\alpha/2+1} - 12$ |
| | $g \equiv 5, 13 \pmod{24}$ | $3 \cdot 2^{\alpha/2+1}$ |
| | $g \equiv 7, 11, 19, 23 \pmod{24}$ | $3 \cdot 2^{\alpha/2+1} - 6$ |
| | *$\alpha$(even) $\beta$(odd)* | |
| $\alpha = 0, \beta \geq 1$ | $g \equiv 1, 5, 11 \pmod{12}$ | 0 |
| | $g \equiv 7 \pmod{12}$ | 12 |
| $\alpha \geq 2, \beta \geq 1$ | $g = 1$ | $3 \cdot 2^{\alpha/2+1} - 6$ |
| | $g \equiv 1, 13 \pmod{24}, g \neq 1$ | $3 \cdot 2^{\alpha/2+2} - 12$ |
| | $g \equiv 5, 11, 17, 23 \pmod{24}$ | 0 |
| | $g \equiv 7 \pmod{24}$ | $3 \cdot 2^{\alpha/2+2}$ |
| | $g \equiv 19 \pmod{24}$ | $9 \cdot 2^{\alpha/2+2} - 24$ |

*Continued on next page*

| $\alpha, \beta$ | $g$ | $k_f(n)$ |
|---|---|---|
| | $\alpha$*(odd)* $\beta$*(even)* | |
| $\alpha = 1, \beta \geq 0$ | | $0$ |
| $\alpha \geq 3, \beta = 0$ | $g \equiv 1 \pmod 6$<br>$g \equiv 5 \pmod 6$ | $0$<br>$3 \cdot 2^{(\alpha-1)/2} - 3$ |
| $\alpha \geq 3, \beta \geq 2$ | | $3 \cdot 2^{(\alpha+1)/2} - 6$ |
| | $\alpha$*(odd)* $\beta$*(odd)* | |
| $\alpha = 1, \beta \geq 1$ | | $0$ |
| $\alpha \geq 3, \beta \geq 1$ | $g \equiv 1 \pmod 6$<br>$g \equiv 5 \pmod 6$ | $0$<br>$3 \cdot 2^{(\alpha+3)/2} - 12$ |

Table 5.5: Values of $k_f(n)$

*If* $(\alpha, \beta, g) = (2k, 0, 1)$*, where* $k \in \mathbb{N}_0$*, we have*

$$r(f;n) = (3 \cdot 2^{\alpha/2} - 2)l(n) - 2^{\alpha/2} \left( \frac{-3}{2^{\alpha/2}h} \right) h.$$

*Proof.* By Theorem 2.1 (xvii) and Proposition 2.1 we have for all $n \in \mathbb{N}$

$$r(f;n) = s(n) r(1,3,9,0,0,0;n) + \frac{1}{2} r\left(1,3,9,0,0,0;\frac{n}{4}\right) + t(n),$$

where

$$\begin{aligned} s(n) &= \frac{1}{64}\left(-\sqrt{2} + i\sqrt{2}\right)\omega_8^n + \frac{1}{32}(4-5i)\omega_8^{2n} + \frac{1}{64}\left(\sqrt{2} + i\sqrt{2}\right)\omega_8^{3n} - \frac{1}{32}\omega_8^{4n} \\ &\quad + \frac{1}{64}\left(\sqrt{2} - i\sqrt{2}\right)\omega_8^{5n} + \frac{1}{32}(4+5i)\omega_8^{6n} + \frac{1}{64}\left(-\sqrt{2} - i\sqrt{2}\right)\omega_8^{7n} + \frac{9}{32} \\ &= \begin{cases} \frac{1}{2} & \text{if } n \equiv 0,1,4 \pmod 8, \\ \frac{3}{4} & \text{if } n \equiv 5 \pmod 8, \\ 0 & \text{if } n \equiv 2,3,6,7 \pmod 8, \end{cases} \\ &= \begin{cases} \frac{1}{2} & \text{if } \alpha = 0, \beta \text{ even, } g \equiv 1 \pmod 8 \\ & \text{or } \alpha = 0, \beta \text{ odd, } g \equiv 3 \pmod 8 \\ & \text{or } \alpha \geq 2, \\ \frac{3}{4} & \text{if } \alpha = 0, \beta \text{ even, } g \equiv 5 \pmod 8 \\ & \text{or } \alpha = 0, \beta \text{ odd, } g \equiv 7 \pmod 8), \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

and

$$t(n) = \begin{cases} -\left(\frac{-3}{\sqrt{n}}\right)\sqrt{n} & \text{if } n = \text{square}, \\ 0 & \text{otherwise.} \end{cases}$$

Appealing to Proposition A.10 for the formula for $r(1,3,9,0,0,0;n)$, we obtain Theorem 5.5. □

From Theorem 5.5 we deduce which positive integers are not represented by the spinor regular ternary form B5. These integers are given in Table A.17. As this determination is a little more complicated we give the details.

Suppose $(\alpha, \beta, g) \neq (2k, 0, 1)$ for all $k \in \mathbb{N}_0$. Then by Theorem 5.5 we see that

$$\begin{aligned}
r(f;n) = 0 \Longleftrightarrow\ & k_f(n) = 0 \\
\Longleftrightarrow\ & (A) \quad \alpha = 0,\ \beta = 0,\ g \equiv 3 \pmod 4 \text{ or} \\
& (B) \quad \alpha = 0,\ \beta = 0,\ g \equiv 5 \pmod 6 \text{ or} \\
& (C) \quad \alpha = 0,\ \beta \text{ (even)} \geq 2,\ g \equiv 3 \pmod 4 \text{ or} \\
& (D) \quad \alpha \text{ (even)} \geq 2,\ \beta = 0,\ g \equiv 5 \pmod 6 \text{ or} \\
& (E) \quad \alpha = 0,\ \beta \text{ (odd)} \geq 1,\ g \equiv 1 \pmod 4 \text{ or} \\
& (F) \quad \alpha = 0,\ \beta \text{ (odd)} \geq 1,\ g \equiv 5 \pmod 6 \text{ or} \\
& (G) \quad \alpha \text{ (even)} \geq 2,\ \beta \text{ (odd)} \geq 1,\ g \equiv 5 \pmod 6 \text{ or} \\
& (H) \quad \alpha = 1,\ \beta = 0,\ g \equiv 1 \pmod 6 \text{ or} \\
& (I) \quad \alpha = 1,\ \beta \text{ (even)} \geq 2,\ g \equiv 1 \pmod 6 \text{ or} \\
& (J) \quad \alpha = 1,\ \beta \text{ (even)} \geq 0,\ g \equiv 5 \pmod 6 \text{ or} \\
& (K) \quad \alpha \text{ (odd)} \geq 3,\ \beta = 0,\ g \equiv 1 \pmod 6 \text{ or} \\
& (L) \quad \alpha = 1,\ \beta \text{ (odd)} \geq 1,\ g \equiv 1 \pmod 6 \text{ or} \\
& (M) \quad \alpha = 1,\ \beta \text{ (odd)} \geq 1,\ g \equiv 5 \pmod 6 \text{ or} \\
& (N) \quad \alpha \text{ (odd)} \geq 3,\ \beta \text{ (odd)} \geq 1,\ g \equiv 1 \pmod 6.
\end{aligned}$$

Now

$$\begin{aligned}
(B) \cup (D) \cup (H) \cup (K) \Longleftrightarrow\ & \alpha \text{ (even)} \geq 0, \beta = 0, g \equiv 5 \pmod 6 \text{ or} \\
& \alpha \text{ (odd)} \geq 1, \beta = 0, g \equiv 1 \pmod 6 \\
\Longleftrightarrow\ & n = 3l + 2, \\
(H) \cup (I) \cup (J) \cup (L) \cup (M) \Longleftrightarrow\ & \alpha = 1 \\
\Longleftrightarrow\ & n = 4l + 2, \\
(A) \cup (C) \cup (E) \Longleftrightarrow\ & \alpha = 0, \beta \text{ (even)} \geq 0, g \equiv 3 \pmod 4 \text{ or} \\
& \alpha = 0, \beta \text{ (odd)} \geq 1, g \equiv 1 \pmod 4 \\
\Longleftrightarrow\ & n = 4l + 3, \\
(F) \cup (G) \cup (L) \cup (N) \Longleftrightarrow\ & \alpha \text{ (even)} \geq 0, \beta \text{ (odd)} \geq 1, g \equiv 5 \pmod 6 \text{ or} \\
& \alpha \text{ (odd)} \geq 1, \beta \text{ (odd)} \geq 1, g \equiv 1 \pmod 6 \\
\Longleftrightarrow\ & n = 9^k(9l + 6).
\end{aligned}$$

Thus, for $(\alpha,\beta,g)\neq(2k,0,1)$ we have

$$r(f;n)=0 \text{ if and only if } n=3l+2,4l+2,4l+3 \text{ or } 9^k(9l+6).$$

Now suppose $(\alpha,\beta,g)=(2k,0,1)$ for some $k\in\mathbb{N}_0$. If $k=0$, then by Theorem 5.5 we have

$$r(f;n)=l(n)-\left(\frac{-3}{h}\right)h.$$

So, by Lemma 3.1 (iii) we deduce that

$$r(f;n)=0 \text{ if and only if } l(n)=\left(\frac{-3}{h}\right)h \text{ if and only if } n\in M_3^2.$$

If $k\geq 1$, then by Theorem 5.5 and Lemma 3.2 (with $m=\frac{\alpha}{2}\geq 1, A=3, B=2, C=1$) we see that

$$\begin{aligned} r(f;n)&=(3\cdot 2^{\alpha/2}-2)l(n)-2^{\alpha/2}\left(\frac{-3}{2^{\alpha/2}h}\right)h\\ &\geq(3\cdot 2^{\alpha/2}-2)l(n)-2^{\alpha/2}h>0.\end{aligned}$$

This completes the proof that $n$ is not represented by $f$ if and only if $n$ belongs to at least one of the progressions $3l+2, 4l+2, 4l+3, 9^k(9l+6)$ $(k,l\in\mathbb{N}_0)$, $M_3^2$. Next, we determine the representation number $r(f;n)$ when $f$ is the form B6.

**Theorem 5.6.** *Let $f$ denote the form B6, that is, $f=3x^2+4y^2+9z^2$. If $(\alpha,\beta,g)\neq(2k,0,1)$ for all $k\in\mathbb{N}_0$, we have*

$$r(f;n)=k_f(n)l(n)h(\mathbb{Q}(\sqrt{-n^*})),$$

*where the values of $k_f(n)$ are given in Table 5.6.*

| $\alpha,\beta$ | $g$ | $k_f(n)$ |
|---|---|---|
| | *$\alpha$(even) $\beta$(even)* | |
| $\alpha=0,\beta=0$ | $g\equiv 1 \pmod{24}, g\neq 1$ | 3 |
| | $g\equiv 5,11,17,23 \pmod{24}$ | 0 |
| | $g\equiv 7,13,19 \pmod{24}$ | 1 |
| $\alpha=0,\beta\geq 2$ | $g=1$ | 2 |
| | $g\equiv 1,17 \pmod{24}, g\neq 1$ | 6 |
| | $g\equiv 5,7,11,13,19,23 \pmod{24}$ | 2 |
| $\alpha\geq 2,\beta=0$ | $g\equiv 1 \pmod{24}, g\neq 1$ | $9\cdot 2^{\alpha/2}-6$ |
| | $g\equiv 5,11,17,23 \pmod{24}$ | 0 |
| | $g\equiv 7,19 \pmod{24}$ | $3\cdot 2^{\alpha/2}-3$ |
| | $g\equiv 13 \pmod{24}$ | $3\cdot 2^{\alpha/2}$ |
| $\alpha\geq 2,\beta\geq 2$ | $g=1$ | $3\cdot 2^{\alpha/2+1}-4$ |
| | $g\equiv 1,17 \pmod{24}, g\neq 1$ | $9\cdot 2^{\alpha/2+1}-12$ |
| | $g\equiv 5,13 \pmod{24}$ | $3\cdot 2^{\alpha/2+1}$ |
| | $g\equiv 7,11,19,23 \pmod{24}$ | $3\cdot 2^{\alpha/2+1}-6$ |
| | | *Continued on next page* |

| $\alpha, \beta$ | $g$ | $k_f(n)$ |
|---|---|---|
| | *$\alpha$(even) $\beta$(odd)* | |
| $\alpha = 0, \beta \geq 1$ | $g = 1$ | 2 |
| | $g \equiv 1, 7, 13 \pmod{24}, g \neq 1$ | 4 |
| | $g \equiv 5, 11, 17, 23 \pmod{24}$ | 0 |
| | $g \equiv 19 \pmod{24}$ | 12 |
| $\alpha \geq 2, \beta \geq 1$ | $g = 1$ | $3 \cdot 2^{\alpha/2+1} - 6$ |
| | $g \equiv 1, 13 \pmod{24}, g \neq 1$ | $3 \cdot 2^{\alpha/2+2} - 12$ |
| | $g \equiv 5, 11, 17, 23 \pmod{24}$ | 0 |
| | $g \equiv 7 \pmod{24}$ | $3 \cdot 2^{\alpha/2+2}$ |
| | $g \equiv 19 \pmod{24}$ | $9 \cdot 2^{\alpha/2+2} - 24$ |
| | *$\alpha$(odd) $\beta$(even)* | |
| $\alpha = 1, \beta \geq 0$ | | 0 |
| $\alpha \geq 3, \beta = 0$ | $g \equiv 1 \pmod 6$ | 0 |
| | $g \equiv 5 \pmod 6$ | $3 \cdot 2^{(\alpha-1)/2} - 3$ |
| $\alpha \geq 3, \beta \geq 2$ | | $3 \cdot 2^{(\alpha+1)/2} - 6$ |
| | *$\alpha$(odd) $\beta$(odd)* | |
| $\alpha = 1, \beta \geq 1$ | | 0 |
| $\alpha \geq 3, \beta \geq 1$ | $g \equiv 1 \pmod 6$ | 0 |
| | $g \equiv 5 \pmod 6$ | $3 \cdot 2^{(\alpha+3)/2} - 12$ |

Table 5.6: Values of $k_f(n)$

*If $(\alpha, \beta, g) = (0, 0, 1)$, we have*

$$r(f; n) = l(n) - \left(\frac{-3}{h}\right) h.$$

*If $(\alpha, \beta, g) = (2k, 0, 1)$, where $k \in \mathbb{N}$, we have*

$$r(f; n) = (3 \cdot 2^{\alpha/2} - 2) l(n) + 2^{\alpha/2} \left(\frac{-3}{2^{\alpha/2} h}\right) h.$$

*Proof.* By Theorem 2.1 (xviii) and Proposition 2.1 we have for all $n \in \mathbb{N}$

$$r(f; n) = s(n) r(1, 3, 9, 0, 0, 0; n) + \frac{1}{2} r\left(1, 3, 9, 0, 0, 0; \frac{n}{4}\right) + t(n),$$

where

$$s(n) = \frac{1}{64}\left(\sqrt{2} - i\sqrt{2}\right)\omega_8^n + \frac{1}{32}(4 + 5i)\omega_8^{2n} + \frac{1}{64}\left(-\sqrt{2} - i\sqrt{2}\right)\omega_8^{3n} - \frac{7}{32}\omega_8^{4n}$$
$$+ \frac{1}{64}\left(-\sqrt{2} + i\sqrt{2}\right)\omega_8^{5n} + \frac{1}{32}(4 - 5i)\omega_8^{6n} + \frac{1}{64}\left(\sqrt{2} + i\sqrt{2}\right)\omega_8^{7n} + \frac{15}{32}$$

$$= \begin{cases} \frac{1}{2} & \text{if } n \equiv 0, 1, 4 \pmod{8}, \\ \frac{1}{4} & \text{if } n \equiv 5 \pmod{8}, \\ 0 & \text{if } n \equiv 2, 6 \pmod{8}, \\ 1 & \text{if } n \equiv 3, 7 \pmod{8}, \end{cases}$$

$$= \begin{cases} \frac{1}{2} & \text{if } \alpha = 0, \beta \text{ even}, g \equiv 1 \pmod{8} \\ & \text{or } \alpha = 0, \beta \text{ odd}, g \equiv 3 \pmod{8} \\ & \text{or } \alpha \geq 2, \\ \frac{1}{4} & \text{if } \alpha = 0, \beta \text{ even}, g \equiv 5 \pmod{8} \\ & \text{or } \alpha = 0, \beta \text{ odd}, g \equiv 7 \pmod{8}, \\ 0 & \text{if } \alpha = 1, \\ 1 & \text{if } \alpha = 0, \beta \text{ even}, g \equiv 3 \pmod{4} \\ & \text{or } \alpha = 0, \beta \text{ odd}, g \equiv 1 \pmod{4}, \end{cases}$$

and

$$t(n) = \begin{cases} -\left(\frac{-3}{\sqrt{n}}\right)\sqrt{n} & \text{if } n = \text{odd square}, \\ \left(\frac{-3}{\sqrt{n}}\right)\sqrt{n} & \text{if } n = \text{even square}, \\ 0 & \text{otherwise}, \end{cases}$$

$$= \begin{cases} -\left(\frac{-3}{h}\right)h & \text{if } (\alpha, \beta, g) = (0, 0, 1), \\ \left(\frac{-3}{2^k h}\right)2^k h & \text{if } (\alpha, \beta, g) = (2k, 0, 1) \ (k \in \mathbb{N}), \\ 0 & \text{otherwise}. \end{cases}$$

Appealing to Proposition A.10 for the formula for $r(1, 3, 9, 0, 0, 0; n)$, we obtain Theorem 5.6. □

From Theorem 5.6 we can determine the positive integers not represented by the form B6 exactly as we did for the form B1. These integers are given in Table A.17. They have been given previously by Lomadze [19, Corollary 2, p. 156] and Berkovich [3, Theorem 4.2]. Next, we determine the representation number $r(f; n)$ when $f$ is the form B7.

**Theorem 5.7.** *Let $f$ denote the form B7, that is, $f = 4x^2 + 9y^2 + 12z^2$. If $(\alpha, \beta, g) \neq (0, 0, 1)$, we have*

$$r(f; n) = k_f(n)l(n)h(\mathbb{Q}(\sqrt{-n^*})),$$

*where the values of $k_f(n)$ are given in Table 5.7.*

| $\alpha, \beta$ | $g$ | $k_f(n)$ |
|---|---|---|
| | *$\alpha$(even) $\beta$(even)* | |
| $\alpha = 0, \beta = 0$ | $g \equiv 1 \pmod{24}, g \neq 1$ | 3 |
| | $g \equiv 5, 7, 11, 17, 19, 23 \pmod{24}$ | 0 |
| | $g \equiv 13 \pmod{24}$ | 1 |
| $\alpha = 0, \beta \geq 2$ | $g = 1$ | 2 |
| | $g \equiv 1, 17 \pmod{24}, g \neq 1$ | 6 |
| | $g \equiv 5, 13 \pmod{24}$ | 2 |
| | $g \equiv 7, 11, 19, 23 \pmod{24}$ | 0 |
| $\alpha \geq 2, \beta = 0$ | $g = 1$ | $2^{\alpha/2+1} - 2$ |
| | $g \equiv 1 \pmod{24}, g \neq 1$ | $3 \cdot 2^{\alpha/2+1} - 6$ |
| | $g \equiv 5, 11, 17, 23 \pmod{24}$ | 0 |
| | $g \equiv 7, 19 \pmod{24}$ | $2^{\alpha/2+1} - 3$ |
| | $g \equiv 13 \pmod{24}$ | $2^{\alpha/2+1}$ |
| $\alpha \geq 2, \beta \geq 2$ | $g = 1$ | $2^{\alpha/2+2} - 4$ |
| | $g \equiv 1, 17 \pmod{24}, g \neq 1$ | $3 \cdot 2^{\alpha/2+2} - 12$ |
| | $g \equiv 5, 13 \pmod{24}$ | $2^{\alpha/2+2}$ |
| | $g \equiv 7, 11, 19, 23 \pmod{24}$ | $2^{\alpha/2+2} - 6$ |
| | *$\alpha$(even) $\beta$(odd)* | |
| $\alpha = 0, \beta \geq 1$ | $g \equiv 1, 5, 11, 13, 17, 23 \pmod{24}$ | 0 |
| | $g \equiv 7 \pmod{24}$ | 4 |
| | $g \equiv 19 \pmod{24}$ | 12 |
| $\alpha \geq 2, \beta \geq 1$ | $g = 1$ | $2^{\alpha/2+2} - 6$ |
| | $g \equiv 1, 13 \pmod{24}, g \neq 1$ | $2^{\alpha/2+3} - 12$ |
| | $g \equiv 5, 11, 17, 23 \pmod{24}$ | 0 |
| | $g \equiv 7 \pmod{24}$ | $2^{\alpha/2+3}$ |
| | $g \equiv 19 \pmod{24}$ | $3 \cdot 2^{\alpha/2+3} - 24$ |
| | *$\alpha$(odd) $\beta$(even)* | |
| $\alpha = 1, \beta \geq 0$ | | 0 |
| $\alpha \geq 3, \beta = 0$ | $g \equiv 1 \pmod{6}$ | 0 |
| | $g \equiv 5 \pmod{6}$ | $2^{(\alpha+1)/2} - 3$ |
| $\alpha \geq 3, \beta \geq 2$ | | $2^{(\alpha+3)/2} - 6$ |
| | *$\alpha$(odd) $\beta$(odd)* | |
| $\alpha = 1, \beta \geq 1$ | | 0 |
| $\alpha \geq 3, \beta \geq 1$ | $g \equiv 1 \pmod{6}$ | 0 |
| | $g \equiv 5 \pmod{6}$ | $2^{(\alpha+5)/2} - 12$ |

Table 5.7: Values of $k_f(n)$

*If* $(\alpha, \beta, g) = (0, 0, 1)$*, we have*

$$r(f; n) = l(n) - \left(\frac{-3}{h}\right) h.$$

*Proof.* By Theorem 2.1 (xix) and Proposition 2.1, we deduce

$$r(f;n) = s(n)r(1,3,9,0,0,0;n) + r\left(1,3,9,0,0,0;\frac{n}{4}\right) + t(n),$$

where

$$\begin{aligned} s(n) = &\frac{\sqrt{2}}{64}(1-i)\omega_8^n - \frac{3i}{32}\omega_8^{2n} - \frac{\sqrt{2}}{64}(1+i)\omega_8^{3n} - \frac{3}{32}\omega_8^{4n} - \frac{\sqrt{2}}{64}(1-i)\omega_8^{5n} \\ &+ \frac{3i}{32}\omega_8^{6n} + \frac{\sqrt{2}}{64}(1+i)\omega_8^{7n} + \frac{3}{32} \end{aligned}$$

and

$$t(n) = \begin{cases} -\left(\frac{-3}{\sqrt{n}}\right)\sqrt{n} & \text{if } n = \text{odd square}, \\ 0 & \text{otherwise.} \end{cases}$$

Now

$$\begin{aligned} \omega_8^n - \omega_8^{3n} - \omega_8^{5n} + \omega_8^{7n} &= 2\left(\frac{2}{n}\right)\sqrt{2}, \\ \omega_8^n + \omega_8^{3n} - \omega_8^{5n} - \omega_8^{7n} &= 2\left(\frac{-2}{n}\right)i\sqrt{2}, \\ \omega_8^{2n} - \omega_8^{6n} = i^n - (-i)^n &= 2\left(\frac{-4}{n}\right)i, \text{ and } \quad \omega_8^{4n} = (-1)^n, \end{aligned}$$

so

$$\begin{aligned} s(n) &= \frac{\sqrt{2}}{64}2\left(\frac{2}{n}\right)\sqrt{2} - \frac{\sqrt{2}}{64}i2\left(\frac{-2}{n}\right)i\sqrt{2} - \frac{3i}{32}2\left(\frac{-4}{n}\right)i - \frac{3}{32}(-1)^n + \frac{3}{32} \\ &= \begin{cases} 0 & \text{if } n \equiv 0 \pmod 2, \\ \frac{1}{16}\left(3 + 3\left(\frac{-4}{n}\right) + \left(\frac{2}{n}\right) + \left(\frac{-2}{n}\right)\right) & \text{if } n \equiv 1 \pmod 2, \end{cases} \\ &= \begin{cases} \frac{1}{2} & \text{if } n \equiv 1 \pmod 8, \\ \frac{1}{4} & \text{if } n \equiv 5 \pmod 8, \\ 0 & \text{if } n \equiv 0 \pmod 2 \text{ or } n \equiv 3 \pmod 4, \end{cases} \\ &= \begin{cases} \frac{1}{2} & \text{if } \alpha = 0, \beta \text{ even}, g \equiv 1 \pmod 8 \\ & \text{or } \alpha = 0, \beta \text{ odd}, g \equiv 3 \pmod 8, \\ \frac{1}{4} & \text{if } \alpha = 0, \beta \text{ even}, g \equiv 5 \pmod 8 \\ & \text{or } \alpha = 0, \beta \text{ odd}, g \equiv 7 \pmod 8, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Also

$$t(n) = \begin{cases} -\left(\frac{-3}{h}\right)h & \text{if } (\alpha,\beta,g) = (0,0,1), \\ 0 & \text{otherwise.} \end{cases}$$

Appealing to Proposition A.10 for the formula for $r(1,3,9,0,0,0;n)$, we obtain Theorem 5.7. □

From Theorem 5.7 using Lemma 3.1(iii), we deduce which positive integers are not represented by the spinor regular ternary form B7. They were first obtained by Lomadze [19, Corollary 2, p. 161], and are given in Table A.17. Next, we determine the representation number $r(f;n)$ when $f$ is the form B8.

**Theorem 5.8.** *Let $f$ denote the form B8, that is, $f = 4x^2 + 9y^2 + 28z^2 + 4zx$. If $(\alpha, \beta, g) \neq (2k, 0, 1)$ for all $k \in \mathbb{N}_0$, we have*

$$r(f;n) = k_f(n)l(n)h(\mathbb{Q}(\sqrt{-n^*})),$$

*where the values of $k_f(n)$ are given in Table 5.8.*

| $\alpha, \beta$ | $g$ | $k_f(n)$ |
|---|---|---|
| | *$\alpha$(even) $\beta$(even)* | |
| $\alpha = 0, \beta = 0$ | $g \equiv 1 \pmod{12}, g \neq 1$ | 1 |
| | $g \equiv 5, 7, 11 \pmod{12}$ | 0 |
| $\alpha = 0, \beta \geq 2$ | $g = 1$ | 2 |
| | $g \equiv 1, 5 \pmod{12}, g \neq 1$ | 6 |
| | $g \equiv 7, 11 \pmod{12}$ | 0 |
| $\alpha \geq 2, \beta = 0$ | $g \equiv 1 \pmod{24}, g \neq 1$ | $3 \cdot 2^{\alpha/2} - 2$ |
| | $g \equiv 5, 11, 17, 23 \pmod{24}$ | 0 |
| | $g \equiv 7, 19 \pmod{24}$ | $2^{\alpha/2} - 1$ |
| | $g \equiv 13 \pmod{24}$ | $2^{\alpha/2}$ |
| $\alpha \geq 2, \beta \geq 2$ | $g = 1$ | $3 \cdot 2^{\alpha/2+1} - 4$ |
| | $g \equiv 1, 17 \pmod{24}, g \neq 1$ | $9 \cdot 2^{\alpha/2+1} - 12$ |
| | $g \equiv 5, 13 \pmod{24}$ | $3 \cdot 2^{\alpha/2+1}$ |
| | $g \equiv 7, 11, 19, 23 \pmod{24}$ | $3 \cdot 2^{\alpha/2+1} - 6$ |
| | *$\alpha$(even) $\beta$(odd)* | |
| $\alpha = 0, \beta = 1$ | | 0 |
| $\alpha = 0, \beta \geq 3$ | $g \equiv 1, 5, 11 \pmod{12}$ | 0 |
| | $g \equiv 7 \pmod{12}$ | 12 |
| $\alpha \geq 2, \beta = 1$ | | 0 |
| $\alpha \geq 2, \beta \geq 3$ | $g = 1$ | $3 \cdot 2^{\alpha/2+1} - 6$ |
| | $g \equiv 1, 13 \pmod{24}, g \neq 1$ | $3 \cdot 2^{\alpha/2+2} - 12$ |
| | $g \equiv 5, 11, 17, 23 \pmod{24}$ | 0 |
| | $g \equiv 7 \pmod{24}$ | $3 \cdot 2^{\alpha/2+2}$ |
| | $g \equiv 19 \pmod{24}$ | $9 \cdot 2^{\alpha/2+2} - 24$ |
| | | *Continued on next page* |

| $\alpha, \beta$ | $g$ | $k_f(n)$ |
|---|---|---|
| | $\alpha$*(odd)* $\beta$*(even)* | |
| $\alpha = 1, \beta \geq 0$ | | $0$ |
| $\alpha \geq 3, \beta = 0$ | $g \equiv 1 \pmod 6$<br>$g \equiv 5 \pmod 6$ | $0$<br>$2^{(\alpha-1)/2} - 1$ |
| $\alpha \geq 3, \beta \geq 2$ | | $3 \cdot 2^{(\alpha+1)/2} - 6$ |
| | $\alpha$*(odd)* $\beta$*(odd)* | |
| $\alpha = 1, \beta \geq 1$ | | $0$ |
| $\alpha \geq 3, \beta = 1$ | | $0$ |
| $\alpha \geq 3, \beta \geq 3$ | $g \equiv 1 \pmod 6$<br>$g \equiv 5 \pmod 6$ | $0$<br>$3 \cdot 2^{(\alpha+3)/2} - 12$ |

Table 5.8: Values of $k_f(n)$

*If* $(\alpha, \beta, g) = (0, 0, 1)$*, we have*

$$r(f;n) = \frac{1}{3} l(n) - \frac{1}{3} \left( \frac{-3}{h} \right) h.$$

*If* $(\alpha, \beta, g) = (2k, 0, 1)$*, where* $k \in \mathbb{N}$*, we have*

$$r(f;n) = \left( 2^{\alpha/2} - \frac{2}{3} \right) l(n) + \frac{1}{3} 2^{\alpha/2} \left( \frac{-3}{2^{\alpha/2} h} \right) h.$$

*Proof.* By Theorem 2.1 (xx) and Proposition 2.1 we have for all $n \in \mathbb{N}$

$$r(f;n) = s(n) r(1,4,4,4,0,0;n) + r\left(1,4,4,4,0,0; \frac{n}{9}\right) + t(n),$$

where

$$\begin{aligned} s(n) &= \frac{1}{36}\left(-1 - i\sqrt{3}\right)\omega_3^n + \frac{1}{36}\left(-1 + i\sqrt{3}\right)\omega_3^{2n} + \frac{1}{18} = \frac{1}{18}\left(\omega_3^{n+2} + \omega_3^{2(n+2)} + 1\right) \\ &= \begin{cases} \frac{1}{6} & \text{if } n \equiv 1 \pmod 3, \\ 0 & \text{if } n \equiv 0, 2 \pmod 3, \end{cases} \\ &= \begin{cases} \frac{1}{6} & \text{if } \alpha \text{ (even)} \geq 0, \beta = 0, g \equiv 1 \pmod 3 \\ & \text{or } \alpha \text{ (odd)} \geq 1, \beta = 0, g \equiv 2 \pmod 3, \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

and

$$\begin{aligned} t(n) &= \begin{cases} -\frac{1}{3}\left(\frac{-3}{\sqrt{n}}\right)\sqrt{n} & \text{if } n = \text{square}, \\ 0 & \text{otherwise,} \end{cases} \\ &= \begin{cases} -\frac{1}{3}\left(\frac{-3}{2^k h}\right) 2^k h & \text{if } (\alpha, \beta, g) = (2k, 0, 1) \ (k \in \mathbb{N}_0), \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Appealing to Proposition A.11 for the formula for $r(1,4,4,4,0,0;n)$, we obtain Theorem 5.8. □

From Theorem 5.8 we deduce using Lemmas 3.1(iii) and 3.2 which positive integers are not represented by the spinor regular ternary form B8. These integers are given in Table A.17. Next, we determine the representation number $r(f;n)$ when $f$ is the form B9.

**Theorem 5.9.** *Let $f$ denote the form B9, that is, $f = 9x^2 + 16y^2 + 16z^2 + 16yz$. If $(\alpha, \beta, g) \neq (2k, 0, 1)$ for all $k \in \mathbb{N}_0$, we have*

$$r(f;n) = k_f(n)l(n)h(\mathbb{Q}(\sqrt{-n^*})),$$

*where the values of $k_f(n)$ are given in Table 5.9.*

| $\alpha, \beta$ | $g$ | $k_f(n)$ |
|---|---|---|
| | *$\alpha$(even) $\beta$(even)* | |
| $\alpha = 0, \beta = 0$ | $g \equiv 1 \pmod{24}, g \neq 1$ | 3 |
| | $g \equiv 5, 7, 11, 13, 17, 19, 23 \pmod{24}$ | 0 |
| $\alpha = 0, \beta \geq 2$ | $g = 1$ | 2 |
| | $g \equiv 1, 17 \pmod{24}, g \neq 1$ | 6 |
| | $g \equiv 5, 7, 11, 13, 19, 23 \pmod{24}$ | 0 |
| $\alpha \geq 2, \beta = 0$ | $g \equiv 1 \pmod{24}, g \neq 1$ | $9 \cdot 2^{\alpha/2-1} - 6$ |
| | $g \equiv 5, 11, 17, 23 \pmod{24}$ | 0 |
| | $g \equiv 7, 19 \pmod{24}$ | $3 \cdot 2^{\alpha/2-1} - 3$ |
| | $g \equiv 13 \pmod{24}$ | $3 \cdot 2^{\alpha/2-1}$ |
| $\alpha \geq 2, \beta \geq 2$ | $g = 1$ | $3 \cdot 2^{\alpha/2} - 4$ |
| | $g \equiv 1, 17 \pmod{24}, g \neq 1$ | $9 \cdot 2^{\alpha/2} - 12$ |
| | $g \equiv 5, 13 \pmod{24}$ | $3 \cdot 2^{\alpha/2}$ |
| | $g \equiv 7, 11, 19, 23 \pmod{24}$ | $3 \cdot 2^{\alpha/2} - 6$ |
| | *$\alpha$(even) $\beta$(odd)* | |
| $\alpha = 0, \beta \geq 1$ | $g \equiv 1, 5, 7, 11, 13, 17, 23 \pmod{24}$ | 0 |
| | $g \equiv 19 \pmod{24}$ | 12 |
| $\alpha \geq 2, \beta \geq 1$ | $g = 1$ | $3 \cdot 2^{\alpha/2} - 6$ |
| | $g \equiv 1, 13 \pmod{24}, g \neq 1$ | $3 \cdot 2^{\alpha/2+1} - 12$ |
| | $g \equiv 5, 11, 17, 23 \pmod{24}$ | 0 |
| | $g \equiv 7 \pmod{24}$ | $3 \cdot 2^{\alpha/2+1}$ |
| | $g \equiv 19 \pmod{24}$ | $9 \cdot 2^{\alpha/2+1} - 24$ |
| | *$\alpha$(odd) $\beta$(even)* | |
| $\alpha = 1, \beta \geq 0$ | | 0 |
| $\alpha \geq 3, \beta = 0$ | $g \equiv 1 \pmod{6}$ | 0 |
| | $g \equiv 5 \pmod{6}$ | $3 \cdot 2^{(\alpha-3)/2} - 3$ |
| $\alpha \geq 3, \beta \geq 2$ | | $3 \cdot 2^{(\alpha-1)/2} - 6$ |
| | | *Continued on next page* |

| $\alpha, \beta$ | $g$ | $k_f(n)$ |
|---|---|---|
| | $\alpha(odd)\ \beta(odd)$ | |
| $\alpha = 1, \beta \geq 1$ | | $0$ |
| $\alpha \geq 3, \beta \geq 1$ | $g \equiv 1 \pmod 6$ | $0$ |
| | $g \equiv 5 \pmod 6$ | $3 \cdot 2^{(\alpha+1)/2} - 12$ |

Table 5.9: Values of $k_f(n)$

*If* $(\alpha, \beta, g) = (0, 0, 1)$*, we have*

$$r(f; n) = l(n) - \left(\frac{-3}{h}\right) h.$$

*If* $(\alpha, \beta, g) = (2k, 0, 1)$*, where* $k \in \mathbb{N}$*, we have*

$$r(f; n) = \left(3 \cdot 2^{\alpha/2-1} - 2\right) l(n) + 2^{\alpha/2-1} \left(\frac{-3}{2^{\alpha/2} h}\right) h.$$

*Proof.* By Theorem 2.1 (xxi) and Proposition 2.1 we have for all $n \in \mathbb{N}$

$$r(f; n) = s(n) r(1, 3, 9, 0, 0, 0; n) + \frac{5}{4} r\left(1, 3, 9, 0, 0, 0; \frac{n}{4}\right) + t(n),$$

where

$$\begin{aligned}
s(n) &= \frac{\sqrt{2} - i\sqrt{2}}{32} \omega_8^n + \left(-\frac{1}{16} - \frac{i}{16}\right) \omega_8^{2n} + \frac{-\sqrt{2} - i\sqrt{2}}{32} \omega_8^{3n} - \frac{1}{8} \omega_8^{4n} \\
&\quad + \frac{-\sqrt{2} + i\sqrt{2}}{32} \omega_8^{5n} + \left(-\frac{1}{16} + \frac{i}{16}\right) \omega_8^{6n} + \frac{\sqrt{2} + i\sqrt{2}}{32} \omega_8^{7n} \\
&= \frac{1}{16} \left(\omega_8^{n+7} + \omega_8^{3(n+7)} + \omega_8^{5(n+7)} + \omega_8^{7(n+7)}\right) \\
&\quad - \frac{1}{16} \left((1+i) i^n + (1-i) i^{3n}\right) - \frac{1}{8} (-1)^n \\
&= \frac{1}{16} \left\{\begin{matrix} (-1)^{(n+7)/4} 4 & \text{if } n \equiv 1 \pmod 4 \\ 0 & \text{otherwise} \end{matrix}\right\} \\
&\quad - \frac{1}{16} \left\{\begin{matrix} 2 & \text{if } n \equiv 0, 3 \pmod 4 \\ -2 & \text{if } n \equiv 1, 2 \pmod 4 \end{matrix}\right\} - \frac{1}{8} (-1)^n \\
&= \begin{cases} \frac{1}{2} & \text{if } n \equiv 1 \pmod 8, \\ 0 & \text{if } n \equiv 2, 3, 5, 6, 7 \pmod 8, \\ -\frac{1}{4} & \text{if } n \equiv 0, 4 \pmod 8, \end{cases} \\
&= \begin{cases} \frac{1}{2} & \text{if } \alpha = 0, \beta \text{ even, } g \equiv 1 \pmod 8 \\ & \text{or } \alpha = 0, \beta \text{ odd, } g \equiv 3 \pmod 8, \\ -\frac{1}{4} & \text{if } \alpha \geq 2, \\ 0 & \text{otherwise,} \end{cases}
\end{aligned}$$

and

$$t(n) = \begin{cases} -\left(\frac{-3}{\sqrt{n}}\right)\sqrt{n} & \text{if } n = \text{odd square,} \\ \frac{1}{2}\left(\frac{-3}{\sqrt{n}}\right)\sqrt{n} & \text{if } n = \text{even square,} \\ 0 & \text{otherwise,} \end{cases}$$
$$= \begin{cases} -\left(\frac{-3}{h}\right)h & \text{if } (\alpha,\beta,g) = (0,0,1), \\ \frac{1}{2}\left(\frac{-3}{2^k h}\right)2^k h & \text{if } (\alpha,\beta,g) = (2k,0,1)\ (k \in \mathbb{N}), \\ 0 & \text{otherwise.} \end{cases}$$

Appealing to Proposition A.10 for the formula for $r(1,3,9,0,0,0;n)$, we obtain Theorem 5.9. □

From Theorem 5.9 we deduce using Lemmas 3.1(iii) and 3.2 the positive integers $n$ not represented by the spinor regular form B9. These integers are listed in Table A.17. As the proof is more complex we give the details.

Suppose that $(\alpha,\beta,g) \neq (2k,0,1)$ for all $k \in \mathbb{N}_0$. Then by Theorem 5.9 we have $r(f;n) = 0$ if and only if $k_f(n) = 0$.

(A) If $\alpha = 0$ and $\beta = 0$, then

$$\begin{aligned} k_f(n) = 0 \text{ if and only if } &(A1)\ g \equiv 1 \pmod 3, g \equiv 3 \pmod 4 \text{ or} \\ &(A2)\ g \equiv 1 \pmod 3, g \equiv 5 \pmod 8 \text{ or} \\ &(A3)\ g \equiv 2 \pmod 3, g \equiv 3 \pmod 4 \text{ or} \\ &(A4)\ g \equiv 2 \pmod 3, g \equiv 1 \pmod 8 \text{ or} \\ &(A5)\ g \equiv 2 \pmod 3, g \equiv 5 \pmod 8. \end{aligned}$$

(B) If $\alpha = 0$ and $\beta$ (odd) $\geq 1$, then

$$\begin{aligned} k_f(n) = 0 \text{ if and only if } &(B1)\ g \equiv 1 \pmod 3, g \equiv 1 \pmod 4 \text{ or} \\ &(B2)\ g \equiv 1 \pmod 3, g \equiv 7 \pmod 8 \text{ or} \\ &(B3)\ g \equiv 2 \pmod 3, g \equiv 1 \pmod 4 \text{ or} \\ &(B4)\ g \equiv 2 \pmod 3, g \equiv 3 \pmod 8 \text{ or} \\ &(B5)\ g \equiv 2 \pmod 3, g \equiv 7 \pmod 8. \end{aligned}$$

(C) If $\alpha = 0$ and $\beta$ (even) $\geq 2$, then

$$\begin{aligned} k_f(n) = 0 \text{ if and only if } &(C1)\ g \equiv 3 \pmod 4 \text{ or} \\ &(C2)\ g \equiv 5 \pmod 8. \end{aligned}$$

(D) If $\alpha = 1$ and $\beta = 0$, then $k_f(n) = 0$. We split this case into 2 subcases as follows:

$$(D1)\ g \equiv 1 \pmod 3,$$
$$(D2)\ g \equiv 2 \pmod 3.$$

(E) If $\alpha = 1$ and $\beta$ (odd) $\geq 1$, then $k_f(n) = 0$. We split this case into 2 subcases as follows:

$$(E1)\ g \equiv 1 \pmod 3,$$
$$(E2)\ g \equiv 2 \pmod 3.$$

(F) If $\alpha = 1$ and $\beta$ (even) $\geq 2$, then $k_f(n) = 0$.

(G) If $\alpha = 2$ and $\beta = 0$, then

$$k_f(n) = 0 \text{ if and only if } \begin{array}{l} (G1)\ g \equiv 1 \pmod 3, g \equiv 3 \pmod 4 \text{ or} \\ (G2)\ g \equiv 2 \pmod 3, g \equiv 1 \pmod 4 \text{ or} \\ (G3)\ g \equiv 2 \pmod 3, g \equiv 3 \pmod 4. \end{array}$$

(H) If $\alpha = 2$ and $\beta$ (odd) $\geq 1$, then

$$k_f(n) = 0 \text{ if and only if } \begin{array}{l} (H1)\ g \equiv 1 \pmod 3, g \equiv 1 \pmod 4 \text{ or} \\ (H2)\ g \equiv 2 \pmod 3, g \equiv 1 \pmod 4 \text{ or} \\ (H3)\ g \equiv 2 \pmod 3, g \equiv 3 \pmod 4. \end{array}$$

(I) If $\alpha = 2$ and $\beta$ (even) $\geq 2$, then

$$k_f(n) = 0 \text{ if and only if } g \equiv 3 \pmod 4.$$

(J) If $\alpha = 3$ and $\beta = 0$, then $k_f(n) = 0$. We split this case into 2 subcases as follows:

$$(J1)\ g \equiv 1 \pmod 3,$$
$$(J2)\ g \equiv 2 \pmod 3.$$

(K) If $\alpha = 3$ and $\beta$ (odd) $\geq 1$, then $k_f(n) = 0$. We split this case into 2 subcases as follows:

$$(K1)\ g \equiv 1 \pmod 3,$$
$$(K2)\ g \equiv 2 \pmod 3.$$

(L) If $\alpha = 3$ and $\beta$ (even) $\geq 2$, then $k_f(n) = 0$.

(M) If $\alpha$ (even) $\geq 4$ and $\beta = 0$, then

$$k_f(n) = 0 \text{ if and only if } g \equiv 2 \pmod{3}.$$

(N) If $\alpha$ (even) $\geq 4$ and $\beta$ (odd) $\geq 1$, then

$$k_f(n) = 0 \text{ if and only if } g \equiv 2 \pmod{3}.$$

(O) If $\alpha$ (even) $\geq 4$ and $\beta$ (even) $\geq 2$, then $k_f(n) \neq 0$.

(P) If $\alpha$ (odd) $\geq 5$ and $\beta = 0$, then

$$k_f(n) = 0 \text{ if and only if } g \equiv 1 \pmod{3}.$$

(Q) If $\alpha$ (odd) $\geq 5$ and $\beta$ (odd) $\geq 1$, then

$$k_f(n) = 0 \text{ if and only if } g \equiv 1 \pmod{3}.$$

(R) If $\alpha$ (odd) $\geq 5$ and $\beta$ (even) $\geq 2$, then $k_f(n) \neq 0$.

Now

$$\begin{aligned}
n = 3l+2 &\iff \alpha \text{ (even)} \geq 0, \beta = 0, g \equiv 2 \pmod{3} \text{ or} \\
&\qquad \alpha \text{ (odd)} \geq 1, \beta = 0, g \equiv 1 \pmod{3} \\
&\iff \text{one of } (A3), (A4), (A5), (D1), (G2), (G3), (J1), (M), (P) \text{ holds}, \\
n = 4l+2 &\iff \alpha = 1 \\
&\iff \text{one of } (D1), (D2), (E1), (E2), (F) \text{ holds}, \\
n = 4l+3 &\iff \alpha = 0, \beta \text{ (even)} \geq 0, g \equiv 3 \pmod{4} \text{ or} \\
&\qquad \alpha = 0, \beta \text{ (odd)} \geq 1, g \equiv 1 \pmod{4} \\
&\iff \text{one of } (A1), (A3), (B1), (B3), (C1) \text{ holds}, \\
n = 8l+5 &\iff \alpha = 0, \beta \text{ (even)} \geq 0, g \equiv 5 \pmod{8} \text{ or} \\
&\qquad \alpha = 0, \beta \text{ (odd)} \geq 1, g \equiv 7 \pmod{8} \\
&\iff \text{one of } (A2), (A5), (B2), (B5), (C2) \text{ holds}, \\
n = 16l+8 &\iff \alpha = 3 \\
&\iff \text{one of } (J1), (J2), (K1), (K2), (L) \text{ holds}, \\
n = 16l+12 &\iff \alpha = 2, \beta \text{ (odd)} \geq 1, g \equiv 1 \pmod{4} \text{ or} \\
&\qquad \alpha = 2, \beta \text{ (even)} \geq 0, g \equiv 3 \pmod{4} \\
&\iff \text{one of } (G1), (G3), (H1), (H2), (I) \text{ holds},
\end{aligned}$$

$$\begin{aligned}n = 9^k(9l+6) &\Longleftrightarrow \alpha \text{ (even)} \geq 0, \beta \text{ (odd)} \geq 1, g \equiv 2 \pmod 3 \text{ or}\\ &\quad\ \alpha \text{ (odd)} \geq 1, \beta \text{ (odd)} \geq 1, g \equiv 1 \pmod 3\\ &\Longleftrightarrow \text{one of } (B3), (B4), (B5), (E1), (H2), (H3), (K1), (N), (Q) \text{ holds.}\end{aligned}$$

Thus, when $(\alpha, \beta, g) \neq (2k, 0, 1)$ $(k \in \mathbb{N}_0)$, we have

$$\begin{aligned}k_f(n) = 0 \Longleftrightarrow &\text{ one of 33 cases } (A1)-(A5), (B1)-(B5), (C1), (C2), (D1), (D2),\\ &(E1), (E2), (F), (G1)-(G3), (H1)-(H3), (I), (J1), (J2), (K1),\\ &(K2), (L), (M), (N), (P), (Q) \text{ holds}\\ \Longleftrightarrow &\ n = 3l+2, 4l+2, 4l+3, 8l+5, 16l+8, 16l+12 \text{ or } 9^k(9l+6).\end{aligned}$$

If $(\alpha, \beta, g) = (0, 0, 1)$, then by Theorem 5.9 and Lemma 3.1 (iii) we have

$$r(f; n) = 0 \text{ if and only if } l(n) = \left(\frac{-3}{h}\right) h \text{ if and only if } n \in M_3^2.$$

If $(\alpha, \beta, g) = (2, 0, 1)$, then by Theorem 5.9 and Lemma 3.1 (iii) we have

$$r(f; n) = l(n) - \left(\frac{-3}{h}\right) h = 0 \text{ if and only if } n \in 4M_3^2.$$

If $(\alpha, \beta, g) = (2k, 0, 1)$ for some $k \in \mathbb{N}\setminus\{1\}$, we have by Theorem 5.9 and Lemma 3.2 (with $A = \frac{3}{2}, B = 2, C = \frac{1}{2}, m = \frac{\alpha}{2} = k \geq 2$)

$$\begin{aligned}r(f; n) &= \left(\frac{3}{2}2^{\alpha/2} - 2\right) l(n) + \frac{1}{2}2^{\alpha/2}\left(\frac{-3}{2^{\alpha/2}h}\right) h\\ &\geq \left(\frac{3}{2}2^{\alpha/2} - 2\right) l(n) - \frac{1}{2}2^{\alpha/2} h > 0.\end{aligned}$$

This completes the proof that the integers $n$ belonging to at least one of the progressions $3l+2, 4l+2, 4l+3, 8l+5, 16l+8, 16l+12, 9^k(9l+4)$ $(k, l \in \mathbb{N}_0), M_3^2, 4M_3^2$ are precisely those not represented by the form B9. Next, we determine the representation number $r(f; n)$ when $f$ is the form B10.

**Theorem 5.10.** *Let $f$ denote the form B10, that is, $f = 13x^2 + 13y^2 + 16z^2 - 8yz + 8zx + 10xy$. If $(\alpha, \beta, g) \neq (2k, 0, 1)$ for all $k \in \mathbb{N}_0$, we have*

$$r(f; n) = k_f(n) l(n) h(\mathbb{Q}(\sqrt{-n^*})),$$

*where the values of $k_f(n)$ are given in Table* 5.10.

| $\alpha, \beta$ | $g$ | $k_f(n)$ |
|---|---|---|
| | *$\alpha$(even) $\beta$(even)* | |
| $\alpha = 0, \beta = 0$ | $g \equiv 1, 5, 7, 11, 17, 19, 23 \pmod{24}, g \neq 1$ | $0$ |
| | $g \equiv 13 \pmod{24}$ | $1$ |
| $\alpha = 0, \beta \geq 2$ | $g \equiv 1, 7, 11, 17, 19, 23 \pmod{24}$ | $0$ |
| | $g \equiv 5, 13 \pmod{24}$ | $2$ |
| $\alpha \geq 2, \beta = 0$ | $g \equiv 1 \pmod{24}, g \neq 1$ | $9 \cdot 2^{\alpha/2-1} - 6$ |
| | $g \equiv 5, 11, 17, 23 \pmod{24}$ | $0$ |
| | $g \equiv 7, 19 \pmod{24}$ | $3 \cdot 2^{\alpha/2-1} - 3$ |
| | $g \equiv 13 \pmod{24}$ | $3 \cdot 2^{\alpha/2-1}$ |
| $\alpha \geq 2, \beta \geq 2$ | $g = 1$ | $3 \cdot 2^{\alpha/2} - 4$ |
| | $g \equiv 1, 17 \pmod{24}, g \neq 1$ | $9 \cdot 2^{\alpha/2} - 12$ |
| | $g \equiv 5, 13 \pmod{24}$ | $3 \cdot 2^{\alpha/2}$ |
| | $g \equiv 7, 11, 19, 23 \pmod{24}$ | $3 \cdot 2^{\alpha/2} - 6$ |
| | *$\alpha$(even) $\beta$(odd)* | |
| $\alpha = 0, \beta \geq 1$ | $g \equiv 1, 5, 11, 13, 17, 19, 23 \pmod{24}$ | $0$ |
| | $g \equiv 7 \pmod{24}$ | $4$ |
| $\alpha \geq 2, \beta \geq 1$ | $g = 1$ | $3 \cdot 2^{\alpha/2} - 6$ |
| | $g \equiv 1, 13 \pmod{24}, g \neq 1$ | $3 \cdot 2^{\alpha/2+1} - 12$ |
| | $g \equiv 5, 11, 17, 23 \pmod{24}$ | $0$ |
| | $g \equiv 7 \pmod{24}$ | $3 \cdot 2^{\alpha/2+1}$ |
| | $g \equiv 19 \pmod{24}$ | $9 \cdot 2^{\alpha/2+1} - 24$ |
| | *$\alpha$(odd) $\beta$(even)* | |
| $\alpha = 1, \beta \geq 0$ | | $0$ |
| $\alpha \geq 3, \beta = 0$ | $g \equiv 1 \pmod{6}$ | $0$ |
| | $g \equiv 5 \pmod{6}$ | $3 \cdot 2^{(\alpha-3)/2} - 3$ |
| $\alpha \geq 3, \beta \geq 2$ | | $3 \cdot 2^{(\alpha-1)/2} - 6$ |
| | *$\alpha$(odd) $\beta$(odd)* | |
| $\alpha = 1, \beta \geq 1$ | | $0$ |
| $\alpha \geq 3, \beta \geq 1$ | $g \equiv 1 \pmod{6}$ | $0$ |
| | $g \equiv 5 \pmod{6}$ | $3 \cdot 2^{(\alpha+1)/2} - 12$ |

Table 5.10: Values of $k_f(n)$

*If* $(\alpha, \beta, g) = (0, 0, 1)$, *we have*

$$r(f;n) = l(n) - \left(\frac{-3}{h}\right) h.$$

*If* $(\alpha, \beta, g) = (2k, 0, 1)$, *where* $k \in \mathbb{N}$, *we have*

$$r(f;n) = \left(3 \cdot 2^{\alpha/2-1} - 2\right) l(n) + 2^{\alpha/2-1} \left(\frac{-3}{2^{\alpha/2} h}\right) h.$$

*Proof.* By Theorem 2.1 (xxii) and Proposition 2.1 we have for all $n \in \mathbb{N}$

$$r(f;n) = s(n)r(1,3,9,0,0,0;n) + \frac{5}{4}r\left(1,3,9,0,0,0;\frac{n}{4}\right) + t(n),$$

where

$$\begin{aligned}
s(n) &= \frac{1}{64}\left(-\sqrt{2}+i\sqrt{2}\right)\omega_8^n + \frac{1}{32}(-2-i)\omega_8^{2n} + \frac{1}{64}\left(\sqrt{2}+i\sqrt{2}\right)\omega_8^{3n} - \frac{3}{32}\omega_8^{4n} \\
&\quad + \frac{1}{64}\left(\sqrt{2}-i\sqrt{2}\right)\omega_8^{5n} + \frac{1}{32}(-2+i)\omega_8^{6n} + \frac{1}{64}\left(-\sqrt{2}-i\sqrt{2}\right)\omega_8^{7n} - \frac{1}{32} \\
&= \frac{1}{32}\left(\omega_8^{n+3} + \omega_8^{3(n+3)} + \omega_8^{5(n+3)} + \omega_8^{7(n+3)}\right) + \frac{1}{32}\left((-2-i)i^n + (-2+i)i^{3n}\right) \\
&\quad - \frac{3}{32}(-1)^n - \frac{1}{32} \\
&= \frac{1}{32}\left\{\begin{matrix} (-1)^{(n+3)/4}4 & \text{if } n \equiv 1 \pmod 4 \\ 0 & \text{otherwise} \end{matrix}\right\} \\
&\quad + \frac{1}{32}\left\{\begin{matrix} -4 & \text{if } n \equiv 0 \pmod 4 \\ 2 & \text{if } n \equiv 1 \pmod 4 \\ 4 & \text{if } n \equiv 2 \pmod 4 \\ -2 & \text{if } n \equiv 3 \pmod 4 \end{matrix}\right\} + \left\{\begin{matrix} -\frac{1}{8} & \text{if } n \equiv 0 \pmod 2 \\ \frac{1}{16} & \text{if } n \equiv 1 \pmod 2 \end{matrix}\right\} \\
&= \begin{cases} 0 & \text{if } n \equiv 1,2,3,6,7 \pmod 8, \\ \frac{1}{4} & \text{if } n \equiv 5 \pmod 8, \\ -\frac{1}{4} & \text{if } n \equiv 0,4 \pmod 8, \end{cases} \\
&= \begin{cases} \frac{1}{4} & \text{if } \alpha = 0, \beta \text{ (even)} \geq 0, g \equiv 5 \pmod 8 \\ & \text{or } \alpha = 0, \beta \text{ (odd)} \geq 1, g \equiv 7 \pmod 8, \\ -\frac{1}{4} & \text{if } \alpha \geq 2, \\ 0 & \text{otherwise,} \end{cases}
\end{aligned}$$

and

$$\begin{aligned}
t(n) &= \begin{cases} \frac{1}{2}\left(\frac{-3}{\sqrt{n}}\right)\sqrt{n} & \text{if } n = \text{even square}, \\ 0 & \text{otherwise,} \end{cases} \\
&= \begin{cases} \frac{1}{2}\left(\frac{-3}{2^k h}\right)2^k h & \text{if } (\alpha,\beta,g) = (2k,0,1)\ (k \in \mathbb{N}), \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

Appealing to Proposition A.10 for the formula for $r(1,3,9,0,0,0;n)$, we obtain Theorem 5.10. □

The positive integers not represented by the spinor regular ternary form B10 can be deduced from Theorem 5.10 in a similar manner to those for B9 using Lemmas 3.1(iii) and 3.2. They are given in Table A.17.

We now consider the representation number $r(f;n)$ when $f$ is the form B11. This spinor regular form is not alone in its spinor genus. We are only able to evaluate $r(f;n)$ for even values of $n \in \mathbb{N}$.

**Theorem 5.11.** *Let $f$ denote the form B11, that is, $f = 9x^2+16y^2+48z^2$. Suppose that $n \in \mathbb{N}$ is even, so that $\alpha \geq 1$. If $(\alpha, \beta, g) \neq (2, 0, 1)$, then*

$$r(f;n) = k_f(n)l(n)h(\mathbb{Q}(\sqrt{-n^*})),$$

*where the values of $k_f(n)$ are given in Table 5.11.*

| $\alpha, \beta$ | $g$ | $k_f(n)$ |
|---|---|---|
| | $\alpha$*(even)* $\beta$*(even)* | |
| $\alpha = 2, \beta = 0$ | $g \equiv 1 \pmod{24}, g \neq 1$ | 3 |
| | $g \equiv 5, 7, 11, 17, 19, 23 \pmod{24}$ | 0 |
| | $g \equiv 13 \pmod{24}$ | 1 |
| $\alpha = 2, \beta \geq 2$ | $g = 1$ | 2 |
| | $g \equiv 1, 17 \pmod{24}, g \neq 1$ | 6 |
| | $g \equiv 5, 13 \pmod{24}$ | 2 |
| | $g \equiv 7, 11, 19, 23 \pmod{24}$ | 0 |
| $\alpha \geq 4, \beta = 0$ | $g = 1$ | $2^{\alpha/2} - 2$ |
| | $g \equiv 1 \pmod{24}, g \neq 1$ | $3 \cdot 2^{\alpha/2} - 6$ |
| | $g \equiv 5, 11, 17, 23 \pmod{24}$ | 0 |
| | $g \equiv 7, 19 \pmod{24}$ | $2^{\alpha/2} - 3$ |
| | $g \equiv 13 \pmod{24}$ | $2^{\alpha/2}$ |
| $\alpha \geq 4, \beta \geq 2$ | $g = 1$ | $2^{\alpha/2+1} - 4$ |
| | $g \equiv 1, 17 \pmod{24}, g \neq 1$ | $3 \cdot 2^{\alpha/2+1} - 12$ |
| | $g \equiv 5, 13 \pmod{24}$ | $2^{\alpha/2+1}$ |
| | $g \equiv 7, 11, 19, 23 \pmod{24}$ | $2^{\alpha/2+1} - 6$ |
| | $\alpha$*(even)* $\beta$*(odd)* | |
| $\alpha = 2, \beta \geq 1$ | $g \equiv 1, 5, 11 \pmod{12}$ | 0 |
| | $g \equiv 7 \pmod{24}$ | 4 |
| | $g \equiv 19 \pmod{24}$ | 12 |
| $\alpha \geq 4, \beta \geq 1$ | $g = 1$ | $2^{\alpha/2+1} - 6$ |
| | $g \equiv 1 \pmod{12}, g \neq 1$ | $2^{\alpha/2+2} - 12$ |
| | $g \equiv 5 \pmod{6}$ | 0 |
| | $g \equiv 7 \pmod{24}$ | $2^{\alpha/2+2}$ |
| | $g \equiv 19 \pmod{24}$ | $3 \cdot 2^{\alpha/2+2} - 24$ |
| | $\alpha$*(odd)* $\beta$*(even)* | |
| $\alpha = 1, 3, \beta \geq 0$ | | 0 |
| $\alpha \geq 5, \beta = 0$ | $g \equiv 1 \pmod{6}$ | 0 |
| | $g \equiv 5 \pmod{6}$ | $2^{(\alpha-1)/2} - 3$ |
| $\alpha \geq 5, \beta \geq 2$ | | $2^{(\alpha+1)/2} - 6$ |
| | | *Continued on next page* |

| $\alpha, \beta$ | $g$ | $k_f(n)$ |
|---|---|---|
| | $\alpha$(odd) $\beta$(odd) | |
| $\alpha = 1, 3, \beta \geq 1$ | | 0 |
| $\alpha \geq 5, \beta \geq 1$ | $g \equiv 1 \pmod 6$<br>$g \equiv 5 \pmod 6$ | 0<br>$2^{(\alpha+3)/2} - 12$ |

Table 5.11: Values of $k_f(n)$

*If* $(\alpha, \beta, g) = (2, 0, 1)$*, we have*

$$r(f; n) = l(n) - \left(\frac{-3}{h}\right) h.$$

*Proof.* If $\alpha = 1$ or 3, it is clear that $r(f; n) = 0$. If $\alpha = 2$, it is easy to show that

$$r(f; n) = r\left(4, 9, 12, 0, 0, 0; \frac{n}{4}\right)$$

and the formulas in these cases follow from Theorem 5.7. If $\alpha \geq 4$, again it is straightforward to show that

$$r(f; n) = r\left(1, 3, 9, 0, 0, 0; \frac{n}{16}\right)$$

and the formulas in these cases follow from Proposition A.10 in the Appendix. □

We determine the even positive integers $n$ which are not represented by $f$. If $n \equiv 2 \pmod 4$, it is clear that $n = 9x^2 + 16y^2 + 48z^2$ is insolvable in integers $x, y, z$ as $9x^2 + 16y^2 + 48z^2 \equiv x^2 \equiv 0, 1 \pmod 4$. Thus, we may suppose that $n \equiv 0 \pmod 4$. Then $n = 9x^2 + 16y^2 + 48z^2$ is solvable in integers $x, y, z$ if and only if $\frac{n}{4} = 4y^2 + 9u^2 + 12z^2$ is solvable in integers $y, u, z$, that is, if and only if $n/4$ is represented by the form B7. This occurs if and only if $n/4$ does not belong to any of the progressions $3l + 2, 4l + 2, 4l + 3, 9^k(9l + 6), M_3^2$. Hence $n \equiv 0 \pmod 2$ is represented by $9x^2 + 16y^2 + 48z^2$ if and only if $n$ does not belong to any of the progressions $4l + 2, 12l + 8, 16l + 8, 16l + 12, 4 \cdot 9^k(9l + 6), 4M_3^2$. Next, we determine the representation number $r(f; n)$ when $f$ is the form B12.

**Theorem 5.12.** *Let $f$ denote the form B12, that is,* $f = 9x^2 + 16y^2 + 112z^2 + 16yz$*. If* $(\alpha, \beta, g) \neq (2k, 0, 1)$ *for all* $k \in \mathbb{N}_0$*, we have*

$$r(f; n) = k_f(n) l(n) h(\mathbb{Q}(\sqrt{-n^*})),$$

*where the values of $k_f(n)$ are given in Table 5.12.*

| $\alpha, \beta$ | $g$ | $k_f(n)$ |
|---|---|---|
| | *$\alpha$(even) $\beta$(even)* | |
| $\alpha = 0, \beta = 0$ | $g \equiv 1 \pmod{24}, g \neq 1$ | 1 |
| | $g \equiv 5, 7, 11, 13, 17, 19, 23 \pmod{24}$ | 0 |
| $\alpha = 0, \beta \geq 2$ | $g = 1$ | 2 |
| | $g \equiv 1, 17 \pmod{24}, g \neq 1$ | 6 |
| | $g \equiv 5, 7, 11, 13, 19, 23 \pmod{24}$ | 0 |
| $\alpha \geq 2, \beta = 0$ | $g \equiv 1 \pmod{24}, g \neq 1$ | $3 \cdot 2^{\alpha/2-1} - 2$ |
| | $g \equiv 5, 11, 17, 23 \pmod{24}$ | 0 |
| | $g \equiv 7, 19 \pmod{24}$ | $2^{\alpha/2-1} - 1$ |
| | $g \equiv 13 \pmod{24}$ | $2^{\alpha/2-1}$ |
| $\alpha \geq 2, \beta \geq 2$ | $g = 1$ | $3 \cdot 2^{\alpha/2} - 4$ |
| | $g \equiv 1, 17 \pmod{24}, g \neq 1$ | $9 \cdot 2^{\alpha/2} - 12$ |
| | $g \equiv 5, 13 \pmod{24}$ | $3 \cdot 2^{\alpha/2}$ |
| | $g \equiv 7, 11, 19, 23 \pmod{24}$ | $3 \cdot 2^{\alpha/2} - 6$ |
| | *$\alpha$(even) $\beta$(odd)* | |
| $\alpha = 0, \beta = 1$ | | 0 |
| $\alpha = 0, \beta \geq 3$ | $g \equiv 1, 5, 7, 11, 13, 17, 23 \pmod{24}$ | 0 |
| | $g \equiv 19 \pmod{24}$ | 12 |
| $\alpha \geq 2, \beta = 1$ | | 0 |
| $\alpha \geq 2, \beta \geq 3$ | $g = 1$ | $3 \cdot 2^{\alpha/2} - 6$ |
| | $g \equiv 1, 13 \pmod{24}, g \neq 1$ | $3 \cdot 2^{\alpha/2+1} - 12$ |
| | $g \equiv 5, 11, 17, 23 \pmod{24}$ | 0 |
| | $g \equiv 7 \pmod{24}$ | $3 \cdot 2^{\alpha/2+1}$ |
| | $g \equiv 19 \pmod{24}$ | $9 \cdot 2^{\alpha/2+1} - 24$ |
| | *$\alpha$(odd) $\beta$(even)* | |
| $\alpha = 1, \beta \geq 0$ | | 0 |
| $\alpha \geq 3, \beta = 0$ | $g \equiv 1 \pmod{6}$ | 0 |
| | $g \equiv 5 \pmod{6}$ | $2^{(\alpha-3)/2} - 1$ |
| $\alpha \geq 3, \beta \geq 2$ | | $3 \cdot 2^{(\alpha-1)/2} - 6$ |
| | *$\alpha$(odd) $\beta$(odd)* | |
| $\alpha = 1, \beta \geq 1$ | | 0 |
| $\alpha \geq 3, \beta = 1$ | | 0 |
| $\alpha \geq 3, \beta \geq 3$ | $g \equiv 1 \pmod{6}$ | 0 |
| | $g \equiv 5 \pmod{6}$ | $3 \cdot 2^{(\alpha+1)/2} - 12$ |

Table 5.12: Values of $k_f(n)$

*If* $(\alpha, \beta, g) = (0, 0, 1)$, *we have*

$$r(f; n) = \frac{1}{3} l(n) - \frac{1}{3} \left( \frac{-3}{h} \right) h.$$

*If* $(\alpha,\beta,g)=(2k,0,1)$, *where* $k\in\mathbb{N}$, *we have*

$$r(f;n)=\left(2^{\alpha/2-1}-\frac{2}{3}\right)l(n)+\frac{1}{3}2^{\alpha/2-1}\left(\frac{-3}{2^{\alpha/2}h}\right)h.$$

*Proof.* By Theorem 2.1 (xxiii) and Proposition 2.1, we obtain for all $n\in\mathbb{N}$

$$r(f;n)=s(n)r(1,16,16,16,0,0;n)+r\left(1,16,16,16,0,0;\frac{n}{9}\right)+t(n),$$

where

$$s(n)=\frac{1}{36}\left(-1-i\sqrt{3}\right)\omega_3^n+\frac{1}{36}\left(-1+i\sqrt{3}\right)\omega_3^{2n}+\frac{1}{18}=\frac{1}{18}\left(\omega_3^{n+2}+\omega_3^{2(n+2)}+1\right)$$

$$=\left.\begin{cases}\frac{1}{6} & \text{if } n\equiv 1 \pmod 3\\ 0 & \text{if } n\equiv 0,2 \pmod 3\end{cases}\right\}=\begin{cases}\frac{1}{6} & \text{if } \alpha \text{ (even)}\geq 0, \beta=0, g\equiv 1 \pmod 3\\ & \text{or } \alpha \text{ (odd)}\geq 1, \beta=0, g\equiv 2 \pmod 3,\\ 0 & \text{otherwise,}\end{cases}$$

and

$$t(n)=\begin{cases}-\frac{1}{3}\left(\frac{-3}{\sqrt{n}}\right)\sqrt{n} & \text{if } n=\text{odd square},\\ \frac{1}{6}\left(\frac{-3}{\sqrt{n}}\right)\sqrt{n} & \text{if } n=\text{even square},\\ 0 & \text{otherwise,}\end{cases}$$

$$=\begin{cases}-\frac{1}{3}\left(\frac{-3}{h}\right)h & \text{if } (\alpha,\beta,g)=(0,0,1),\\ \frac{1}{6}\left(\frac{-3}{2^kh}\right)2^kh & \text{if } (\alpha,\beta,g)=(2k,0,1)\ (k\in\mathbb{N}),\\ 0 & \text{otherwise.}\end{cases}$$

Appealing to Proposition A.12 for the formula for $r(1,16,16,16,0,0;n)$, we obtain Theorem 5.12. □

The positive integers which are not represented by the spinor regular form B12 can be deduced from Theorem 5.12 using Lemmas 3.1 and 3.2 in a manner similar to that for the form B9. They are given in Table A.17.

## 6. Spinor Regular Ternaries with Discriminant $2^r\cdot 7^s$

The four spinor regular positive-definite ternary quadratic forms $f=f(x,y,z)$ which are not regular and have discriminant $\Delta=2^r\cdot 7^s$ for some $r,s\in\mathbb{N}_0$ (with $r$ even and $s=3$) are those with identification numbers C1–C4 in Table 1.1. We determine their representation numbers in this section.

When considering the representation of $n\in\mathbb{N}$ by $f(x,y,z)$ we use the integers $\alpha=\nu_2(n)$, $\gamma=\nu_7(n)$, as well as $g$, $h$ and $n^*$, which are defined uniquely in terms of $n$ by (1.6), (1.7) and (1.5), respectively. We have

$$n=2^\alpha 7^\gamma gh^2, \tag{6.1}$$

where

$$\alpha, \gamma \in \mathbb{N}_0, \quad g, h \in \mathbb{N}, \quad g \text{ squarefree}, \quad (gh, 14) = 1, \tag{6.2}$$

and

$$n^* = 2^{\alpha - 2[\alpha/2]} 7^{\gamma+1-2[(\gamma+1)/2]} g. \tag{6.3}$$

We now state and prove formulas for the representation numbers of the forms C1–C4. All of these formulas involve the quantities defined in (6.1)–(6.3) as well as $l(n)$, which is defined in (1.8). We begin with C1.

**Theorem 6.1.** *Let $f$ denote the form C1, that is, $f = 2x^2 + 7y^2 + 8z^2 + 7yz + zx$. If $(\alpha, \gamma, g) \neq (2k, 0, 1)$ for all $k \in \mathbb{N}_0$, we have*

$$r(f; n) = k_f(n) l(n) h(\mathbb{Q}(\sqrt{-n^*})),$$

*where the values of $k_f(n)$ are given in Table 6.1.*

| $\alpha, \gamma$ | $g$ | $k_f(n)$ |
|---|---|---|
| | $\alpha$*(even)* $\gamma$*(even)* | |
| $\alpha \geq 0, \gamma = 0$ | $g \equiv 1 \pmod 8, g \equiv 1, 2, 4 \pmod 7, g \neq 1$ | $2^{\alpha/2}$ |
| | $g \equiv 3, 5, 6 \pmod 7$ | $0$ |
| | $g \equiv 3 \pmod 4, g \equiv 1, 2, 4 \pmod 7$ | $2^{\alpha/2} - \frac{1}{2}$ |
| | $g \equiv 5 \pmod 8, g \equiv 1, 2, 4 \pmod 7$ | $3 \cdot 2^{\alpha/2} - 1$ |
| $\alpha \geq 0, \gamma \geq 2$ | $g \equiv 1 \pmod 8$ | $2^{\alpha/2+2}$ |
| | $g \equiv 3 \pmod 4$ | $2^{\alpha/2+2} - 2$ |
| | $g \equiv 5 \pmod 8$ | $3 \cdot 2^{\alpha/2+2} - 4$ |
| | $\alpha$*(even)* $\gamma$*(odd)* | |
| $\alpha \geq 0, \gamma \geq 1$ | $g = 1$ | $2^{\alpha/2+2} - 2$ |
| | $g \equiv 1 \pmod 4, g \equiv 1, 2, 4 \pmod 7, g \neq 1$ | $2^{\alpha/2+3} - 4$ |
| | $g \equiv 3, 5, 6 \pmod 7$ | $0$ |
| | $g \equiv 3 \pmod 8, g \equiv 1, 2, 4 \pmod 7$ | $3 \cdot 2^{\alpha/2+3} - 8$ |
| | $g \equiv 7 \pmod 8, g \equiv 1, 2, 4 \pmod 7$ | $2^{\alpha/2+3}$ |
| | $\alpha$*(odd)* $\gamma$*(even)* | |
| $\alpha \geq 1, \gamma = 0$ | $g \equiv 1, 2, 4 \pmod 7$ | $2^{(\alpha-1)/2} - \frac{1}{2}$ |
| | $g \equiv 3, 5, 6 \pmod 7$ | $0$ |
| $\alpha \geq 1, \gamma \geq 2$ | | $2^{(\alpha+3)/2} - 2$ |
| | $\alpha$*(odd)* $\gamma$*(odd)* | |
| $\alpha \geq 1, \gamma \geq 1$ | $g \equiv 1, 2, 4 \pmod 7$ | $2^{(\alpha+5)/2} - 4$ |
| | $g \equiv 3, 5, 6 \pmod 7$ | $0$ |

Table 6.1: Values of $k_f(n)$

*If* $(\alpha, \gamma, g) = (2k, 0, 1)$*, where* $k \in \mathbb{N}_0$*, we have*

$$r(f;n) = 2^{\alpha/2} l(n) - 2^{\alpha/2} \left(\frac{-7}{h}\right) h.$$

*Proof.* By Theorem 2.1 (xxiv) and Proposition 2.1 we have

$$r(f;n) = s(n) r(1,1,2,0,1,0;n) + t(n),$$

where

$$\begin{aligned} s(n) &= \frac{1}{56}\left(7 - i\sqrt{7}\right)\left(\omega_7^n + \omega_7^{2n} + \omega_7^{4n}\right) + \frac{1}{56}\left(7 + i\sqrt{7}\right)\left(\omega_7^{3n} + \omega_7^{5n} + \omega_7^{6n}\right) + \frac{1}{4} \\ &= \left.\begin{cases} \frac{1}{4} & \text{if } n \equiv 1,2,4 \pmod 7 \\ 0 & \text{if } n \equiv 3,5,6 \pmod 7 \\ 1 & \text{if } n \equiv 0 \pmod 7 \end{cases}\right\} = \begin{cases} \frac{1}{4} & \text{if } \gamma = 0, g \equiv 1,2,4 \pmod 7, \\ 0 & \text{if } \gamma = 0, g \equiv 3,5,6 \pmod 7, \\ 1 & \text{if } \gamma \geq 1, \end{cases} \end{aligned}$$

and

$$\begin{aligned} t(n) &= \begin{cases} -\left(\frac{-7}{\sqrt{n}}\right)\sqrt{n} & \text{if } n = \text{square}, \\ 0 & \text{otherwise}, \end{cases} \\ &= \begin{cases} -\left(\frac{-7}{h}\right) 2^k h & \text{if } (\alpha,\gamma,g) = (2k,0,1) \ (k \in \mathbb{N}_0), \\ 0 & \text{otherwise}. \end{cases} \end{aligned}$$

Appealing to Proposition A.13 for the formula for $r(1,1,2,0,1,0;n)$, we obtain Theorem 6.1. □

From Theorem 6.1, and making use of Lemma 3.1(iv), we obtain the positive integers not represented by the spinor regular form C1. These integers are given in Table A.17. We give the details of the proof.

We begin by recalling that $(gh, 14) = 1$ so that $h^2 \equiv 1,2,4 \pmod 7$. Also $2^k \equiv 1,2,4 \pmod 7, k \in \mathbb{N}_0$. If $(\alpha,\gamma,g) \neq (2k,0,1)$ for any $k \in \mathbb{N}_0$, by Theorem 6.1 we have

$$\begin{aligned} r(f;n) = 0 &\iff k_f(n) = 0 \\ &\iff \alpha \text{ (even)} \geq 0, \gamma = 0, g \equiv 3,5,6 \pmod 7 \text{ or} \\ &\qquad \alpha \text{ (even)} \geq 0, \gamma \text{ (odd)} \geq 1, g \equiv 3,5,6 \pmod 7 \text{ or} \\ &\qquad \alpha \text{ (odd)} \geq 1, \gamma = 0, g \equiv 3,5,6 \pmod 7 \text{ or} \\ &\qquad \alpha \text{ (odd)} \geq 1, \gamma \text{ (odd)} \geq 1, g \equiv 3,5,6 \pmod 7 \\ &\iff \gamma = 0, g \equiv 3,5,6 \pmod 7 \text{ or} \\ &\qquad \gamma \text{ (odd)} \geq 1, g \equiv 3,5,6 \pmod 7 \\ &\iff n = 7l+3, 7l+5, 7l+6 \text{ or} \\ &\qquad n = 49^k(49l+21), 49^k(49l+35), 49^k(49l+42). \end{aligned}$$

If $(\alpha, \gamma, g) = (2k, 0, 1)$ for some $k \in \mathbb{N}_0$, then by Theorem 6.1 and Lemma 3.1 (iv) we obtain

$$\begin{aligned} r(f;n) = 0 \text{ if and only if } & \ 2^{\alpha/2} l(n) - 2^{\alpha/2}\left(\frac{-7}{h}\right) h = 0 \\ \text{if and only if } & \ l(n) = \left(\frac{-7}{h}\right) h \\ \text{if and only if } & \ n \in 2^{\alpha} M_7^2 \\ \text{if and only if } & \ n \in 2^{2k} M_7^2 \\ \text{if and only if } & \ n \in M_7^2. \end{aligned}$$

This completes the determination of the integers $n$ not represented by $f$. Next, we determine the representation number $r(f;n)$ when $f$ is the form C2.

**Theorem 6.2.** *Let $f$ denote the form C2, that is, $f = 7x^2 + 8y^2 + 9z^2 + 6yz + 7zx$. If $(\alpha, \gamma, g) \neq (2k, 0, 1)$ for all $k \in \mathbb{N}_0$, we have*

$$r(f;n) = k_f(n) l(n) h(\mathbb{Q}(\sqrt{-n^*})),$$

*where the values of $k_f(n)$ are given in Table* 6.2.

| $\alpha, \gamma$ | $g$ | $k_f(n)$ |
|---|---|---|
| | *$\alpha$(even) $\gamma$(even)* | |
| $\alpha = 0, \gamma = 0$ | $g \equiv 1, 3, 7 \pmod 8, g \equiv 1, 2, 4 \pmod 7, g \neq 1$ | $\frac{1}{2}$ |
| | $g \equiv 3, 5, 6 \pmod 7$ | $0$ |
| | $g \equiv 5 \pmod 8, g \equiv 1, 2, 4 \pmod 7$ | $\frac{3}{2}$ |
| $\alpha = 0, \gamma \geq 2$ | $g \equiv 1, 3, 7 \pmod 8$ | $2$ |
| | $g \equiv 5 \pmod 8$ | $6$ |
| $\alpha \geq 2, \gamma = 0$ | $g \equiv 1 \pmod 8, g \equiv 1, 2, 4 \pmod 7, g \neq 1$ | $2^{\alpha/2-1}$ |
| | $g \equiv 3, 5, 6 \pmod 7$ | $0$ |
| | $g \equiv 3 \pmod 4, g \equiv 1, 2, 4 \pmod 7$ | $2^{\alpha/2-1} - \frac{1}{2}$ |
| | $g \equiv 5 \pmod 8, g \equiv 1, 2, 4 \pmod 7$ | $3 \cdot 2^{\alpha/2-1} - 1$ |
| $\alpha \geq 2, \gamma \geq 2$ | $g \equiv 1 \pmod 8$ | $2^{\alpha/2+1}$ |
| | $g \equiv 3 \pmod 4$ | $2^{\alpha/2+1} - 2$ |
| | $g \equiv 5 \pmod 8$ | $3 \cdot 2^{\alpha/2+1} - 4$ |
| | *$\alpha$(even) $\gamma$(odd)* | |
| $\alpha = 0, \gamma \geq 1$ | $g = 1$ | $2$ |
| | $g \equiv 1, 5, 7 \pmod 8, g \equiv 1, 2, 4 \pmod 7, g \neq 1$ | $4$ |
| | $g \equiv 3, 5, 6 \pmod 7$ | $0$ |
| | $g \equiv 3 \pmod 8, g \equiv 1, 2, 4 \pmod 7$ | $12$ |
| | | *Continued on next page* |

| $\alpha,\gamma$ | $g$ | $k_f(n)$ |
|---|---|---|
| $\alpha\geq 2,\gamma\geq 1$ | $g=1$ | $2^{\alpha/2+1}-2$ |
| | $g\equiv 1 \pmod 4, g\equiv 1,2,4 \pmod 7, g\neq 1$ | $2^{\alpha/2+2}-4$ |
| | $g\equiv 3,5,6 \pmod 7$ | $0$ |
| | $g\equiv 3 \pmod 8, g\equiv 1,2,4 \pmod 7$ | $3\cdot 2^{\alpha/2+2}-8$ |
| | $g\equiv 7 \pmod 8, g\equiv 1,2,4 \pmod 7$ | $2^{\alpha/2+2}$ |
| | *$\alpha$(odd) $\gamma$(even)* | |
| $\alpha=1,\gamma\geq 0$ | | $0$ |
| $\alpha\geq 3,\gamma=0$ | $g\equiv 1,2,4 \pmod 7$ | $2^{(\alpha-3)/2}-\frac{1}{2}$ |
| | $g\equiv 3,5,6 \pmod 7$ | $0$ |
| $\alpha\geq 3,\gamma\geq 2$ | | $2^{(\alpha+1)/2}-2$ |
| | *$\alpha$(odd) $\gamma$(odd)* | |
| $\alpha=1,\gamma\geq 1$ | | $0$ |
| $\alpha\geq 3,\gamma\geq 1$ | $g\equiv 1,2,4 \pmod 7$ | $2^{(\alpha+3)/2}-4$ |
| | $g\equiv 3,5,6 \pmod 7$ | $0$ |

Table 6.2: Values of $k_f(n)$

*If* $(\alpha,\gamma,g)=(2k,0,1)$*, where* $k\in\mathbb{N}_0$*, we have*

$$r(f;n)=2^{\alpha/2-1}l(n)-2^{\alpha/2-1}\left(\frac{-7}{h}\right)h.$$

*Proof.* By Theorem 2.1 (xxv) and Proposition 2.1 we have

$$r(f;n)=s(n)r(1,3,3,2,1,1;n)+t(n),$$

where

$$\begin{aligned} s(n)&=\frac{1}{56}\left(7-i\sqrt{7}\right)\left(\omega_7^n+\omega_7^{2n}+\omega_7^{4n}\right)+\frac{1}{56}\left(7+i\sqrt{7}\right)\left(\omega_7^{3n}+\omega_7^{5n}+\omega_7^{6n}\right)+\frac{1}{4}\\ &=\left\{\begin{array}{ll}\frac{1}{4} & \text{if } n\equiv 1,2,4 \pmod 7\\ 0 & \text{if } n\equiv 3,5,6 \pmod 7\\ 1 & \text{if } n\equiv 0 \pmod 7\end{array}\right\}=\begin{cases}\frac{1}{4} & \text{if } \gamma=0, g\equiv 1,2,4 \pmod 7,\\ 0 & \text{if } \gamma=0, g\equiv 3,5,6 \pmod 7,\\ 1 & \text{if } \gamma\geq 1,\end{cases}\end{aligned}$$

and

$$\begin{aligned} t(n)&=\begin{cases}-\frac{1}{2}\left(\frac{-7}{\sqrt{n}}\right)\sqrt{n} & \text{if } n=\text{square},\\ 0 & \text{otherwise},\end{cases}\\ &=\begin{cases}-\frac{1}{2}\left(\frac{-7}{h}\right)2^k h & \text{if } (\alpha,\gamma,g)=(2k,0,1)\ (k\in\mathbb{N}_0),\\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

Appealing to Proposition A.14 for the formula for $r(1,3,3,2,1,1;n)$, we obtain Theorem 6.2. □

From Theorem 6.2, and making use of Lemma 3.1(iv), we obtain the positive integers not represented by the spinor regular form C2. These are listed in Table A.17. Next, we determine the representation number $r(f;n)$ when $f$ is the form C3.

**Theorem 6.3.** *Let $f$ denote the form C3, that is, $f = 8x^2 + 9y^2 + 25z^2 + 2yz + 4zx + 8xy$. If $(\alpha,\gamma,g) \neq (2k,0,1)$ for all $k \in \mathbb{N}_0$, we have*

$$r(f;n) = k_f(n)l(n)h(\mathbb{Q}(\sqrt{-n^*})),$$

*where the values of $k_f(n)$ are given in Table* 6.3.

| $\alpha,\gamma$ | $g$ | $k_f(n)$ |
|---|---|---|
| | *$\alpha$(even) $\gamma$(even)* | |
| $\alpha=0,\gamma=0$ | $g\equiv 1 \pmod 4, g\equiv 1,2,4 \pmod 7, g\neq 1$ | $\frac{1}{2}$ |
| | $g\equiv 1 \pmod 4, g\equiv 3,5,6 \pmod 7$ *or* $g\equiv 3 \pmod 4$ | $0$ |
| $\alpha=0,\gamma\geq 2$ | $g\equiv 1 \pmod 4$ | $2$ |
| | $g\equiv 3 \pmod 4$ | $0$ |
| $\alpha\geq 2,\gamma=0$ | $g\equiv 1 \pmod 8, g\equiv 1,2,4 \pmod 7, g\neq 1$ | $2^{\alpha/2-1}$ |
| | $g\equiv 3,5,6 \pmod 7$ | $0$ |
| | $g\equiv 3 \pmod 4, g\equiv 1,2,4 \pmod 7$ | $2^{\alpha/2-1}-\frac{1}{2}$ |
| | $g\equiv 5 \pmod 8, g\equiv 1,2,4 \pmod 7$ | $3\cdot 2^{\alpha/2-1}-1$ |
| $\alpha\geq 2,\gamma\geq 2$ | $g\equiv 1 \pmod 8$ | $2^{\alpha/2+1}$ |
| | $g\equiv 3 \pmod 4$ | $2^{\alpha/2+1}-2$ |
| | $g\equiv 5 \pmod 8$ | $3\cdot 2^{\alpha/2+1}-4$ |
| | *$\alpha$(even) $\gamma$(odd)* | |
| $\alpha=0,\gamma\geq 1$ | $g\equiv 1 \pmod 4, g\equiv 1,2,4 \pmod 7$ *or* $g\equiv 3,5,6 \pmod 7$ | $0$ |
| | $g\equiv 3 \pmod 4, g\equiv 1,2,4 \pmod 7$ | $4$ |
| $\alpha\geq 2,\gamma\geq 1$ | $g=1$ | $2^{\alpha/2+1}-2$ |
| | $g\equiv 1 \pmod 4, g\equiv 1,2,4 \pmod 7, g\neq 1$ | $2^{\alpha/2+2}-4$ |
| | $g\equiv 3,5,6 \pmod 7$ | $0$ |
| | $g\equiv 3 \pmod 8, g\equiv 1,2,4 \pmod 7$ | $3\cdot 2^{\alpha/2+2}-8$ |
| | $g\equiv 7 \pmod 8, g\equiv 1,2,4 \pmod 7$ | $2^{\alpha/2+2}$ |
| | *$\alpha$(odd) $\gamma$(even)* | |
| $\alpha=1,\gamma\geq 0$ | | $0$ |
| $\alpha\geq 3,\gamma=0$ | $g\equiv 1,2,4 \pmod 7$ | $2^{(\alpha-3)/2}-\frac{1}{2}$ |
| | $g\equiv 3,5,6 \pmod 7$ | $0$ |
| $\alpha\geq 3,\gamma\geq 2$ | | $2^{(\alpha+1)/2}-2$ |
| | | *Continued on next page* |

| $\alpha, \gamma$ | $g$ | $k_f(n)$ |
|---|---|---|
| | $\alpha$(odd) $\gamma$(odd) | |
| $\alpha = 1, \gamma \geq 1$ | | 0 |
| $\alpha \geq 3, \gamma \geq 1$ | $g \equiv 1, 2, 4 \pmod 7$ | $2^{(\alpha+3)/2} - 4$ |
| | $g \equiv 3, 5, 6 \pmod 7$ | 0 |

Table 6.3: Values of $k_f(n)$

*If* $(\alpha, \gamma, g) = (2k, 0, 1)$, *where* $k \in \mathbb{N}_0$, *we have*

$$r(f; n) = 2^{\alpha/2-1} l(n) - 2^{\alpha/2-1} \left(\frac{-7}{h}\right) h.$$

*Proof.* By Theorem 2.1 (xxvi) and Proposition 2.1 we have for all $n \in \mathbb{N}$

$$r(f; n) = s(n) r(1, 4, 8, 4, 0, 0; n) + t(n),$$

where

$$\begin{aligned} s(n) &= \frac{1}{56}\left(7 - i\sqrt{7}\right)\left(\omega_7^n + \omega_7^{2n} + \omega_7^{4n}\right) + \frac{1}{56}\left(7 + i\sqrt{7}\right)\left(\omega_7^{3n} + \omega_7^{5n} + \omega_7^{6n}\right) + \frac{1}{4} \\ &= \left.\begin{cases} \frac{1}{4} & \text{if } n \equiv 1, 2, 4 \pmod 7 \\ 0 & \text{if } n \equiv 3, 5, 6 \pmod 7 \\ 1 & \text{if } n \equiv 0 \pmod 7 \end{cases}\right\} = \begin{cases} \frac{1}{4} & \text{if } \gamma = 0, g \equiv 1, 2, 4 \pmod 7, \\ 0 & \text{if } \gamma = 0, g \equiv 3, 5, 6 \pmod 7, \\ 1 & \text{if } \gamma \geq 1, \end{cases} \end{aligned}$$

and

$$\begin{aligned} t(n) &= \begin{cases} -\frac{1}{2}\left(\frac{-7}{\sqrt{n}}\right)\sqrt{n} & \text{if } n = \text{square}, \\ 0 & \text{otherwise}, \end{cases} \\ &= \begin{cases} -\frac{1}{2}\left(\frac{-7}{h}\right) 2^k h & \text{if } (\alpha, \gamma, g) = (2k, 0, 1) \ (k \in \mathbb{N}_0), \\ 0 & \text{otherwise}. \end{cases} \end{aligned}$$

Appealing to Proposition A.15 for the formula for $r(1, 4, 8, 4, 0, 0; n)$, we obtain Theorem 6.3. □

From Theorem 6.3 making use of Lemma 3.1(iv), we deduce which positive integers are not represented by the spinor regular ternary form C3. These are given in Table A.17. Finally, we determine the representation number $r(f; n)$ when $f$ is the form C4.

**Theorem 6.4.** *Let* $f$ *denote the form C4, that is,* $f = 29x^2 + 32y^2 + 36z^2 + 32yz + 12zx + 24xy$. *If* $(\alpha, \gamma, g) \neq (2k, 0, 1)$ *for all* $k \in \mathbb{N}$, *we have*

$$r(f; n) = k_f(n) l(n) h(\mathbb{Q}(\sqrt{-n^*})),$$

*where the values of* $k_f(n)$ *are given in Table* 6.4.

| $\alpha, \gamma$ | $g$ | $k_f(n)$ |
|---|---|---|
| | *$\alpha$(even) $\gamma$(even)* | |
| $\alpha = 0, \gamma = 0$ | $g \equiv 1, 3, 7 \pmod 8, g \equiv 1, 2, 4 \pmod 7$ *or* $g \equiv 3, 5, 6 \pmod 7$ | 0 |
| | $g \equiv 5 \pmod 8, g \equiv 1, 2, 4 \pmod 7$ | $\frac{1}{2}$ |
| $\alpha = 0, \gamma \geq 2$ | $g \equiv 1, 3, 7 \pmod 8$ | 0 |
| | $g \equiv 5 \pmod 8$ | 2 |
| $\alpha = 2, \gamma = 0$ | $g \equiv 1 \pmod 4, g \equiv 1, 2, 4 \pmod 7, g \neq 1$ | $\frac{1}{2}$ |
| | $g \equiv 1 \pmod 4, g \equiv 3, 5, 6 \pmod 7$ *or* $g \equiv 3 \pmod 4$ | 0 |
| $\alpha = 2, \gamma \geq 2$ | $g \equiv 1 \pmod 4$ | 2 |
| | $g \equiv 3 \pmod 4$ | 0 |
| $\alpha \geq 4, \gamma = 0$ | $g \equiv 1 \pmod 8, g \equiv 1, 2, 4 \pmod 7, g \neq 1$ | $2^{\alpha/2-2}$ |
| | $g \equiv 3, 5, 6 \pmod 7$ | 0 |
| | $g \equiv 3 \pmod 4, g \equiv 1, 2, 4 \pmod 7$ | $2^{\alpha/2-2} - \frac{1}{2}$ |
| | $g \equiv 5 \pmod 8, g \equiv 1, 2, 4 \pmod 7$ | $3 \cdot 2^{\alpha/2-2} - 1$ |
| $\alpha \geq 4, \gamma \geq 2$ | $g \equiv 1 \pmod 8$ | $2^{\alpha/2}$ |
| | $g \equiv 3 \pmod 4$ | $2^{\alpha/2} - 2$ |
| | $g \equiv 5 \pmod 8$ | $3 \cdot 2^{\alpha/2} - 4$ |
| | *$\alpha$(even) $\gamma$(odd)* | |
| $\alpha = 0, \gamma \geq 1$ | $g \equiv 1, 5, 7 \pmod 8, g \equiv 1, 2, 4 \pmod 7$ *or* $g \equiv 3, 5, 6 \pmod 7$ | 0 |
| | $g \equiv 3 \pmod 8, g \equiv 1, 2, 4 \pmod 7$ | 4 |
| $\alpha = 2, \gamma \geq 1$ | $g \equiv 1 \pmod 4, g \equiv 1, 2, 4 \pmod 7$ *or* $g \equiv 3, 5, 6 \pmod 7$ | 0 |
| | $g \equiv 3 \pmod 4, g \equiv 1, 2, 4 \pmod 7$ | 4 |
| $\alpha \geq 4, \gamma \geq 1$ | $g = 1$ | $2^{\alpha/2} - 2$ |
| | $g \equiv 1 \pmod 4, g \equiv 1, 2, 4 \pmod 7, g \neq 1$ | $2^{\alpha/2+1} - 4$ |
| | $g \equiv 3, 5, 6 \pmod 7$ | 0 |
| | $g \equiv 3 \pmod 8, g \equiv 1, 2, 4 \pmod 7$ | $3 \cdot 2^{\alpha/2+1} - 8$ |
| | $g \equiv 7 \pmod 8, g \equiv 1, 2, 4 \pmod 7$ | $2^{\alpha/2+1}$ |
| | *$\alpha$(odd) $\gamma$(even)* | |
| $\alpha = 1, 3, \gamma \geq 0$ | | 0 |
| $\alpha \geq 5, \gamma = 0$ | $g \equiv 1, 2, 4 \pmod 7$ | $2^{(\alpha-5)/2} - \frac{1}{2}$ |
| | $g \equiv 3, 5, 6 \pmod 7$ | 0 |
| $\alpha \geq 5, \gamma \geq 2$ | | $2^{(\alpha-1)/2} - 2$ |
| | | *Continued on next page* |

| $\alpha, \gamma$ | $g$ | $k_f(n)$ |
|---|---|---|
| | $\alpha$ (odd) $\gamma$ (odd) | |
| $\alpha = 1, 3, \gamma \geq 1$ | | 0 |
| $\alpha \geq 5, \gamma \geq 1$ | $g \equiv 1, 2, 4 \pmod 7$ | $2^{(\alpha+1)/2} - 4$ |
| | $g \equiv 3, 5, 6 \pmod 7$ | 0 |

Table 6.4: Values of $k_f(n)$

*If* $(\alpha, \gamma, g) = (2k, 0, 1)$*, where* $k \in \mathbb{N}$*, we have*

$$r(f; n) = 2^{\alpha/2-2} l(n) - 2^{\alpha/2-2} \left( \frac{-7}{h} \right) h.$$

*Proof.* By Theorem 2.1 (xxvii) and Proposition 2.1 we have for all $n \in \mathbb{N}$

$$r(f; n) = s(n) r(4, 5, 29, 2, 4, 4; n) + t(n),$$

where

$$s(n) = \frac{1}{56} \left( 7 - i\sqrt{7} \right) \left( \omega_7^n + \omega_7^{2n} + \omega_7^{4n} \right) + \frac{1}{56} \left( 7 + i\sqrt{7} \right) \left( \omega_7^{3n} + \omega_7^{5n} + \omega_7^{6n} \right) + \frac{1}{4}$$
$$= \left. \begin{cases} \frac{1}{4} & \text{if } n \equiv 1, 2, 4 \pmod 7 \\ 0 & \text{if } n \equiv 3, 5, 6 \pmod 7 \\ 1 & \text{if } n \equiv 0 \pmod 7 \end{cases} \right\} = \begin{cases} \frac{1}{4} & \text{if } \gamma = 0, g \equiv 1, 2, 4 \pmod 7, \\ 0 & \text{if } \gamma = 0, g \equiv 3, 5, 6 \pmod 7, \\ 1 & \text{if } \gamma \geq 1, \end{cases}$$

and, as $\left( \frac{-7}{2} \right) = 1$ and $\left( \frac{-7}{7} \right) = 0$, we have

$$t(n) = \begin{cases} -\frac{1}{2} \left( \frac{-7}{\sqrt{n/4}} \right) \sqrt{n/4} & \text{if } n = 4 \times \text{square}, \\ 0 & \text{otherwise}, \end{cases}$$
$$= \begin{cases} -\left( \frac{-7}{h} \right) 2^{\alpha/2-2} h & \text{if } \alpha \text{ (even)} \geq 2, \gamma = 0, g = 1, \\ 0 & \text{otherwise.} \end{cases}$$

If $\gamma \geq 1$ then $s(n) = 1$ and $t(n) = 0$, so $r(f; n) = r(4, 5, 29, 2, 4, 4; n)$ and the result follows from Proposition A.16. If $\alpha \geq 0, \gamma = 0, g \equiv 3, 5, 6 \pmod 7$, we have $s(n) = 0$ and $t(n) = 0$, so $r(f; n) = 0$. If $\alpha \geq 0, \gamma = 0, g \equiv 1, 2, 4 \pmod 7$, we have $s(n) = \frac{1}{4}$ and

$$t(n) = \begin{cases} -\left( \frac{-7}{h} \right) 2^{\alpha/2-2} h & \text{if } \alpha \text{ (even)} \geq 2, \gamma = 0, g = 1, \\ 0 & \text{otherwise,} \end{cases}$$

so

$$r(f; n) = \frac{1}{4} r(4, 5, 29, 2, 4, 4; n)$$

$$+\begin{cases} -\left(\frac{-7}{h}\right)2^{\alpha/2-2}h & \text{if } \alpha \text{ (even)} \geq 2, \gamma = 0, g = 1, \\ 0 & \text{if } \alpha = 0, \gamma = 0, g \equiv 1,2,4 \pmod 7 \text{ or} \\ & \quad \alpha \text{ (odd)} \geq 1, \gamma = 0, g \equiv 1,2,4 \pmod 7 \text{ or} \\ & \quad \alpha \text{ (even)} \geq 2, \gamma = 0, g \equiv 1,2,4 \pmod 7, g \neq 1, \end{cases}$$

and the result follows from Proposition A.16. □

From Theorem 6.4 making use of Lemma 3.1(iv), we deduce which positive integers are not represented by the spinor regular ternary form C4. These are listed in Table A.17.

## 7. Concluding Remarks

It would be interesting to extend Theorems 5.4 and 5.11 to all positive integers $n$. Almost certainly coefficients of more complicated cusp forms are required. For related work on representation numbers of ternary quadratic forms the reader should consult [6], [10], [14], [19] and [23].

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## A. Appendix

In this appendix we give formulas for the number of representations of $n$ by the ternary quadratic forms listed in Table A.0. These formulas are given in Propositions A.1–A.16. The fourth column of Table A.0 indicates the theorems in which these representation numbers are used in their proofs. In Propositions A.1–A.6 the quantities $\alpha, g, h$ and $n^*$ are defined in (4.1)–(4.3). In Propositions A.7–A.12 the quantities $\alpha, \beta, g, h$ and $n^*$ are defined in (5.1)–(5.3). In Propositions A.13–A.16 the quantities $\alpha, \gamma, g, h$ and $n^*$ are defined in (6.1)–(6.3). For all Propositions A.1–A.16 the integer $l(n)$ is defined in (1.8).

| Proposition | Regular Form | Disc | Theorem(s) |
|---|---|---|---|
| A.1 | $x^2 + y^2 + 4z^2$ | $2^4$ | 4.1,3,4,7,9,11–13 |
| A.2 | $x^2 + y^2 + 8z^2$ | $2^5$ | 4.2 |
| A.3 | $x^2 + 4y^2 + 4z^2$ | $2^6$ | 4.5 |
| A.4 | $x^2 + 4y^2 + 5z^2 + 4yz$ | $2^6$ | 4.6 |
| A.5 | $x^2 + 2y^2 + 16z^2$ | $2^7$ | 4.8 |
| A.6 | $x^2 + 8y^2 + 8z^2$ | $2^8$ | 4.10 |
| A.7 | $x^2 + y^2 + 4z^2 + xy$ | $2^2 \cdot 3$ | 5.1 |
| A.8 | $x^2 + 2y^2 + 2z^2 + yz + zx + xy$ | $2^2 \cdot 3$ | 5.2 |
| A.9 | $x^2 + y^2 + 12z^2 + xy$ | $2^2 \cdot 3^2$ | 5.3 |
| A.10 | $x^2 + 3y^2 + 9z^2$ | $2^2 \cdot 3^3$ | 5.5–7,9–11 |
| A.11 | $x^2 + 4y^2 + 4z^2 + 4yz$ | $2^4 \cdot 3$ | 5.8 |
| A.12 | $x^2 + 16y^2 + 16z^2 + 16yz$ | $2^8 \cdot 3$ | 5.12 |
| A.13 | $x^2 + y^2 + 2z^2 + zx$ | $7$ | 6.1 |
| A.14 | $x^2 + 3y^2 + 3z^2 + 2yz + zx + xy$ | $2^2 \cdot 7$ | 6.2 |
| A.15 | $x^2 + 4y^2 + 8z^2 + 4yz$ | $2^4 \cdot 7$ | 6.3 |
| A.16 | $4x^2 + 5y^2 + 29z^2 + 2yz + 4zx + 4xy$ | $2^8 \cdot 7$ | 6.4 |

Table A.0: Regular positive-definite ternary quadratic forms used in proofs of Theorems 4.1–6.4

**Proposition A.1.** (Lomadze [18, Theorem 4, p. 149]) *(A1,A3,A4,A7,A9,A11–A13) Let $f = x^2 + y^2 + 4z^2$. We have*

$$r(f;n) = k_f(n)l(n)h(\mathbb{Q}(\sqrt{-n^*})),$$

*where the values of $k_f(n)$ are given in Table* A.1.

| $\alpha$ | $g$ | $k_f(n)$ |
|---|---|---|
| 0 | $g = 1$ | 4 |
| | $g \equiv 1 \pmod 4, g \neq 1$ | 8 |
| | $g \equiv 3 \pmod 4$ | 0 |
| | | *Continued on next page* |

| $\alpha$ | $g$ | $k_f(n)$ |
|---|---|---|
| 1 | | 4 |
| *even* $\geq 2$ | $g = 1$ | 6 |
| | $g \equiv 1 \pmod 4, g \neq 1$ | 12 |
| | $g = 3$ | 8 |
| | $g \equiv 3 \pmod 8, g \neq 3$ | 24 |
| | $g \equiv 7 \pmod 8$ | 0 |
| *odd* $\geq 3$ | | 12 |

Table A.1: Values of $k_f(n)$

**Proposition A.2.** (Lomadze [18, Theorem 7, p. 150]) *(A2) Let* $f = x^2 + y^2 + 8z^2$*. We have*

$$r(f; n) = k_f(n)l(n)h(\mathbb{Q}(\sqrt{-n^*})),$$

*where the values of* $k_f(n)$ *are given in Table* A.2.

| $\alpha$ | $g$ | $k_f(n)$ |
|---|---|---|
| 0 | $g \equiv 1 \pmod 4$ | 4 |
| | $g \equiv 3 \pmod 4$ | 0 |
| 1 | $g = 1$ | 4 |
| | $g \equiv 1 \pmod 4, g \neq 1$ | 8 |
| | $g \equiv 3 \pmod 4$ | 0 |
| 2 | | 4 |
| *odd* $\geq 3$ | $g = 1$ | 6 |
| | $g \equiv 1 \pmod 4, g \neq 1$ | 12 |
| | $g = 3$ | 8 |
| | $g \equiv 3 \pmod 8, g \neq 3$ | 24 |
| | $g \equiv 7 \pmod 8$ | 0 |
| *even* $\geq 4$ | | 12 |

Table A.2: Values of $k_f(n)$

**Proposition A.3.** (Lomadze [18, Theorem 28, p. 158]) *(A5) Let* $f = x^2+4y^2+4z^2$*. We have*

$$r(f; n) = k_f(n)l(n)h(\mathbb{Q}(\sqrt{-n^*})),$$

*where the values of* $k_f(n)$ *are given in Table* A.3.

| $\alpha$ | $g$ | $k_f(n)$ |
|---|---|---|
| 0 | $g = 1$ | 2 |
| | $g \equiv 1 \pmod 4, g \neq 1$ | 4 |
| | $g \equiv 3 \pmod 4$ | 0 |
| | | *Continued on next page* |

| $\alpha$ | $g$ | $k_f(n)$ |
|---|---|---|
| 1 | | 0 |
| *even* $\geq 2$ | $g = 1$ | 6 |
| | $g \equiv 1 \pmod 4, g \neq 1$ | 12 |
| | $g = 3$ | 8 |
| | $g \equiv 3 \pmod 8, g \neq 3$ | 24 |
| | $g \equiv 7 \pmod 8$ | 0 |
| *odd* $\geq 3$ | | 12 |

Table A.3: Values of $k_f(n)$

**Proposition A.4.** *(A6) Let* $f = x^2 + 4y^2 + 5z^2 + 4yz$*. We have*

$$r(f; n) = k_f(n)l(n)h(\mathbb{Q}(\sqrt{-n^*})),$$

*where the values of* $k_f(n)$ *are given in Table* A.4.

| $\alpha$ | $g$ | $k_f(n)$ |
|---|---|---|
| 0 | $g = 1$ | 2 |
| | $g \equiv 1 \pmod 4, g \neq 1$ | 4 |
| | $g \equiv 3 \pmod 4$ | 0 |
| 1 | $g \equiv 1 \pmod 4$ | 0 |
| | $g \equiv 3 \pmod 4$ | 4 |
| 2 | $g = 1$ | 4 |
| | $g \equiv 1 \pmod 4, g \neq 1$ | 8 |
| | $g \equiv 3 \pmod 4$ | 0 |
| 3 | | 4 |
| *even* $\geq 4$ | $g = 1$ | 6 |
| | $g \equiv 1 \pmod 4, g \neq 1$ | 12 |
| | $g = 3$ | 8 |
| | $g \equiv 3 \pmod 8, g \neq 3$ | 24 |
| | $g \equiv 7 \pmod 8$ | 0 |
| *odd* $\geq 5$ | | 12 |

Table A.4: Values of $k_f(n)$

**Proposition A.5.** (Lomadze [18, Theorem 19, p. 155]) *(A8) Let* $f = x^2 + 2y^2 + 16z^2$*. We have*

$$r(f; n) = k_f(n)l(n)h(\mathbb{Q}(\sqrt{-n^*})),$$

*where the values of* $k_f(n)$ *are given in Table* A.5.

| $\alpha$ | $g$ | $k_f(n)$ |
|---|---|---|
| 0 | $g \equiv 1, 3 \pmod 8$ | 2 |
| | $g \equiv 5, 7 \pmod 8$ | 0 |
| 1 | $g = 1$ | 2 |
| | $g \equiv 1 \pmod 8, g \neq 1$ | 4 |
| | $g \equiv 5 \pmod 8$ | 0 |
| | $g = 3$ | 4 |
| | $g \equiv 3 \pmod 8, g \neq 3$ | 12 |
| | $g \equiv 7 \pmod 8$ | 0 |
| 2 | | 2 |
| 3 | $g = 1$ | 2 |
| | $g \equiv 1 \pmod 4, g \neq 1$ | 4 |
| | $g = 3$ | 8 |
| | $g \equiv 3 \pmod 8, g \neq 3$ | 24 |
| | $g \equiv 7 \pmod 8$ | 0 |
| 4 | | 4 |
| *odd* $\geq 5$ | $g = 1$ | 6 |
| | $g \equiv 1 \pmod 4, g \neq 1$ | 12 |
| | $g = 3$ | 8 |
| | $g \equiv 3 \pmod 8, g \neq 3$ | 24 |
| | $g \equiv 7 \pmod 8$ | 0 |
| *even* $\geq 6$ | | 12 |

Table A.5: Values of $k_f(n)$

**Proposition A.6.** (Lomadze [18, Theorem 43, p. 164]) *(A10) Let $f = x^2 + 8y^2 + 8z^2$. We have*

$$r(f; n) = k_f(n)l(n)h(\mathbb{Q}(\sqrt{-n^*})),$$

*where the values of $k_f(n)$ are given in Table* A.6*.*

| $\alpha$ | $g$ | $k_f(n)$ |
|---|---|---|
| 0 | $g = 1$ | 2 |
| | $g \equiv 1 \pmod 8, g \neq 1$ | 4 |
| | $g \equiv 3, 5, 7 \pmod 8$ | 0 |
| 1 | | 0 |
| 2 | $g = 1$ | 2 |
| | $g \equiv 1 \pmod 4, g \neq 1$ | 4 |
| | $g = 3$ | 8 |
| | $g \equiv 3 \pmod 8, g \neq 3$ | 24 |
| | $g \equiv 7 \pmod 8$ | 0 |
| 3 | | 4 |
| *even* $\geq 4$ | $g = 1$ | 6 |

*Continued on next page*

| $\alpha$ | $g$ | $k_f(n)$ |
|---|---|---|
| | $g \equiv 1 \pmod 4, g \neq 1$ | 12 |
| | $g = 3$ | 8 |
| | $g \equiv 3 \pmod 8, g \neq 3$ | 24 |
| | $g \equiv 7 \pmod 8$ | 0 |
| $odd \geq 5$ | | 12 |

Table A.6: Values of $k_f(n)$

**Proposition A.7.** *(B1) Let $f = x^2 + y^2 + 4z^2 + xy$. We have*

$$r(f;n) = k_f(n)l(n)h(\mathbb{Q}(\sqrt{-n^*})),$$

*where the values of $k_f(n)$ are given in Table* A.7*.*

| $\alpha, \beta$ | $g$ | $k_f(n)$ |
|---|---|---|
| | $\alpha$*(even)* $\beta$*(even)* | |
| $\alpha = 0, \beta \geq 0$ | $g = 1$ | 6 |
| | $g \equiv 1 \pmod 8, g \neq 1$ | 18 |
| | $g \equiv 3, 5, 7 \pmod 8$ | 6 |
| $\alpha \geq 2, \beta \geq 0$ | $g = 1$ | $3 \cdot 2^{\alpha/2+1} - 4$ |
| | $g \equiv 1 \pmod 8, g \neq 1$ | $9 \cdot 2^{\alpha/2+1} - 12$ |
| | $g \equiv 5 \pmod 8$ | $3 \cdot 2^{\alpha/2+1}$ |
| | $g \equiv 3 \pmod 4$ | $3 \cdot 2^{\alpha/2+1} - 6$ |
| | $\alpha$*(even)* $\beta$*(odd)* | |
| $\alpha = 0, \beta \geq 1$ | $g = 1$ | 6 |
| | $g \equiv 1, 7, 13 \pmod{24}, g \neq 1$ | 12 |
| | $g \equiv 5 \pmod 6$ | 0 |
| | $g \equiv 19 \pmod{24}$ | 36 |
| $\alpha \geq 2, \beta \geq 1$ | $g = 1$ | $3 \cdot 2^{\alpha/2+1} - 6$ |
| | $g \equiv 1, 13 \pmod{24}, g \neq 1$ | $3 \cdot 2^{\alpha/2+2} - 12$ |
| | $g \equiv 5 \pmod 6$ | 0 |
| | $g \equiv 7 \pmod{24}$ | $3 \cdot 2^{\alpha/2+2}$ |
| | $g \equiv 19 \pmod{24}$ | $9 \cdot 2^{\alpha/2+2} - 24$ |
| | $\alpha$*(odd)* $\beta$*(even)* | |
| $\alpha \geq 1, \beta \geq 0$ | | $3 \cdot 2^{(\alpha+1)/2} - 6$ |
| | $\alpha$*(odd)* $\beta$*(odd)* | |
| $\alpha \geq 1, \beta \geq 1$ | $g \equiv 1 \pmod 6$ | 0 |
| | $g \equiv 5 \pmod 6$ | $3 \cdot 2^{(\alpha+3)/2} - 12$ |

Table A.7: Values of $k_f(n)$

**Proposition A.8.** *(B2) Let* $f = x^2 + 2y^2 + 2z^2 + yz + zx + xy$. *We have*

$$r(f;n) = k_f(n)l(n)h(\mathbb{Q}(\sqrt{-n^*})),$$

*where the values of* $k_f(n)$ *are given in Table* A.8.

| $\alpha, \beta$ | $g$ | $k_f(n)$ |
|---|---|---|
| | $\alpha$*(even)* $\beta$*(even)* | |
| $\alpha = 0, \beta \geq 0$ | $g = 1$ | 2 |
| | $g \equiv 1 \pmod 8, g \neq 1$ | 6 |
| | $g \equiv 3, 5, 7 \pmod 8$ | 2 |
| $\alpha \geq 2, \beta \geq 0$ | $g = 1$ | $5 \cdot 2^{\alpha/2+1} - 4$ |
| | $g \equiv 1 \pmod 8, g \neq 1$ | $15 \cdot 2^{\alpha/2+1} - 12$ |
| | $g \equiv 5 \pmod 8$ | $5 \cdot 2^{\alpha/2+1}$ |
| | $g \equiv 3 \pmod 4$ | $5 \cdot 2^{\alpha/2+1} - 6$ |
| | $\alpha$*(even)* $\beta$*(odd)* | |
| $\alpha = 0, \beta \geq 1$ | $g = 1$ | 2 |
| | $g \equiv 1, 7, 13 \pmod{24}, g \neq 1$ | 4 |
| | $g \equiv 5 \pmod 6$ | 0 |
| | $g \equiv 19 \pmod{24}$ | 12 |
| $\alpha \geq 2, \beta \geq 1$ | $g = 1$ | $5 \cdot 2^{\alpha/2+1} - 6$ |
| | $g \equiv 1, 13 \pmod{24}, g \neq 1$ | $5 \cdot 2^{\alpha/2+2} - 12$ |
| | $g \equiv 5 \pmod 6$ | 0 |
| | $g \equiv 7 \pmod{24}$ | $5 \cdot 2^{\alpha/2+2}$ |
| | $g \equiv 19 \pmod{24}$ | $15 \cdot 2^{\alpha/2+2} - 24$ |
| | $\alpha$*(odd)* $\beta$*(even)* | |
| $\alpha \geq 1, \beta \geq 0$ | | $5 \cdot 2^{(\alpha+1)/2} - 6$ |
| | $\alpha$*(odd)* $\beta$*(odd)* | |
| $\alpha \geq 1, \beta \geq 1$ | $g \equiv 1 \pmod 6$ | 0 |
| | $g \equiv 5 \pmod 6$ | $5 \cdot 2^{(\alpha+3)/2} - 12$ |

Table A.8: Values of $k_f(n)$

**Proposition A.9.** *(B3) Let* $f = x^2 + y^2 + 12z^2 + xy$. *We have*

$$r(f;n) = k_f(n)l(n)h(\mathbb{Q}(\sqrt{-n^*})),$$

*where the values of* $k_f(n)$ *are given in Table* A.9.

| $\alpha, \beta$ | $g$ | $k_f(n)$ |
|---|---|---|
| | $\alpha$*(even)* $\beta$*(even)* | |
| $\alpha = 0, \beta \geq 0$ | $g = 1$ | 6 |
| | $g \equiv 1, 7, 13 \pmod{24}, g \neq 1$ | 12 |

*Continued on next page*

| $\alpha, \beta$ | $g$ | $k_f(n)$ |
|---|---|---|
| | $g \equiv 5 \pmod 6$ | 0 |
| | $g \equiv 19 \pmod{24}$ | 36 |
| $\alpha \geq 2, \beta \geq 0$ | $g = 1$ | $3 \cdot 2^{\alpha/2+1} - 6$ |
| | $g \equiv 1, 13 \pmod{24}, g \neq 1$ | $3 \cdot 2^{\alpha/2+2} - 12$ |
| | $g \equiv 5 \pmod 6$ | 0 |
| | $g \equiv 7 \pmod{24}$ | $3 \cdot 2^{\alpha/2+2}$ |
| | $g \equiv 19 \pmod{24}$ | $9 \cdot 2^{\alpha/2+2} - 24$ |
| | *$\alpha$(even) $\beta$(odd)* | |
| $\alpha = 0, \beta \geq 1$ | $g = 1$ | 6 |
| | $g \equiv 1 \pmod 8, g \neq 1$ | 18 |
| | $g \equiv 3, 5, 7 \pmod 8$ | 6 |
| $\alpha \geq 2, \beta \geq 1$ | $g = 1$ | $3 \cdot 2^{\alpha/2+1} - 4$ |
| | $g \equiv 1, 17 \pmod{24}, g \neq 1$ | $9 \cdot 2^{\alpha/2+1} - 12$ |
| | $g \equiv 5, 13 \pmod{24}$ | $3 \cdot 2^{\alpha/2+1}$ |
| | $g \equiv 7, 11, 19, 23 \pmod{24}$ | $3 \cdot 2^{\alpha/2+1} - 6$ |
| | *$\alpha$(odd) $\beta$(even)* | |
| $\alpha \geq 1, \beta \geq 0$ | $g \equiv 1 \pmod 6$ | 0 |
| | $g \equiv 5 \pmod 6$ | $3 \cdot 2^{(\alpha+3)/2} - 12$ |
| | *$\alpha$(odd) $\beta$(odd)* | |
| $\alpha \geq 1, \beta \geq 1$ | | $3 \cdot 2^{(\alpha+1)/2} - 6$ |

Table A.9: Values of $k_f(n)$

**Proposition A.10.** (Lomadze [18, Theorem 23, p. 156]) *(B5,B6,B7,B9,B10,B11) Let $f = x^2 + 3y^2 + 9z^2$. We have*

$$r(f; n) = k_f(n) l(n) h(\mathbb{Q}(\sqrt{-n^*})),$$

*where the values of $k_f(n)$ are given in Table* A.10.

| $\alpha, \beta$ | $g$ | $k_f(n)$ |
|---|---|---|
| | *$\alpha$(even) $\beta$(even)* | |
| $\alpha \geq 0, \beta = 0$ | $g = 1$ | $2^{\alpha/2+2} - 2$ |
| | $g \equiv 1 \pmod{24}, g \neq 1$ | $3 \cdot 2^{\alpha/2+2} - 6$ |
| | $g \equiv 5, 11, 17, 23 \pmod{24}$ | 0 |
| | $g \equiv 7, 19 \pmod{24}$ | $2^{\alpha/2+2} - 3$ |
| | $g \equiv 13 \pmod{24}$ | $2^{\alpha/2+2}$ |
| $\alpha \geq 0, \beta \geq 2$ | $g = 1$ | $2^{\alpha/2+3} - 4$ |
| | $g \equiv 1 \pmod 8, g \neq 1$ | $3 \cdot 2^{\alpha/2+3} - 12$ |
| | $g \equiv 5 \pmod 8$ | $2^{\alpha/2+3}$ |
| | $g \equiv 3 \pmod 4$ | $2^{\alpha/2+3} - 6$ |

*Continued on next page*

| $\alpha, \beta$ | $g$ | $k_f(n)$ |
|---|---|---|
| | $\alpha$*(even)* $\beta$*(odd)* | |
| $\alpha \geq 0, \beta \geq 1$ | $g = 1$ | $2^{\alpha/2+3} - 6$ |
| | $g \equiv 1, 13 \pmod{24}, g \neq 1$ | $2^{\alpha/2+4} - 12$ |
| | $g \equiv 5 \pmod{6}$ | $0$ |
| | $g \equiv 7 \pmod{24}$ | $2^{\alpha/2+4}$ |
| | $g \equiv 19 \pmod{24}$ | $3 \cdot 2^{\alpha/2+4} - 24$ |
| | $\alpha$*(odd)* $\beta$*(even)* | |
| $\alpha \geq 1, \beta = 0$ | $g \equiv 1 \pmod{6}$ | $0$ |
| | $g \equiv 5 \pmod{6}$ | $2^{(\alpha+3)/2} - 3$ |
| $\alpha \geq 1, \beta \geq 2$ | | $2^{(\alpha+5)/2} - 6$ |
| | $\alpha$*(odd)* $\beta$*(odd)* | |
| $\alpha \geq 1, \beta \geq 1$ | $g \equiv 1 \pmod{6}$ | $0$ |
| | $g \equiv 5 \pmod{6}$ | $2^{(\alpha+7)/2} - 12$ |

Table A.10: Values of $k_f(n)$

**Proposition A.11.** *(B8) Let* $f = x^2 + 4y^2 + 4z^2 + 4yz$*. We have*

$$r(f; n) = k_f(n)l(n)h(\mathbb{Q}(\sqrt{-n^*})),$$

*where the values of* $k_f(n)$ *are given in Table* A.11.

| $\alpha, \beta$ | $g$ | $k_f(n)$ |
|---|---|---|
| | $\alpha$*(even)* $\beta$*(even)* | |
| $\alpha \geq 0, \beta \geq 0$ | $g = 1$ | $3 \cdot 2^{\alpha/2+1} - 4$ |
| | $g \equiv 1 \pmod{8}, g \neq 1$ | $9 \cdot 2^{\alpha/2+1} - 12$ |
| | $g \equiv 5 \pmod{8}$ | $3 \cdot 2^{\alpha/2+1}$ |
| | $g \equiv 3 \pmod{4}$ | $3 \cdot 2^{\alpha/2+1} - 6$ |
| | $\alpha$*(even)* $\beta$*(odd)* | |
| $\alpha \geq 0, \beta \geq 1$ | $g = 1$ | $3 \cdot 2^{\alpha/2+1} - 6$ |
| | $g \equiv 1, 13 \pmod{24}, g \neq 1$ | $3 \cdot 2^{\alpha/2+2} - 12$ |
| | $g \equiv 5 \pmod{6}$ | $0$ |
| | $g \equiv 7 \pmod{24}$ | $3 \cdot 2^{\alpha/2+2}$ |
| | $g \equiv 19 \pmod{24}$ | $9 \cdot 2^{\alpha/2+2} - 24$ |
| | $\alpha$*(odd)* $\beta$*(even)* | |
| $\alpha \geq 1, \beta \geq 0$ | | $3 \cdot 2^{(\alpha+1)/2} - 6$ |
| | $\alpha$*(odd)* $\beta$*(odd)* | |
| $\alpha \geq 1, \beta \geq 1$ | $g \equiv 1 \pmod{6}$ | $0$ |
| | $g \equiv 5 \pmod{6}$ | $3 \cdot 2^{(\alpha+3)/2} - 12$ |

Table A.11: Values of $k_f(n)$

**Proposition A.12.** *(B12) Let* $f = x^2 + 16y^2 + 16z^2 + 16yz$*. We have*

$$r(f;n) = k_f(n)l(n)h(\mathbb{Q}(\sqrt{-n^*})),$$

*where the values of* $k_f(n)$ *are given in Table* A.12.

| $\alpha, \beta$ | $g$ | $k_f(n)$ |
|---|---|---|
| | $\alpha$(even) $\beta$(even) | |
| $\alpha = 0, \beta \geq 0$ | $g = 1$ | 2 |
| | $g \equiv 1 \pmod 8, g \neq 1$ | 6 |
| | $g \equiv 3, 5, 7 \pmod 8$ | 0 |
| $\alpha \geq 2, \beta \geq 0$ | $g = 1$ | $3 \cdot 2^{\alpha/2} - 4$ |
| | $g \equiv 1 \pmod 8, g \neq 1$ | $9 \cdot 2^{\alpha/2} - 12$ |
| | $g \equiv 5 \pmod 8$ | $3 \cdot 2^{\alpha/2}$ |
| | $g \equiv 3 \pmod 4$ | $3 \cdot 2^{\alpha/2} - 6$ |
| | $\alpha$(even) $\beta$(odd) | |
| $\alpha = 0, \beta \geq 1$ | $g \equiv 1, 5, 7, 11, 13, 17, 23 \pmod{24}$ | 0 |
| | $g \equiv 19 \pmod{24}$ | 12 |
| $\alpha \geq 2, \beta \geq 1$ | $g = 1$ | $3 \cdot 2^{\alpha/2} - 6$ |
| | $g \equiv 1, 13 \pmod{24}, g \neq 1$ | $3 \cdot 2^{\alpha/2+1} - 12$ |
| | $g \equiv 5 \pmod 6$ | 0 |
| | $g \equiv 7 \pmod{24}$ | $3 \cdot 2^{\alpha/2+1}$ |
| | $g \equiv 19 \pmod{24}$ | $9 \cdot 2^{\alpha/2+1} - 24$ |
| | $\alpha$(odd) $\beta$(even) | |
| $\alpha = 1, \beta \geq 0$ | | 0 |
| $\alpha \geq 3, \beta \geq 0$ | | $3 \cdot 2^{(\alpha-1)/2} - 6$ |
| | $\alpha$(odd) $\beta$(odd) | |
| $\alpha = 1, \beta \geq 1$ | | 0 |
| $\alpha \geq 3, \beta \geq 1$ | $g \equiv 1 \pmod 6$ | 0 |
| | $g \equiv 5 \pmod 6$ | $3 \cdot 2^{(\alpha+1)/2} - 12$ |

Table A.12: Values of $k_f(n)$

**Proposition A.13.** *(C1) Let* $f = x^2 + y^2 + 2z^2 + zx$*. We have*

$$r(f;n) = k_f(n)l(n)h(\mathbb{Q}(\sqrt{-n^*})),$$

*where the values of* $k_f(n)$ *are given in Table* A.13.

| $\alpha, \gamma$ | $g$ | $k_f(n)$ |
|---|---|---|
| | $\alpha$(even) $\gamma$(even) | |
| $\alpha \geq 0, \gamma \geq 0$ | $g \equiv 1 \pmod 8$ | $2^{\alpha/2+2}$ |
| | $g \equiv 3 \pmod 4$ | $2^{\alpha/2+2} - 2$ |

*Continued on next page*

| $\alpha, \gamma$ | $g$ | $k_f(n)$ |
|---|---|---|
| | $g \equiv 5 \pmod 8$ | $3 \cdot 2^{\alpha/2+2} - 4$ |
| | $\alpha$*(even)* $\gamma$*(odd)* | |
| $\alpha \geq 0, \gamma \geq 1$ | $g = 1$ | $2^{\alpha/2+2} - 2$ |
| | $g \equiv 1 \pmod 4, g \equiv 1, 2, 4 \pmod 7, g \neq 1$ | $2^{\alpha/2+3} - 4$ |
| | $g \equiv 3, 5, 6 \pmod 7$ | 0 |
| | $g \equiv 3 \pmod 8, g \equiv 1, 2, 4 \pmod 7$ | $3 \cdot 2^{\alpha/2+3} - 8$ |
| | $g \equiv 7 \pmod 8, g \equiv 1, 2, 4 \pmod 7$ | $2^{\alpha/2+3}$ |
| | $\alpha$*(odd)* $\gamma$*(even)* | |
| $\alpha \geq 1, \gamma \geq 0$ | | $2^{(\alpha+3)/2} - 2$ |
| | $\alpha$*(odd)* $\gamma$*(odd)* | |
| $\alpha \geq 1, \gamma \geq 1$ | $g \equiv 1, 2, 4 \pmod 7$ | $2^{(\alpha+5)/2} - 4$ |
| | $g \equiv 3, 5, 6 \pmod 7$ | 0 |

Table A.13: Values of $k_f(n)$

**Proposition A.14.** *(C2) Let* $f = x^2 + 3y^2 + 3z^2 + 2yz + zx + xy$*. We have*

$$r(f; n) = k_f(n)l(n)h(\mathbb{Q}(\sqrt{-n^*})),$$

*where the values of* $k_f(n)$ *are given in Table* A.14.

| $\alpha, \gamma$ | $g$ | $k_f(n)$ |
|---|---|---|
| | $\alpha$*(even)* $\gamma$*(even)* | |
| $\alpha = 0, \gamma \geq 0$ | $g \equiv 1, 3, 7 \pmod 8$ | 2 |
| | $g \equiv 5 \pmod 8$ | 6 |
| $\alpha \geq 2, \gamma \geq 0$ | $g \equiv 1 \pmod 8$ | $2^{\alpha/2+1}$ |
| | $g \equiv 3 \pmod 4$ | $2^{\alpha/2+1} - 2$ |
| | $g \equiv 5 \pmod 8$ | $3 \cdot 2^{\alpha/2+1} - 4$ |
| | $\alpha$*(even)* $\gamma$*(odd)* | |
| $\alpha = 0, \gamma \geq 1$ | $g = 1$ | 2 |
| | $g \equiv 1, 5, 7 \pmod 8, g \equiv 1, 2, 4 \pmod 7, g \neq 1$ | 4 |
| | $g \equiv 3, 5, 6 \pmod 7$ | 0 |
| | $g \equiv 3 \pmod 8, g \equiv 1, 2, 4 \pmod 7$ | 12 |
| $\alpha \geq 2, \gamma \geq 1$ | $g = 1$ | $2^{\alpha/2+1} - 2$ |
| | $g \equiv 1 \pmod 4, g \equiv 1, 2, 4 \pmod 7, g \neq 1$ | $2^{\alpha/2+2} - 4$ |
| | $g \equiv 3, 5, 6 \pmod 7$ | 0 |
| | $g \equiv 3 \pmod 8, g \equiv 1, 2, 4 \pmod 7$ | $3 \cdot 2^{\alpha/2+2} - 8$ |
| | $g \equiv 7 \pmod 8, g \equiv 1, 2, 4 \pmod 7$ | $2^{\alpha/2+2}$ |
| | $\alpha$*(odd)* $\gamma$*(even)* | |
| $\alpha \geq 1, \gamma \geq 0$ | | $2^{(\alpha+1)/2} - 2$ |
| | | *Continued on next page* |

| $\alpha, \gamma$ | $g$ | $k_f(n)$ |
|---|---|---|
| | $\alpha$*(odd)* $\gamma$*(odd)* | |
| $\alpha \geq 1, \gamma \geq 1$ | $g \equiv 1, 2, 4 \pmod 7$ | $2^{(\alpha+3)/2} - 4$ |
| | $g \equiv 3, 5, 6 \pmod 7$ | $0$ |

Table A.14: Values of $k_f(n)$

**Proposition A.15.** *(C3) Let* $f = x^2 + 4y^2 + 8z^2 + 4yz$. *We have*

$$r(f; n) = k_f(n)l(n)h(\mathbb{Q}(\sqrt{-n^*})),$$

*where the values of* $k_f(n)$ *are given in Table* A.15.

| $\alpha, \gamma$ | $g$ | $k_f(n)$ |
|---|---|---|
| | $\alpha$*(even)* $\gamma$*(even)* | |
| $\alpha \geq 0, \gamma \geq 0$ | $g \equiv 1 \pmod 8$ | $2^{\alpha/2+1}$ |
| | $g \equiv 3 \pmod 4$ | $2^{\alpha/2+1} - 2$ |
| | $g \equiv 5 \pmod 8$ | $3 \cdot 2^{\alpha/2+1} - 4$ |
| | $\alpha$*(even)* $\gamma$*(odd)* | |
| $\alpha \geq 0, \gamma \geq 1$ | $g = 1$ | $2^{\alpha/2+1} - 2$ |
| | $g \equiv 1 \pmod 4, g \equiv 1, 2, 4 \pmod 7, g \neq 1$ | $2^{\alpha/2+2} - 4$ |
| | $g \equiv 3, 5, 6 \pmod 7$ | $0$ |
| | $g \equiv 3 \pmod 8, g \equiv 1, 2, 4 \pmod 7$ | $3 \cdot 2^{\alpha/2+2} - 8$ |
| | $g \equiv 7 \pmod 8, g \equiv 1, 2, 4 \pmod 7$ | $2^{\alpha/2+2}$ |
| | $\alpha$*(odd)* $\gamma$*(even)* | |
| $\alpha \geq 1, \gamma \geq 0$ | | $2^{(\alpha+1)/2} - 2$ |
| | $\alpha$*(odd)* $\gamma$*(odd)* | |
| $\alpha \geq 1, \gamma \geq 1$ | $g \equiv 1, 2, 4 \pmod 7$ | $2^{(\alpha+3)/2} - 4$ |
| | $g \equiv 3, 5, 6 \pmod 7$ | $0$ |

Table A.15: Values of $k_f(n)$

**Proposition A.16.** *(C4) Let* $f = 4x^2 + 5y^2 + 29z^2 + 2yz + 4zx + 4xy$. *We have*

$$r(f; n) = k_f(n)l(n)h(\mathbb{Q}(\sqrt{-n^*})),$$

*where the values of* $k_f(n)$ *are given in Table* A.16.

| $\alpha, \gamma$ | $g$ | $k_f(n)$ |
|---|---|---|
| | $\alpha$*(even)* $\gamma$*(even)* | |
| $\alpha = 0, \gamma \geq 0$ | $g \equiv 1, 3, 7 \pmod 8$ | $0$ |
| | $g \equiv 5 \pmod 8$ | $2$ |

*Continued on next page*

| $\alpha,\gamma$ | $g$ | $k_f(n)$ |
|---|---|---|
| $\alpha\geq 2,\gamma\geq 0$ | $g\equiv 1 \pmod 8$ | $2^{\alpha/2}$ |
| | $g\equiv 3 \pmod 4$ | $2^{\alpha/2}-2$ |
| | $g\equiv 5 \pmod 8$ | $3\cdot 2^{\alpha/2}-4$ |
| | *$\alpha$(even) $\gamma$(odd)* | |
| $\alpha=0,\gamma\geq 1$ | $g\equiv 1,5,7 \pmod 8, g\equiv 1,2,4 \pmod 7$ *or* $g\equiv 3,5,6 \pmod 7$ | 0 |
| | $g\equiv 3 \pmod 8, g\equiv 1,2,4 \pmod 7$ | 4 |
| $\alpha=2,\gamma\geq 1$ | $g\equiv 1 \pmod 4, g\equiv 1,2,4 \pmod 7$ *or* $g\equiv 3,5,6 \pmod 7$ | 0 |
| | $g\equiv 3 \pmod 4, g\equiv 1,2,4 \pmod 7$ | 4 |
| $\alpha\geq 4,\gamma\geq 1$ | $g=1$ | $2^{\alpha/2}-2$ |
| | $g\equiv 1 \pmod 4, g\equiv 1,2,4 \pmod 7, g\neq 1$ | $2^{\alpha/2+1}-4$ |
| | $g\equiv 3,5,6 \pmod 7$ | 0 |
| | $g\equiv 3 \pmod 8, g\equiv 1,2,4 \pmod 7$ | $3\cdot 2^{\alpha/2+1}-8$ |
| | $g\equiv 7 \pmod 8, g\equiv 1,2,4 \pmod 7$ | $2^{\alpha/2+1}$ |
| | *$\alpha$(odd) $\gamma$(even)* | |
| $\alpha=1,3,\gamma\geq 0$ | | 0 |
| $\alpha\geq 5,\gamma\geq 0$ | | $2^{(\alpha-1)/2}-2$ |
| | *$\alpha$(odd) $\gamma$(odd)* | |
| $\alpha=1,3,\gamma\geq 1$ | | 0 |
| $\alpha\geq 5,\gamma\geq 1$ | $g\equiv 1,2,4 \pmod 7$ | $2^{(\alpha+1)/2}-4$ |
| | $g\equiv 3,5,6 \pmod 7$ | 0 |

Table A.16: Values of $k_f(n)$

Propositions A.1, A.2, A.3, A.5 and A.6 can be deduced from (2.3) using results in [1]. For example, from Appendix B of [1] we have (writing $r(a,b,c;n)$ for $r(a,b,c,0,0,0;n)$)

$$r(1,1,8;n)=\begin{cases} r(1,1,2;n) & \text{if } n\equiv 1 \pmod 4,\\ \frac{2}{3}r(1,1,2;n) & \text{if } n\equiv 2 \pmod 8,\\ r(1,1,2;n/4) & \text{if } n\equiv 0 \pmod 4,\\ 0 & \text{if } n\equiv 3,6,7 \pmod 8,\end{cases}$$

and

$$r(1,1,2;n)=\begin{cases} \frac{1}{3}r(1,1,1;2n) & \text{if } n\equiv 1 \pmod 2,\\ r(1,1,1;2n) & \text{if } n\equiv 0 \pmod 2,\end{cases}$$

so

$$r(1,1,8;n)=\begin{cases}\frac{1}{3}r(1,1,1;2n) & \text{if } n\equiv 1 \pmod 4,\\ \frac{2}{3}r(1,1,1;2n) & \text{if } n\equiv 2 \pmod 8,\\ r(1,1,1;n/2) & \text{if } n\equiv 0 \pmod 4,\\ 0 & \text{if } n\equiv 3,6,7 \pmod 8,\end{cases}$$

and Proposition A.2 follows from (2.3).

In Table A.17 we list the positive integers not represented by the twenty-seven spinor regular positive-definite ternary quadratic forms A1–A13, B1–B3, B5–B10, B12 and C1–C4, which are alone in their spinor genus. In addition, the table gives the even positive integers not represented by the two spinor regular forms B4 and B11, which are not alone in their spinor genus.

| no. | $(a,b,c,d,e,f)$ | non-represented integers |
|---|---|---|
| A1 | (2,2,5,2,2,0) | $8l+3, 8l+6, 32l+12, 4^k(8l+7)(k,l\in\mathbb{N}_0), M_4^2$ |
| A2 | (1,4,9,4,0,0) | $4l+3, 16l+6, 16l+12, 64l+24,$ $4^k(16l+14)(k,l\in\mathbb{N}_0), 2M_4^2$ |
| A3 | (2,5,8,4,0,2) | $8l+3, 8l+6, 32l+12, 32l+24, 128l+48,$ $4^k(8l+7)(k,l\in\mathbb{N}_0), M_4^2, 4M_4^2$ |
| A4 | (4,4,5,0,4,0) | $4l+2, 8l+3, 32l+12,$ $4^k(8l+7)(k,l\in\mathbb{N}_0), M_4^2$ |
| A5 | (4,9,9,2,4,4) | $4l+2, 8l+3, 8l+5, 16l+8, 32l+12,$ $4^k(8l+7)(k,l\in\mathbb{N}_0), M_4^2$ |
| A6 | (4,5,13,2,0,0) | $4l+2, 8l+3, 32l+8, 32l+12, 128l+48,$ $4^k(8l+7)(k,l\in\mathbb{N}_0), M_4^2$ |
| A7 | (5,8,8,0,4,4) | $4l+2, 8l+3, 32l+12, 32l+24, 128l+48,$ $4^k(8l+7)(k,l\in\mathbb{N}_0), M_4^2, 4M_4^2$ |
| A8 | (4,8,17,0,4,0) | $4l+3, 8l+2, 8l+5, 16l+6, 32l+20, 32l+28,$ $64l+40, 4^k(16l+14)(k,l\in\mathbb{N}_0), M_8^2$ |
| A9 | (9,9,16,8,8,2) | $4l+2, 8l+3, 8l+5, 16l+8, 32l+12, 128l+48,$ $4^k(8l+7)(k,l\in\mathbb{N}_0), M_4^2, 4M_4^2$ |
| A10 | (4,9,32,0,0,4) | $4l+2, 8l+3, 8l+5, 16l+8, 32l+12, 32l+20,$ $4^k(8l+7)(k,l\in\mathbb{N}_0), M_4^2$ |
| A11 | (5,13,16,0,0,2) | $4l+2, 8l+1, 8l+3, 16l+8, 32l+12, 128l+48,$ $4^k(8l+7)(k,l\in\mathbb{N}_0), 4M_4^2$ |
| A12 | (9,17,32,-8,8,6) | $4l+2, 8l+3, 8l+5, 16l+8, 32l+12, 128l+48,$ $128l+96, 512l+192, 4^k(8l+7)(k,l\in\mathbb{N}_0),$ $M_4^2, 4M_4^2, 16M_4^2$ |
| A13 | (9,16,36,16,4,8) | $4l+2, 8l+3, 8l+5, 16l+8, 32l+12, 32l+20,$ $64l+32, 128l+48, 4^k(8l+7)(k,l\in\mathbb{N}_0), M_4^2, 4M_4^2$ |
| B1 | (3,3,4,0,0,3) | $3l+2, 4l+2, 9^k(9l+6)(k,l\in\mathbb{N}_0), M_3^2$ |
| B2 | (3,4,4,4,3,3) | $3l+2, 9^k(9l+6)(k,l\in\mathbb{N}_0), M_3^2$ |
| | | Continued on next page |

| no. | $(a, b, c, d, e, f)$ | non-represented integers |
|---|---|---|
| B3 | (1,7,12,0,0,1) | $4l+2, 9l+6, 9^k(3l+2)(k,l \in \mathbb{N}_0), 3M_3^2$ |
| B4* | (3,7,7,5,3,3) | $4l+2, 12l+8, 16l+8, 4\cdot 9^k(9l+6)(k,l \in \mathbb{N}_0), 4M_3^2$ |
| B5 | (4,4,9,0,0,4) | $3l+2, 4l+2, 4l+3, 9^k(9l+6)(k,l \in \mathbb{N}_0), M_3^2$ |
| B6 | (3,4,9,0,0,0) | $3l+2, 4l+2, 9^k(9l+6)(k,l \in \mathbb{N}_0), M_3^2$ |
| B7 | (4,9,12,0,0,0) | $3l+2, 4l+2, 4l+3, 9^k(9l+6)(k,l \in \mathbb{N}_0), M_3^2$ |
| B8 | (4,9,28,0,4,0) | $3l+2, 4l+2, 4l+3, 9l+3, 9^k(9l+6)(k,l \in \mathbb{N}_0), M_3^2$ |
| B9 | (9,16,16,16,0,0) | $3l+2, 4l+2, 4l+3, 8l+5, 16l+8, 16l+12,$ $9^k(9l+6)(k,l \in \mathbb{N}_0), M_3^2, 4M_3^2$ |
| B10 | (13,13,16,-8,8,10) | $3l+2, 4l+2, 4l+3, 8l+1, 16l+8, 16l+12,$ $9^k(9l+6)(k,l \in \mathbb{N}_0), M_3^2, 4M_3^2$ |
| B11* | (9,16,48,0,0,0) | $4l+2, 12l+8, 16l+8, 16l+12,$ $4\cdot 9^k(9l+6)(k,l \in \mathbb{N}_0), 4M_3^2$ |
| B12 | (9,16,112,16,0,0) | $3l+2, 4l+2, 4l+3, 8l+5, 9l+3, 16l+8,$ $16l+12, 9^k(9l+6)(k,l \in \mathbb{N}_0), M_3^2, 4M_3^2$ |
| C1 | (2,7,8,7,1,0) | $7l+3, 7l+5, 7l+6, 49^k(49l+21),$ $49^k(49l+35), 49^k(49l+42)(k,l \in \mathbb{N}_0), M_7^2$ |
| C2 | (7,8,9,6,7,0) | $4l+2, 7l+3, 7l+5, 7l+6, 49^k(49l+21),$ $49^k(49l+35), 49^k(49l+42)(k,l \in \mathbb{N}_0), M_7^2$ |
| C3 | (8,9,25,2,4,8) | $4l+2, 4l+3, 7l+3, 7l+5, 7l+6, 49^k(49l+21),$ $49^k(49l+35), 49^k(49l+42)(k,l \in \mathbb{N}_0), M_7^2$ |
| C4 | (29,32,36,32,12,24) | $4l+2, 4l+3, 7l+3, 7l+5, 7l+6, 8l+1, 16l+8,$ $16l+12, 49^k(49l+21), 49^k(49l+35),$ $49^k(49l+42)(k,l \in \mathbb{N}_0), M_7^2$ |

Table A.17: Positive integers not represented by spinor regular positive-definite ternary quadratic forms $(a, b, c, d, e, f) = ax^2 + by^2 + cz^2 + dyz + ezx + fxy$ which are not regular (*: Only positive even integers, which are not represented)