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ADVANCES IN FINDING IDEAL PLAY ON POSET GAMES

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Abstract

Poset games are a class of combinatorial games that remain unsolved. Soltys and Wilson proved that computing winning strategies is in **PSPACE** and aside from special cases such as NIM and N-Free games, **P** time algorithms for finding ideal play are unknown. This paper presents methods to calculate the nimber of poset games allowing for the classification of winning or losing positions. The results present an equivalence of ideal strategies on posets that are seemingly unrelated.

1. Introduction

Poset games are impartial combinatorial games whose game boards are partially ordered sets (posets) P on which players take turns removing an element $p \in P$ and every element $p' \geq_P p$ from P. We define $P - p_{\leq} = P \setminus \{p' : p \leq p'\}$. Each turn a player must remove an element if they can. An example of some moves in a poset game is given in Figure 1.

In this paper we consider *normal play* games, where the first player to have no move loses. Poset games include NIM, CHOMP, SUBSET TAKE-AWAY, DIVISORS and GREEN HACKENBUSH on trees. As an example, Figure 2 shows a game of CHOMP and the equivalent poset game.

NIM, central to the theory of all impartial games, is a poset game played on any number of disjoint totally ordered sets of finite or transfinite cardinality called piles (see Figure 3) and as a result serves as one of the simplest poset games. It is well known that a game of NIM is in $\mathcal{P}$ (the set of previous player win games, also called $\mathcal{P}$-positions) if and only if the binary XOR sum of the pile's heights is 0 [3].

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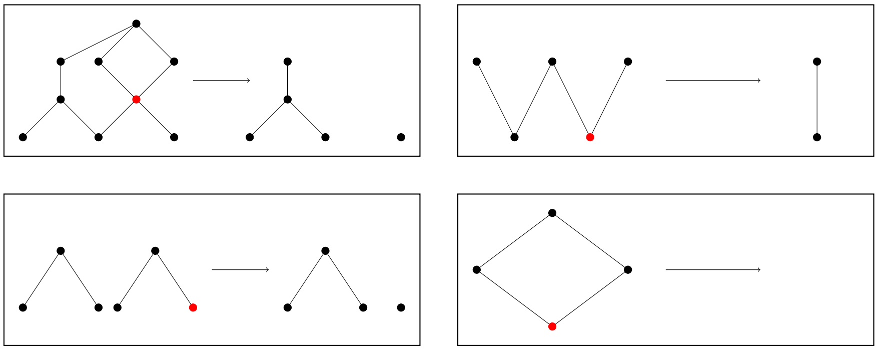

Figure 1: Examples of moves in poset games

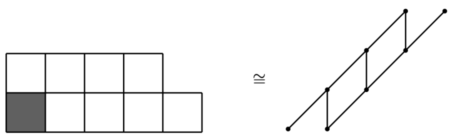

Figure 2: A CHOMP position described as a poset game

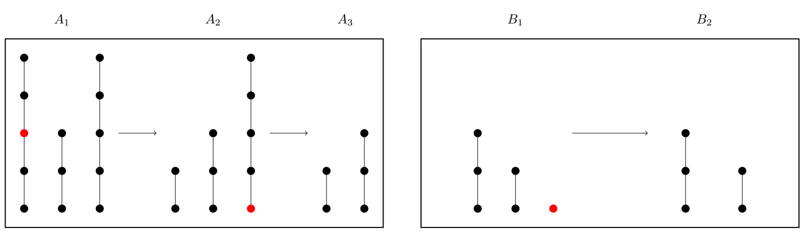

Figure 3: Examples of moves in NIM

Aside from NIM very little is known about ideal play on general poset games. Soltys and Wilson [7] proved that computing a winning strategy is in **PSPACE** and aside from special cases such as NIM and N-Free games [4], which are not the focus of this paper, **P** time algorithms for finding ideal play are unknown. Byrnes [2] also proved non-constructively that local periodicity exists in CHOMP and poset games that resemble CHOMP. Attempts at constructive results have thus far been largely unsuccessful [8]. Computational efforts, like those of Zeilburger [9] demonstrate that this local periodicity leads to no discernible global pattern even in cases as small as 3 by n CHOMP.

Partisan games are games where players may have different sets of moves. We can describe a game G as the set of its left and right options (or moves). This is often denoted $G = \{G^{\mathcal{L}}|G^{\mathcal{R}}\}$ where $G^{\mathcal{L}}$ and $G^{\mathcal{R}}$ are the sets of left and right options respectively. Impartial games are exactly those games where the left and right options are the same for G and all followers of G. Two impartial games G and H are equal, $G = H$, if $G + H$ is a $\mathcal{P}$-position. To describe this another way, playing on the games G and H where each turn a player chooses to move on G or on H is a $\mathcal{P}$-position. Sprague and Grundy independently proved the following result, which is vital to the study of impartial games in normal play.

The Sprague-Grundy Theorem ([1])**.** *For all impartial combinatorial games G, there exists a* Nim *pile which is equivalent to G.*

The game with a Nim pile of size n is denoted as $*n$ and $G = *n$ denotes that the game G is equivalent to a Nim pile of size n. The $\mathcal{G}$-value (Grundy value or Grundy number) of an impartial game G is exactly the n such that $G = *n$. Equivallently, we write $\mathcal{G}(G) = n$. We define the option value set of an impartial game G, which we denote G^*, to be the set of $\mathcal{G}$-values of all the options of G. An important tool in determining $\mathcal{G}$-values is the *mex*-rule (minimal excluded value rule). Formally, the *mex* of a set of non-negative integers is the smallest non-negative integer not in the set. For example $mex\{0, 2\} = 1$. From Sprague and Grundy's theory [1], $\mathcal{G}(G) = mex(G^*)$ for all impartial games G. We will write $G \equiv H$ if $G^* = H^*$.

For impartial games, the canonical form of a value $*n$ is exactly the Nim pile of size n (or any other game with the same game tree). We will call G *weakly-canonical* if $G \equiv *n$ which is equivalent to $|G^*| = \mathcal{G}(G)$. An example of a game that is weakly canonical is $G = \{\{*|*\}, *2 + *3, *||\{* *\}, *2 + *3, *\} \equiv *2$. An example of a game that is not weakly canonical is $H = \{0, *2|0, *2\}$.

A *fence* F is a poset $F = \{f_0, f_1, f_2, .., f_n\}$ such that $f_0 > f_1, f_1 < f_2, f_2 > f_3, f_3 < f_4, ...f_{n-1} < f_n$ or $f_0 < f_1, f_1 > f_2, f_2 < f_3, f_3 > f_4, ...f_{n-1} > f_n$ if n is even and $f_0 > f_1, f_1 < f_2, f_2 > f_3, f_3 < f_4, ...f_{n-1} > f_n$ or $f_0 < f_1, f_1 > f_2, f_2 < f_3, f_3 > f_4, ...f_{n-1} < f_n$ if n is odd, and such that these are all comparabilities between the points. The points f_0 and f_n are the *endpoints* of the fence. A poset P is *connected* if and only if for all $p_1, p_2 \in P$, there is a fence $F \subset P$ with endpoints p_1, p_2. A poset that is not connected is called *disconnected* [6].

In this paper we provide insights into how to play on connected poset games. We do so by factoring/partitioning a connected poset A into subposets that give meaningful information about $\mathcal{G}(A)$ and the subgames of A. The work is related to playing games on the ordinal sum of posets (which is exactly the ordinal sum of poset games), but generalizes this idea. Let A and B be posets with disjoint underlying sets. Then the ordinal sum $A : B$ is the poset on $A \cup B$ with $x \leq y$ if, either $x, y \in A$ and $x \leq_A y$; or $x, y \in B$ and $x \leq_B y$; or $x \in A$ and $y \in B$. In other words any move in A, eliminates all possible moves in B for the remainder for the game. This concept has been generalized to impartial games. The following result is due to Fisher, Nowakowski and Santos [5] where, $mex(S, 0) = mex(S)$ and for $k \geq 1$, $mex(S, k) = mex(S')$, when

$$S' = S \cup \{mex(S, 0), mex(S, 1), ..., mex(S, k-1)\}.$$

Theorem 1 ([5])**.** *Let G, H be impartial games. Then, $\mathcal{G}(G : H) = mex(G^*, \mathcal{G}(H))$.*

In Section 2, we define a class of mapping between posets which preserves the underlying order of the poset in a useful way for studying poset games. This map is used to partition (or factor) the poset and establish a relationship between this partition and the nimbers in Section 3. Section 4 provides examples of applications of the results and Section 5 provides some concluding remarks and directions for future research.

2. Order Compressing Map

We define a function $f : P \to Q$ such that P and Q are partially ordered sets, to be *order compressing* if for all $x, y \in P$, $f(x) = f(y) = q \in Q$ if and only if for every $z \in P$:

- if $z <_P x$ and $z <_P y$, then $f(z) \leq_Q q$;
- if $z \not<_P x$ and $z \not<_P y$, then $f(z) \not<_Q q$; and
- if $\left\{ \begin{array}{c} z <_P x \text{ and } z \not<_P y; \\ z \not<_P x \text{ and } z <_P y \end{array} \right\}$, then $f(z) = q$.

Clearly each order compressing function is a homomorphism; however, every order compressing function is also order reflecting. The converse of this statement is false. Figure 5 gives an example of a homomorphism which is not order compressing.

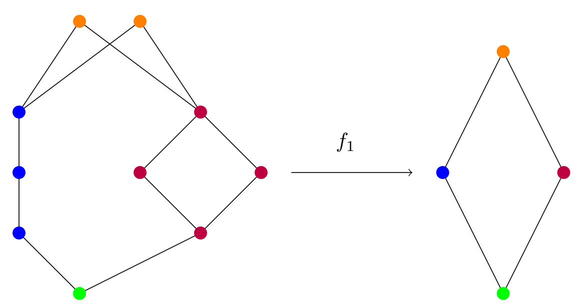

Figure 4: An order compressing homomorphism

When $f : P \to Q$ is order compressing, an *f-factor* of P is a subposet of P defined by $f^{-1}(x)$ for some $x \in Q$. The set of f-factors is a *Q-factorization* of P.

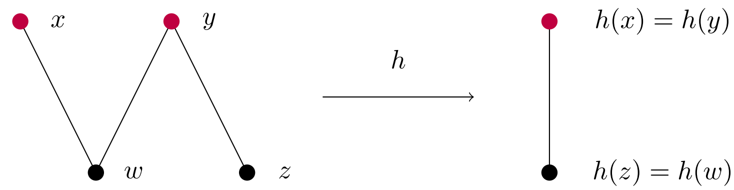

Figure 5: A map which is a homomorphism, but not order compressing

When the choice of f is clear from the context we call an f-factor of P a *factor of* P and Q-factorization of P a *factorization* of P.

3. Equivalencies of Games

In this section we establish two results allowing for the reduction of poset games to simpler poset games using order compressions. The first (Theorem 3) is a generalization of the colon principle originally given in [3]. The second (Theorem 4) is a result that deals with the interchangeability of special classes of subposets which are option equivalent.

Theorem 2. *Let A, B be posets, and $f : A \to Q$ and $g : B \to Q$ be order compressing maps. If x is maximal in A, $f(x) = \alpha$ and for all $\beta \in Q$ with $\beta \neq \alpha \in Q$, $f^{-1}(\beta) \cong g^{-1}(\beta)$, then $f^{-1}(\alpha) = g^{-1}(\alpha)$ implies $A = B$.*

Proof. Assume $f^{-1}(\alpha) = g^{-1}(\alpha)$ and consider $A + B$. It is sufficient to show $o(A + B) = \mathcal{P}$. Without loss of generality, for all moves on A not on $f^{-1}(\alpha)$ the second player will mirror the move in B. If a player moves on $f^{-1}(\alpha), g^{-1}(\alpha)$, then respond as if you were playing $f^{-1}(\alpha) + g^{-1}(\alpha)$. By the maximality of x any such move will not effect any $y \in f^{-1}(\beta)$ or $z \in g^{-1}(\beta)$. By induction, all of the above counter-move are winning moves. Thus, $o(A + B) = \mathcal{P}$. □

Theorem 3. *Let A, B be posets and $f : A \to Q$ and $g : B \to Q$ be order compressing maps. If x is maximal in A, $f(x) = \alpha$ and for all $\beta \in Q$ with $\beta \neq \alpha \in Q$, $f^{-1}(\beta) \cong g^{-1}(\beta)$, then $f^{-1}(\alpha) = g^{-1}(\alpha)$ if and only if $A = B$*

Proof. By Theorem 2 we only need to show if $A = B$, then $f^{-1}(\alpha) = g^{-1}(\alpha)$. Assume $A = B$ and consider $f^{-1}(\alpha) + g^{-1}(\alpha)$. It suffices to show $o(f^{-1}(\alpha) + g^{-1}(\alpha)) = \mathcal{P}$. Assume for the sake of contradiction that $o(f^{-1}(\alpha) + g^{-1}(\alpha)) = \mathcal{N}$. Then there exists an element z on $f^{-1}(\alpha)$ or $g^{-1}(\alpha)$ such that without loss of generality $o((f^{-1}(\alpha) - z_{\leq}) + g^{-1}(\alpha)) = \mathcal{P}$. By Theorem 2, this implies that playing z on $A + B$ is a winning move, which contradicts our assumption that $A = B$. Hence, no such z exists and as a result $o(f^{-1}(\alpha) + g^{-1}(\alpha)) = \mathcal{P}$. □

An example of an application of Theorem 3 is given in Figure 6, which depicts two Green Hackenbush positions (depicted as poset games). Theorem 3 implies that both of these positions have the same $\mathcal{G}$-value.

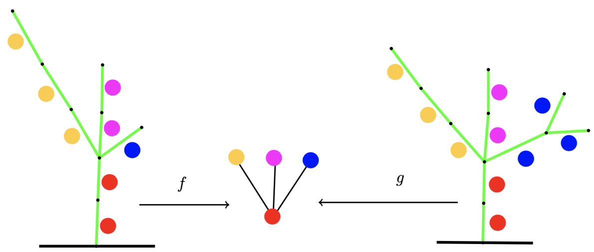

Figure 6: An example of Theorem 3 applied to a game of Green Hackenbush

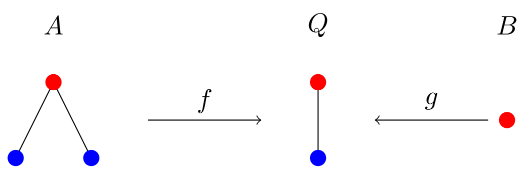

Figure 7: An example of why Theorem 3 requires x to be maximal

Figure 7 gives an example of a poset where the assumptions of Theorem 3 are satisfied except for the maximality of x in A. In this example, $A = *2$ and $B = *$ and hence the assumption that x is maximal may not be relaxed in Theorem 3. Figure 8, gives an example of a non-trival poset. The following Corollary implies this poset is a previous player win position.

Corollary 1. *Let $f : A \to Q$ be an order compressing map, if for all $\beta \in Q$ $\mathcal{G}(f^{-1}(\beta)) = 0$, then $A = 0$*

Proof. Let $A_1 = A \setminus f^{-1}(\alpha)$ where α is maximal in Q. By definition $\emptyset = 0$ so by Theorem 3, $A_1 = A$. We may now repeat this process on any maximal element of A_1. It follows by induction that $A = 0$. □

Theorem 4. *Let A, B be posets and let $f : A \to Q$ and $g : B \to Q$ be order compressing maps so that for all $\beta \in Q$, $f^{-1}(\beta) \equiv g^{-1}(\beta)$. Then $A \equiv B$.*

Proof. Let $\alpha_1, \alpha_2, \ldots, \alpha_d$ be the set of elements in Q such that $f^{-1}(\alpha_i) \not\cong g^{-1}(\alpha_i)$. Let A_i be the poset such that $h_i : A_i \to Q$ is an order compression and for all $j \leq i$, $h_i^{-1}(\alpha_j) \cong g^{-1}(\alpha_j)$ while for all $k > i$, $h_i^{-1}(\alpha_j) \cong f^{-1}(\alpha_j)$. We say $A \cong A_0$ and $B \cong A_d$.

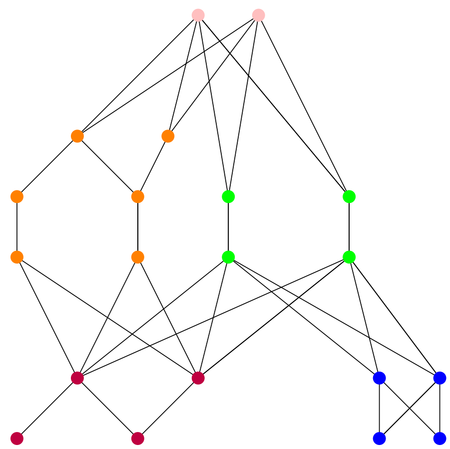

Figure 8: A $\mathcal{P}$-position given by Theorem 3

Consider A_i and A_{i+1}. If $|Q| = 1$, then the statement is trivial. Otherwise, without lose of generality for all moves on $h_i^{-1}(\alpha_{i+1})$ in A there exists a move on $h_{i+1}^{-1}(\alpha_{i+1})$ in B such that the two resulting games are equal by our assumption that $f^{-1}(\beta) \equiv g^{-1}(\beta)$ and Theorem 3. If either player moves anywhere else on A_i or A_{i+1} the resulting games are equal by induction as $A_i \equiv A_{i+1}$ implies $A_i = A_{i+1}$. Thus, $A_i \equiv A_{i+1}$. Hence, by the transitivity of $\equiv$, $A_0 \equiv A_d$. $\square$

4. Applications

In this section we provide two examples of applications of the results of the previous section. Consider the posets drawn in Figure 9. We claim these all have the same nimber.

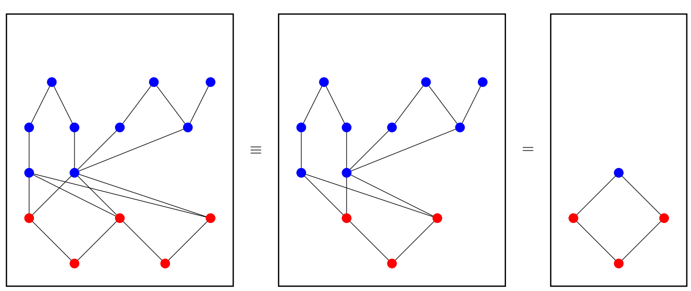

Figure 9: Three posets with nimber 3, given by a special case of Theorem 3 (Colon Principle) and Theorem 4

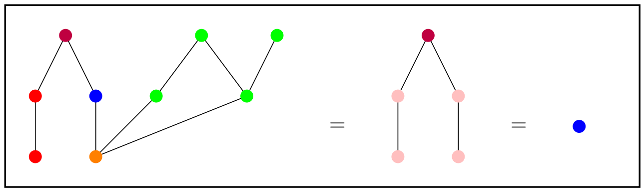

Figure 10: Equality of the subposet of blue elements from Figure 9 by Theorem 1 and Theorem 3

To establish the first equivalence shown in Figure 9, consider the subposets consisting of red elements. The option value set of these subposets is $\{0, 2\}$ and hence the equivalence follows from Theorem 4. The second equivalence in Figure 9 follows from applying Theorem 3 as we claim the subposets consisting of blue elements have nimber 1. To demostrate this, the subposet of blue elements is redrawn and recoloured in Figure 10. Observe that the subset of green elements in Figure 10 has nimber 0 and contains a maximal element of the poset. Theorem 3 now implies the first equivalence in Figure 10. The second equivalence in Figure 10 can be found by applying Theorem 1.

Theorem 4 points to the importance of determining the set of posets with a given option value set. A natural avenue of investigation is given a set of non-negative integers S, enumerate the number of games that have option value set S.

Theorem 5. *For all $S \subset \mathbb{N}_0$ such that $S \neq \{\}$, $\{P : P^* = S\} \neq \emptyset$ if and only if $|\{P : P^* = S\}|$ is non-finite.*

Proof. If $\{P : P^* = S\} = \emptyset$, then $|\{P : P^* = S\}| = 0$ trivially. Assume $Q \in \{P \in \mathbf{P} : P^* = S\}$. Note the identity map on Q is an order compressing function. For a given positive integer n, create Q_n by replacing a fixed but arbitrary $x \in Q$ with $2n + 1$ copies of itself, so that the copies form an antichain and when Q_n is order compressed to Q each element is sent to x. It follows from Theorem 4, that $Q \equiv Q_n$. As n can be any integer, this concludes the proof. □

5. Conclusion

This paper establishes equivalencies in ideal play between poset games that have obvious and not so obvious similarities. In particular, this work contributes to looking at ideal play on connected poset games. Whether the converse of Theorem 4 is true remains an open question. We end the paper asking a related question.

Let the *lexicographic product* of two posets, A and B, denoted here $A \otimes B$, be the poset given by the Cartesian Product $A \times B$ ordered by the following rule:

$(a, b) \leq (a', b')$ if and only if

$$\bullet a \leq_A a' \qquad \text{and} \qquad \bullet \text{ if } a = a', \text{then } b \leq_B b'.$$

We note that there is a natural order compressing function $f : A \otimes B \to A$ where each f-factor of $A \otimes B$ is isomorphic to B.

Conjecture 1. Let A, B be poset games, such that $\mathcal{G}(B) = 2^n$ where $n \in \mathbb{N}_0$ and B is weakly-canonical. Then, for all $(a, b) \in A \otimes B$,

$$\mathcal{G}(A \otimes B - (a, b)_{\leq}) = \mathcal{G}(B)\mathcal{G}(A - a_{\leq}) + \mathcal{G}(B - b_{\leq})$$

where multiplication and addition are standard for integers.

If true, this would imply that for the lexicographic product, $A \otimes B$, of any two posets that satisfied the assumptions of Conjecture 1, $\mathcal{G}(A \otimes B) = \mathcal{G}(A)\mathcal{G}(B)$. Moreover, this implies that if the left factor A has nimber 0, then $\mathcal{G}(A \otimes B) = 0$. Note that if $B = 0$ and B are weakly-canonical, $A \otimes B = \emptyset$. Verifying Conjecture 1 would point to the possibility of the existence of other equations like that of the ordinal sum, such as Theorem 1, existing.

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