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CIRCULAR NIM GAMES CN(7,4)

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Abstract

Circular Nim is a two-player impartial combinatorial game consisting of n stacks of tokens placed in a circle. A move consists of choosing k consecutive stacks and taking at least one token from one or more of the stacks. The last player able to make a move wins. The question of interest is: Who can win from a given position if both players play optimally? This question is answered by determining the set of $\mathcal{P}$-positions, those from which the next player is bound to lose, no matter what moves s/he makes. We will completely characterize the set of $\mathcal{P}$-positions for $n = 7$ and $k = 4$, adding to the known results for other games in this family. The interesting feature of the set of $\mathcal{P}$-positions of this game is that it splits into different subsets, unlike the structures for the previously solved games in this family.

1. Introduction

The game of Nim has been played since ancient times, and the earliest European references to Nim are from the beginning of the sixteenth century. Its current name was coined by Charles L. Bouton of Harvard University, who also developed the complete theory of the game in 1902 [3]. Nim plays a central role among impartial games as any such game is equivalent to a Nim stack [2]. Many variations and generalizations of Nim have been analyzed. They include subtraction games, Wythoff's game, Nim on graphs and on simplicial complexes, Take-away games, Fibonacci Nim, etc. [1, 5, 6, 8–13, 15, 16]. We will study a particular case of another variation, called Circular Nim, which was introduced in [4]. This game imposes a geometric structure on Nim heaps which gives rise to interesting features in the set of $\mathcal{P}$-positions.

Definition 1. In Circular Nim, n stacks of tokens are arranged in a circle. A move consists of choosing k consecutive stacks and then removing at least one token from at least one of the k stacks. Players alternate moves and the last player who is able to make a legal move wins. We denote this game by $\mathrm{CN}(n,k)$. A *position* in $\mathrm{CN}(n,k)$ is represented by the vector $\boldsymbol{p} = (p_1, p_2, \ldots, p_n)$ of non-negative entries indicating the heights of the stacks in order around the circle. An *option* of $\boldsymbol{p}$ is a position to which there is a legal move from $\boldsymbol{p}$. We denote an option of $\boldsymbol{p}$ by $\boldsymbol{p}' = (p'_1, p'_2, \ldots, p'_n)$, and use the notation $\boldsymbol{p} \to \boldsymbol{p}'$ to denote a legal move from $\boldsymbol{p}$ to $\boldsymbol{p}'$.

Note that a position in Circular Nim is determined only up to rotational symmetry and reflection (reading the position forward or backward). The only *terminal position* of $\mathrm{CN}(n,k)$ is $\mathbf{0} := (0,0,\ldots,0)$, for all n and k. Figure 1 shows an example of the position $\boldsymbol{p} = (1,7,5,6,2,3,6) \in \mathrm{CN}(7,4)$ and one possible move, to option $\boldsymbol{p}' = (0,1,5,4,2,3,6)$, where the four stacks enclosed by squares are the stacks that were selected for play. Note that no tokens were taken from the stack of height 5. We will use *play on a stack* to mean that the stack is one of the selected ones, whether actual tokens were removed or not.

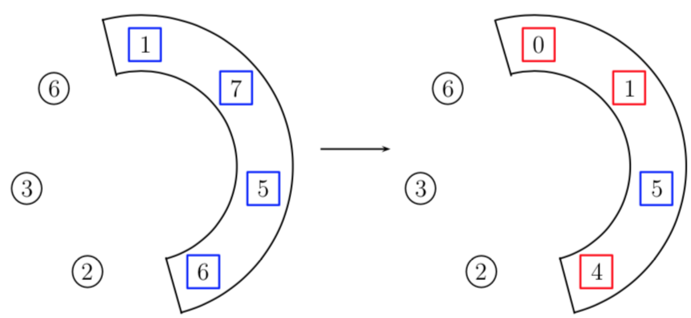


Figure 1: A move from $\boldsymbol{p} = (1,7,5,6,2,3,6)$ to $\boldsymbol{p}' = (0,1,5,4,2,3,6)$.

Circular Nim is an example of an *impartial* combinatorial game, one for which both players have the same moves. For impartial games, there are only two types of positions (= outcomes classes). The outcome classes are described from the standpoint of which player will win when playing from the given position. An $\mathcal{N}$-position indicates that the **N**ext player to play from the current position can win, while a $\mathcal{P}$-position indicates that the **P**revious player, the one who made the move to the current position, is the one to win. Thus, the current player is bound to lose from this position, no matter what moves she or he makes. A winning strategy for a player in an $\mathcal{N}$-position is to move to one of the $\mathcal{P}$-positions. More background on combinatorial games can be found in [1, 2].

Dufour and Heubach [4] proved general results on the set of $\mathcal{P}$-positions of $\mathrm{CN}(n, 1)$, $\mathrm{CN}(n, n)$, and $\mathrm{CN}(n, n-1)$ for all n. These general cases cover all games for $n \leq 3$. They also gave results for all games with $n \leq 6$ except for $\mathrm{CN}(6, 2)$, and also solved the game $\mathrm{CN}(8, 6)$. In this paper, the main result is on the $\mathcal{P}$-positions for $\mathrm{CN}(7, 4)$. One sign of the increase in complexity as n and k increase is that, unlike in the results for the cases already proved, we no longer can describe the set of $\mathcal{P}$-positions as a single set, which makes the proofs more complicated.

To prove our main result, we use the following theorem.

Theorem 1 (Theorem 2.13, [1]). *Suppose the positions of a finite impartial game can be partitioned into mutually exclusive sets A and B with these properties:*

I. *Every option of a position in A is in B; and*

II. *Every position in B has at least one option in A.*

Then A is the set of $\mathcal{P}$-positions and B is the set of $\mathcal{N}$-positions.

2. The Game CN(7, 4)

In the discussion of $\mathrm{CN}(7, 4)$, we will use the generic position $\boldsymbol{p} = (a, b, c, d, e, f, g)$. Since positions of $\mathrm{CN}(7, 4)$ are only determined up to rotation and reflection, we will assume without loss of generality that in a generic position a is a minimum. Figure 2 shows a generic position (a, b, c, d, e, f, g) where the minimum stack is rendered in red (gray). Note that to avoid cumbersome notation, we will use the label, say a, to refer to either the stack itself or to its number of tokens - which one it is will be clear from the context.

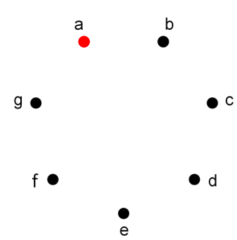
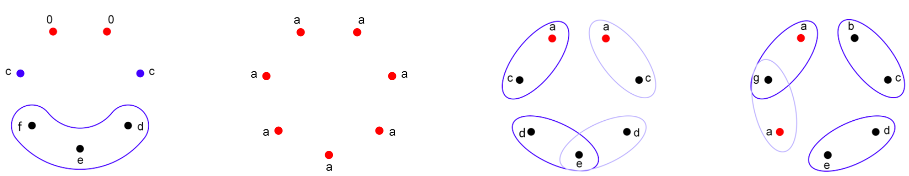


Figure 2: A generic position in the game $\mathrm{CN}(7, 4)$, with $a = \min(\boldsymbol{p})$.

Here is our main result, with a visualization of the $\mathcal{P}$-positions of $\mathrm{CN}(7, 4)$ given in Figure 3. In this figure, we highlight the sum conditions by encircling stacks whose sums have to be equal in dark blue, and color any stacks that equal the sums in the same color. Pairs of stacks that also have the same sum, but for which this is true due to some symmetry, are encircled in lighter blue.

Theorem 2. *Let* $\boldsymbol{p} = (a, b, c, d, e, f, g)$ *with* $a = \min(\boldsymbol{p})$. *The* $\mathcal{P}$*-positions of* $\mathrm{CN}(7, 4)$ *are given by* $S = S_1 \cup S_2 \cup S_3 \cup S_4$, *where:*

- $S_1 = \{\boldsymbol{p} \mid a = b = 0,\ c = g > 0,\ d + e + f = c\}$.
- $S_2 = \{\boldsymbol{p} \mid \boldsymbol{p} = (a, a, a, a, a, a, a)\}$.
- $S_3 = \{\boldsymbol{p} \mid a = b,\ c = g,\ d = f,\ a + c = d + e,\ \ 0 < a < e\}$, *and*
- $S_4 = \{\boldsymbol{p} \mid a = f,\ b + c = d + e = g + a,\ a < \min\{b,\ e\}, a < \max\{c, d\}\}$.



(a) $\mathcal{P}$-positions in S_1 (b) $\mathcal{P}$-positions in S_2 (c) $\mathcal{P}$-positions in S_3 (d) $\mathcal{P}$-positions in S_4

Figure 3: Visualization of the $\mathcal{P}$-positions of $\mathrm{CN}(7, 4)$.

Note that all the subsets of S are disjoint. The condition $a < \max\{c, d\}$ of S_4 prohibits a pair of adjacent minima, which all other sets have. Also, S_2 is disjoint from the other sets because they all have a strict inequality condition. Finally, $S_1 \cap S_3 = \varnothing$ since $a > 0$ for S_3.

The following definitions and remarks will aid us in the proofs of our results. Note that we assume a to be the minimum, not necessarily unique. In the proofs, we will typically denote the minimal and maximal values of a target position $\boldsymbol{p}$' by m and M, respectively.

Definition 2. A *tub configuration* $xaax$ is a set of four adjacent stacks that consists of a pair of adjacent minima (of the position) surrounded by two stacks of equal height. There are three other stacks in the position, which we denote by $x_1x_2x_3$ unless we know the actual stack heights. A *peak configuration* is of the form xXx, a set of three adjacent stacks with $x < X$. If x and X are the minimum and the maximum, respectively, of the position, then we call this configuration a *minmax peak*. A position with a peak contains four other stacks which we denote by $x_1x_2x_3x_4$

unless we know the actual stack heights. Finally, a position has the *common sum requirement* if consecutive disjoint pairs of adjacent stacks have to have the same sum, with one exceptional *overlap stack* contributing to two sums.

With these definitions, we can make the following remarks regarding the specific features of each subset of S.

Remark 1.

(1) In S_3, $a < e$ and the sum conditions imply that $c > \max\{a, d\}$ and $c \geq e$.

(2) In S_4, we have the following inequalities: $a < \min\{b, e\}$ implies that $g > \max\{c, d\}$ due to the common sum requirement. Furthermore, $g > a$.

(3) Positions in $S_1 \cup S_3$ contain a tub configuration. We have

- $\boldsymbol{p} \in S_1$ needs to satisfy the *tri-sum condition*: $x_1 + x_2 + x_3 = x$
- $\boldsymbol{p} \in S_3$ needs to satisfy: $x_1 = x_3$ and $x_2 + x_3 = a + x$.

(4) When trying to move from a position $\boldsymbol{p}$ with $a = \min(\boldsymbol{p}) > 0$ to $\boldsymbol{p}' \in S_1 \cup S_3$, we can create a tub configuration with minimum $a' < a$ by selecting a pair of stacks, making them height a' and then reducing the larger of the two stacks adjacent to the pair of a'-stacks to the height of the smaller one (if needed). This height gives the value of x in the tub configuration $xa'a'x$. Any remaining play has to occur on x_1, the stack adjacent to the stack that was decreased to x. In labeling the three non-tub configuration stacks, we read $\boldsymbol{p}'$ starting from the minima a' in the direction of the stack whose height was reduced to x. (If the two stacks adjacent to the pair $a'a'$ were already of equal height then either one of the stacks adjacent to the x stacks can play the role of x_1.) Note that we cannot play on the remaining two stacks x_2 and x_3, so we may not be able to meet the remaining conditions of S_1 or S_3, respectively.

(5) Positions in S_4 always contain a minmax peak, while positions in S_3 may contain a peak. In either case, the remaining four stacks have to satisfy that $x_1 + x_2 = x_3 + x_4 = x + X$.

(6) Positions in S_4 have either two or three minima. If $c = a = m$, then $\boldsymbol{p} = (m, M, m, d, e, m, M)$, that is, two maxima alternate with three minima. Otherwise, the two minima are separated by the maximum.

(7) The common sum requirement is a condition of $S_3 \cup S_4$ and is trivially satisfied for positions in S_2. It is relatively easy to see that we cannot make a move from a position $\boldsymbol{p}$ that satisfies the common sum requirement to $\boldsymbol{p}'$ which also satisfies the common sum requirement if the location of the overlap stack remains the **same**. In that case, at least one sum remains unchanged while

at least one other sum is decreased. Specifically, there is no move from S_2 to $S_3 \cup S_4$.

We are now ready to embark on the proofs.

2.1. There Is No Move from $\boldsymbol{p} \in \boldsymbol{S}$ to $\boldsymbol{p}' \in \boldsymbol{S}$

Proposition 1. *If* $\boldsymbol{p} \in S$*, then* $\boldsymbol{p}' \notin S$*.*

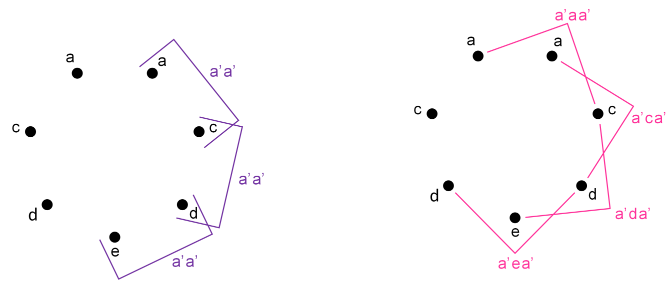
Proof. To prove condition (I) of Theorem 1 we will use the equivalent statement that there is no move from a $\mathcal{P}$-position to another $\mathcal{P}$-position. For each of the four subsets of S, we consider moves to all the other sets. Note that the only terminal position is $\mathbf{0} \in S_2$.

Moves from S_1**:** We start with $\boldsymbol{p} = (0, c, d, e, f, c, 0) \in S_1$, with $d + e + f = c$. Note that we cannot move to $\boldsymbol{p}' \in S_1 \cup S_2$ because in either case, we would have to play on the five stacks $cdefc$ to simultaneously reduce the c stacks and the sum to a new value $c' < c$ in the case of S_1 and $c' = 0$ in the case of S_2. A move to S_3 is not possible since the minimum in S_3 is greater than zero. A move to $\boldsymbol{p}' \in S_4$ is not possible because S_4 does not have adjacent minima by Remark 1(6). Thus, no move is possible from S_1 to S.

Moves from S_2**:** Now assume that $\boldsymbol{p} = (a, a, a, a, a, a, a) \in S_2$ with $a > 0$ because $\boldsymbol{p}$ is the terminal position for $a = 0$. To move to S_1, we have to create a tub configuration of the form $x00x$, which requires play on three stacks (even though we remove tokens from only two stacks). We can at most reduce one of the three remaining stacks $x_1x_2x_3 = aaa$, so the sum $x_1 + x_2 + x_3 \geq 2a$, while $x = a$, so there is no move from S_2 to S_1. Clearly, one cannot move from S_2 to S_2. By Remark 1(7) there is no move from S_2 to $S_3 \cup S_4$.

Moves from S_3**:** Let $\boldsymbol{p} = (a, a, c, d, e, d, c) \in S_3$ with $a + c = d + e$. To move to $S_1 \cup S_3$, we have to create a tub configuration of the form $xa'a'x$, with $a' = 0$ for $\boldsymbol{p}' \in S_1$ and $a' \leq a$ for $\boldsymbol{p}' \in S_3$. First we consider play when the minima a' of $\boldsymbol{p}'$ are located at the a stacks. For a move to S_1, we play on both a stacks making them zero, and then either reduce both c stacks or one of the d stacks, but not both. In either case, we have that $x \leq c$ and the tri-sum $d' + e + d \geq d + e = a + c > c$, so the tri-sum condition is not satisfied. For a move to S_3, the overlap stack remains at the same location, and by Remark 1(7), there is no move to S_3.

Now we look at the cases where we create a tub configuration $xa'a'x$ elsewhere. In each case, we use play on three stacks as described in Remark 1(4). By symmetry of positions in S_3 we have to consider the three possibilities indicated in Figure 4a. They are $x = a$ with $x_1x_2x_3 = edc$, $x = a$ with $x_1x_2x_3 = dca$, or $x = d$ with $x_1x_2x_3 = aac$ (since $c > d$ by Remark 1(1), so we read counter-clockwise). By Remark 1(3), we need to satisfy the conditions $x_1 + x_2 + x_3 = x + 0 = x + a'$ for $\boldsymbol{p} \in S_1$ and both $x_1 = x_3$ and $x_2 + x_3 = x + a'$ for $\boldsymbol{p} \in S_3$. We will show that even if we reduce x_1 to zero, we will not be able to satisfy the respective sum conditions. When $x = a$, then $x_2 + x_3 \geq \min\{d + c, c + a\} > a + a' = x + a'$, and for $x = d$, $a + c = d + e > d + a' = x + a'$. Thus, $\boldsymbol{p}' \notin S_1 \cup S_3$. It is also not possible to move to $\boldsymbol{p}' \in S_2$, since by Remark 1(1), $\min\{c, e\} > a$, so we would need to play on five stacks to reduce $cdedc$ to $aaaaa$.



(a) (b)

Figure 4: Visualization of moves from S_3 to (a) $S_1 \cup S_3$ (b) S_4.

To show that we cannot move from S_3 to S_4, we consider the possible locations of the minmax peak of $\boldsymbol{p}'$. Due to symmetry of positions in S_3, we need to consider the four possible peak configurations shown in Figure 4b: $a'aa'$ with sums $d+e \leq d+c$, $a'ca'$ with sums $e + d = c + a$, $a'da'$ with sums $d + c > a + a$, or $a'ea'$ with sums $c + a$ (in both cases). Note that in the first three cases, we have $a' < a$ because the minimum of the minmax peak in S_4 has to be strictly less than the adjacent stacks, and in each of these cases, the a stack is one of them. We can play on one more stack adjacent to the a' stacks and we play on the stack that affects the larger sum. In the first two cases, the peak sum is smaller than the smaller of the two sums, and since we can adjust only one sum, we cannot legally move to $\boldsymbol{p}' \in S_4$. For the third case, equality with the peak sum requires that $d' + c = d + a'$ and hence $d' = d - c + a' < a'$ because $c > d$ by Remark 1(1). For the last case, the overlap stack is at the same location in $\boldsymbol{p}$ and $\boldsymbol{p}'$, so by Remark 1(7), we cannot adjust all four sums with play on only four stacks. This shows that we cannot move to $\boldsymbol{p}' \in S_4$.

Moves from S_4: Finally, we check whether we can move from $\boldsymbol{p} = (a, b, c, d, e, a, g) \in S_4$ to $\boldsymbol{p}' \in S$. The approach is similar to the one taken when $\boldsymbol{p} \in S_3$. For a move to $\boldsymbol{p}' \in S_1 \cup S_3$, we once more need to create a tub configuration $xa'a'x$, where $a' \leq a$, and $a' = 0$ for moves to S_1. Due to the semi-symmetric nature of positions in S_4, we now need to consider all seven placements of the new pair of minima. We start by putting them at stacks a and b and get the following cases: $x = c, x_1x_2x_3 = aed$ (since we have to reduce g), $x = a, x_1x_2x_3 = eag$, $x = \min\{b, e\}, x_1x_2x_3 = aga$ (no matter which side we need to play on), $x = a, x_1x_2x_3 = bag$ (since we need to play on c), $x = d, x_1x_2x_3 = abc$, $x = a, x_1x_2x_3 = dcb$, and $x = a, x_1x_2x_3 = cde$.

First we look at the cases where $x = a$. Reducing x_1 to zero, we have that $x_2+x_3 = a+g = c+b = e+d > a+a \geq a+a'$, so the sum conditions of S_1 and S_3 are not satisfied. Likewise, for $x = c$, we have that $x_2+x_3 = e+d = c+b > c+a \geq c+a'$, and for $x = d$, we obtain $x_2 + x_3 = b + c = d + e > d + a \geq d + a'$. Finally, for $x = \min\{b, e\}$, we have that $x_2+x_3 = g+a = \min\{b, e\}+\max\{d, c\} > \min\{b, e\}+a \geq \min\{b, e\} + a'$, so we cannot move to $\boldsymbol{p}' \in S_1 \cup S_3$.

Next we look at moves from S_4 to S_2. Since $a < \min\{b, e, g\}$, we have to reduce at least those three stacks to a which requires play on five stacks. Therefore we cannot move from S_4 to S_2.

Finally, we look at moves from S_4 to S_4. If we keep the location of the minima and hence the overlap stack, then by Remark 1(7) there is no move to $\boldsymbol{p}' \in S_4$. Thus we need to consider whether we can create a minmax peak $a'Xa'$ with $a' < a$ and remaining stacks $x_1x_2x_3x_4$ which satisfy $x_1 + x_2 = x_3 + x_4 = a' + X$ by Remark 1(5). We can play on either x_1 or x_4, but in either case we can only modify one of the two sums $x_1 + x_2$ and $x_3 + x_4$. The common sum for $\boldsymbol{p}$ is $s = g + a$, while for $\boldsymbol{p}'$ it is $s' = X + a' < s$. Furthermore, x_2 and x_3 cannot be adjusted. Let us look at the possible cases, going clockwise and starting with new minimia at the g and b stacks, for a total of six cases: (1) $X \leq a$ and $x_1x_2x_3x_4 = cdea$; (2) $X \leq b$ and $x_1x_2x_3x_4 = deag$; (3) $X \leq c$ and $x_1x_2x_3x_4 = eaga$; (4) $X \leq d$ and $x_1x_2x_3x_4 = agab$; (5) $X \leq e$ and $x_1x_2x_3x_4 = gabc$; and (6) $X \leq a$ and $x_1x_2x_3x_4 = abcd$. In cases (1) and (3), $x_3 > X$, while in cases (4) and (6), $x_2 > X$, either directly from the definition of positions in S_4 or by Remark 1(2). For the remaining two cases, (2) and (5), we have that $x_1 + x_2 = x_3 + x_4 = g + a = s > s'$ and we can adjust only one of the two sums. This shows that there is no move from S_4 to S_4, and thus shows that there is no move from S to S, completing the proof. □

2.2. There Always Is a Move from $\boldsymbol{p} \in S^c$ to $\boldsymbol{p}' \in S$

We now show the second part of Theorem 1.

Proposition 2. *If $\boldsymbol{p} \in S^c$, then there is a move to $\boldsymbol{p}' \in S$.*

To show that we can make a legal move from any position $\boldsymbol{p} \in S^c$ to a position $\boldsymbol{p}' \in S$, we partition the set S^c according to the number of zeros of $\boldsymbol{p}$ and, for positions without a zero stack, according to the locations of the maximal stacks. We will only need to distinguish between the case of exactly one zero and the case of at least two zeros. Note that in [14], S^c was partitioned according to the exact number of minima of $\boldsymbol{p}$. The proof presented here is shorter and uses some of the ideas from [14], such as Definition 3 and Lemma 1. We call out these structures and CN$(3, 2)$-equivalence (defined below) because they give insight into stack configurations from which it is easy to move to $\mathcal{P}$-positions.

Definition 3. A position $\boldsymbol{p}$ is called *deep-valley* if and only if five consecutive stacks $p_1p_2p_3p_4p_5$ satisfy $p_2+p_3+p_4 \le \min\{p_1,p_5\}$. It is called *shallow-valley* if and only if $p_1 \le p_5$ and $p_2+p_3 \le p_1 < p_2+p_3+p_4$.

Lemma 1 (Valley Lemma)**.** *If* $\boldsymbol{p} = (p_1,p_2,p_3,p_4,p_5,p_6,p_7)$ *is* deep-valley *with stacks* $p_1p_2p_3p_4p_5$ *and* $s = p_2+p_3+p_4$*, then there is a move to* $\boldsymbol{p}' = (s,p_2,p_3,p_4,s,0,0) \in S_1$*. On the other hand, if* $\boldsymbol{p}$ *is* shallow-valley *with* $p_1p_2p_3p_4p_5$*, then there is a move to* $\boldsymbol{p}' = (p_1,p_2,p_3,p_1-(p_2+p_3),p_1,0,0) \in S_1$*.*

Proof. If $\boldsymbol{p}$ is deep-valley, then $p'_1 = p'_5 = p_2+p_3+p_4 \le \min\{p_1,p_5\}$, so it follows that $\boldsymbol{p} \to \boldsymbol{p}' \in S_1$ is a legal move. If $\boldsymbol{p}$ is shallow-valley, then $p'_1 = p'_5 = p_1 \le p_5$, $p_1-(p_2+p_3) \ge 0$, and $p_4 \ge p'_4 = p_1-(p_2+p_3)$. Also, $p_1-(p_2+p_3)+p_2+p_3 = p_1$, $\boldsymbol{p} \to \boldsymbol{p}' \in S_1$ is a legal move. □

The notion of CN$(3,2)$-equivalence comes into play when $\boldsymbol{p}$ contains zero stacks. It builds on the structure of the $\mathcal{P}$-positions of CN$(3,2)$, which are those with equal stack heights (see either [4] or convince yourself easily with a one-line proof). Note that the definition below is not specific to the game CN$(7,4)$.

Definition 4. A position $\boldsymbol{p}$ of a CN(n,k) game is CN$(3,2)$-*equivalent* if the stacks of $\boldsymbol{p}$ can be partitioned into (mutually exclusive) subsets A_1, A_2, and A_3 together with a set (or sets) of consecutive zero stacks, where A_1, A_2, and A_3 satisfy the following conditions:

(1) Any pair of the three sets A_1, A_2, and A_3, together with any zero stacks that are between them, are contained in k consecutive stacks;

(2) Any move that involves at least one stack from each of the three sets A_1, A_2, and A_3 requires play on at least $k+1$ consecutive stacks, thus is not allowed.

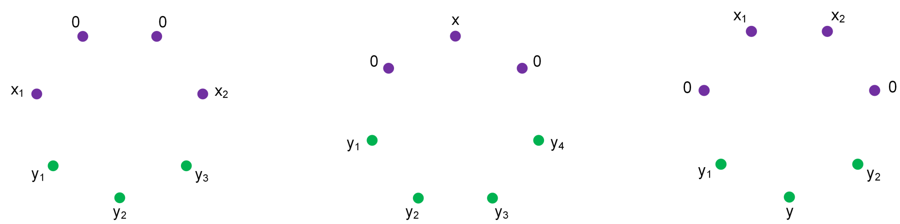
We define the *set sums* $\tilde{p}_i = \sum_{p_j \in A_i} p_j$ and call a move a CN$(3,2)$ *winning move* if play on the stacks in the sets A_i results in equal set sums in $\boldsymbol{p}'$. A CN$(3,2)$-equivalent position that has equal set sums is called a CN$(3,2)$-*equivalent* $\mathcal{P}$*-position.*

CN$(3,2)$-equivalent positions are perfectly suited to moves to S_1 since the conditions on the non-zero stacks in S_1 require equality of the tri-sum and the two adjacent stack heights (set sum of a single stack). But we will also see that a CN$(3,2)$ winning move can be used when there are additional inequality conditions on some of the stacks as long as those conditions can be maintained. In other instances, the sum conditions may involve a stack outside the three sets, but the sum condition can be achieved without play on that "outside" stack.

The proof of Proposition 2 will proceed as a sequence of lemmas where we will consider the individual cases according to the number of zeros and the location(s) of the maximum values in the case when the position does not have a zero. We start by dealing with positions that have at least two zero stacks.

Lemma 2 (Multiple Zeros Lemma)**.** *If $\boldsymbol{p} \in S^c$ and $\boldsymbol{p}$ has at least two stacks without tokens, then there is a move to $\boldsymbol{p}' \in S_1 \cup S_2 \cup S_4$.*

Proof. Note that we will label the individual stacks as x, x_i, y, and y_j depending on the symmetry of the position as well as the role the different stacks play. Typically, stacks labeled x or x_i are between zeros (short distance) or adjacent to zeros. Since the positions in $S_1 \cup S_3 \cup S_4$ all have sum conditions that need to be satisfied, we will typically use s to denote this target sum. We consider the case of two adjacent zeros, two zeros separated by one stack and finally two zeros separated by two (or three) stacks, where distance is always assumed to be the shorter distance between any stacks. Figure 5 shows the generic positions in each of the cases. Any position with at least three zeros falls into either case (a) or case (b).



(a) (b) (c)

Figure 5: Generic positions with at least two zeros. (a) Two consecutive zeros. (b) Two zeros separated by one stack. (c) Two zeros separated by two stacks.

First, suppose there are two consecutive zeros in the position as shown in Figure 5a. Note that $\boldsymbol{p} = (x_1, 0, 0, x_2, y_3, y_2, y_1)$ is CN(3, 2)-equivalent with sets $A_1 = \{x_1\}$, $A_2 = \{x_2\}$, and $A_3 = \{y_1, y_2, y_3\}$. Thus we can make the CN(3, 2) winning move to $\boldsymbol{p}' \in S_1 \cup S_2$ by adjusting the stacks in two of the A_i to make the set sums in $\boldsymbol{p}'$ equal to the minimal set sum in $\boldsymbol{p}$. This can be achieved with play on four stacks or fewer. Note that we move to S_2 when either x_1 or x_2 (or both) equal zero, that is, we have at least three consecutive zeros, and $\boldsymbol{p}'$ is the terminal position in that case.

Now we can assume that any zeros in $\boldsymbol{p}$ are isolated, that is, they are either separated by one stack or by two stacks. Let us first consider the case of two zeros separated by a single stack, that is, $\boldsymbol{p} = (0, x, 0, y_1, y_2, y_3, y_4)$ with $\min\{x, y_1, y_4\} > 0$ because of the isolated zero condition (see Figure 5b). Our goal is to move to S_4. Due to the zeros (which will also be the minima in $\boldsymbol{p}'$), the sum conditions of S_4

reduce to $x' = y_1' + y_2' = y_3' + y_4'$, with $\min\{y_1', y_4'\} > 0$. Since $\boldsymbol{p}$ is CN$(3,2)$-equivalent with sets $A_1 = \{x\}$, $A_2 = \{y_1, y_2\}$, and $A_3 = \{y_3, y_4\}$, and we can make the CN$(3,2)$ winning move to $\boldsymbol{p}'$. Note that we can achieve the condition $\min\{y_1', y_4'\} > 0$ because the original stacks were non-zero, and any set of two stacks that is being played on can be adjusted to achieve the desired sum without making y_1 or y_4 equal to zero because $x > 0$ by assumption of the isolated zeros. However, if in the process we need to make $y_2' = y_3' = 0$, then the resulting position is in S_1.

Now we turn to the case where the zeros are separated by two stacks, that is, $\boldsymbol{p} = (0, x_1, x_2, 0, y_2, y, y_1)$, with $\min\{x_1, x_2, y_1, y_2\} > 0$ since we assume isolated zeros (see Figure 5c). We also assume without loss of generality that $y_2 \geq y_1$. Now we need to consider two subcases: $y_1 \geq x_1$ and $y_1 < x_1$. Note that for each of the subcases, the sum s will be defined on a case-by-case basis.

In the first case, we let $s = \min\{x_1+x_2, y_1\}$ and move to $\boldsymbol{p}' = (0, x_1, x_2', 0, s, 0, s) \in S_4$ with $x_1 + x_2' = s$. While this looks like there is play on five stacks, either x_2 or y_1 will remain the same. If $s = y_1$, then play is on the $x_2, 0, y_2$ and y stacks, and because $x_1 \leq y_1$, we have $x_2' = s - x_1 = y_1 - x_1 \geq 0$. If $s = x_1 + x_2$, then play is on the three y stacks.

Now we look at $y_1 < x_1$, which is a little bit more involved. Here our goal is to move to S_1, so we need to create a pair of zeros. Since $y_2 \geq y_1$, we choose $x_2' = 0$ and show that we can make x_1', y_2', and the tri-sum $0 + y_1' + y'$ equal in $\boldsymbol{p}'$. Let $s = \min\{x_1, y_1 + y, y_2\}$. Unless $s = y_2$ with $y > y_2$, we can move to $\boldsymbol{p}' = (s, 0, 0, s, y', y_1', 0)$ with $y' + y_1' = s$ by playing on at most four stacks as follows: If $s = x_1$, then play is on stacks x_2, y_2, and y, with $y' = s - y_1 = x_1 - y_1 > 0$. If $s = y_1 + y$, then play is on stacks x_1, x_2, and y_2. Finally, if $s = y_2 \geq y$, then play is on stacks x_2, x_1, and y_1, with $y_1' = y_2 - y \geq 0$.

This leaves the case of $y_1 < x_1$, $y_1 \leq y_2$, $y_2 < \{x_1, y_1 + y\}$ with $y > y_2$ unresolved. This set of inequalities can be simplified to $y_1 \leq y_2$, $y_1 < x_1$, and $y_2 < \{x_1, y\}$. Note specifically that $y > y_i$ for $i = 1, 2$. We need to make further distinctions as to where the maximal value occurs. In all cases we will move to S_1, but the location of the maximal value determines where the pair of adjacent zeros is created. Let $M = \max(\boldsymbol{p}) = \max\{x_1, x_2, y\}$ (all other stacks cannot be maximal due to the inequalities).

First we consider the case where the maximal value occurs next to a zero, that is, $M = x_1$ or $M = x_2$. Let $s = \min\{x_1 + y_1, x_2 + y_2, y\}$ and assume that $M = x_1$. We claim that there is a legal move to $\boldsymbol{p}' \in S_1$ where $\boldsymbol{p}' = (0, s, x_2', 0, y_2, s, 0)$ with $x_2' + y_2 = s$. Note that $M = x_1$ implies that $s < x_1 + y_1$ because $s = x_1 + y_1$ leads to a contradiction. Specifically, because $y_i > 0$ due to isolated zeros, we would have $x_1 < x_1 + y_1 = s \leq y \leq M = x_1$. If $s = x_2 + y_2$, then play is on stacks x_1, 0, y_1, and y and it is a legal move since $x_1 = M \geq y \geq s$. If $s = y$, then play is on stacks y_1, 0, x_1, and x_2, with $x_2' = s - y_2 = y - y_2 > 0$. Since $y > y_i$, the same proof, except with subscripts 1 and 2 changing places, applies when $M = x_2$.

The final case is when $M = y > \max\{x_1, x_2\}$. We first consider $x_1 > x_2$ and let $s = \min\{x_1, x_2 + y_2\}$. Then the move is to $\boldsymbol{p}' = (0, s, x_2, 0, y_2', s, 0) \in S_1$ with $x_2 + y_2' = s$. If $s = x_1$, then play is on y_1, y, and y_2. The move is legal since $y > x_1$ and $y_2' = x_1 - x_2 > 0$. On the other hand, if $s = x_2 + y_2$, then play is on stacks y, y_1, 0, and x_1 and $y > x_1 > s$. When $x_1 \le x_2$, then the move is to $\boldsymbol{p}' = (0, x_1, s, 0, 0, s, y_1') \in S_1$ with $y_1' + x_1 = s$ and $s = \min\{x_2, x_1 + y_1\}$. The proof follows like in the case $x_1 > x_2$. This completes the case of two zeros that are two stacks apart, and therefore, the case of more than two zeros. □

We next consider the case of a single isolated zero.

Lemma 3 (Unique Zero Lemma)**.** *If a position* $\boldsymbol{p} \in S^c$ *has a unique zero, then there is a move to* $\boldsymbol{p}' \in S$.

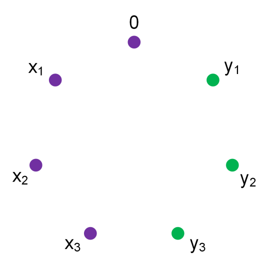


Figure 6: Generic position with a unique zero.

Proof. The generic position for this case is shown in Figure 6. Note that due to the assumption of the unique zero, we have that all other stack heights are non-zero, so $x_i > 0$ and $y_i > 0$ for $i = 1, 2, 3$. We may also assume without loss of generality that $x_2 \ge y_2$. We will see that in almost all cases, we can move to S_1; there is a single subcase where we will move to S_4. Table 1 gives a quick overview of the structure of the subcases.

$x_1 + y_1 \le \min\{x_2, y_2\} = y_2$			(a)	$\boldsymbol{p}' \in S_1$
$x_1 + y_1 > y_2$	$y_2 \ge y_1$		(b)	$\boldsymbol{p}' \in S_1$
	$y_2 < y_1$	$x_2 \ge y_1$	(c)	$\boldsymbol{p}' \in S_1$
		$x_2 < y_1$	(d)	$\boldsymbol{p}' \in S_1 \cup S_4$

Table 1: Subcases for unique zero.

(a) If $s = x_1 + y_1 \leq \min\{x_2, y_2\}$, then we can move to $\boldsymbol{p'} = (0, x_1, s, 0, 0, s, y_1) \in S_1$.

(b) When $x_1 + y_1 > y_2 \geq y_1$, we have that $y_2y_10x_1x_2$ is a shallow valley and by the Valley Lemma, there is a move to S_1.

(c) Since $y_1 > y_2$ implies that $x_1 + y_1 > y_2$, the conditions of this case reduce to $y_1 > y_2$, $x_2 \geq y_1$ and $x_2 \geq y_2$. Let $s = \min\{y_1, y_2 + y_3 + x_3\}$. The goal is to keep stacks y_2 and 0 and then adjust the other stacks according to the value of s. If $s = y_1$, then we move to $\boldsymbol{p'} = (0, y_1, y_2, y_3', x_3', s, 0) \in S_1$ with $y_2 + y_3' + x_3' = s = y_1$, otherwise, we move to $\boldsymbol{p'} = (0, y_1', y_2, y_3, x_3, s, 0) \in S_1$ with $y_1' = s = y_2 + y_3 + x_3$. These two moves are legal because $x_2 \geq y_1 \geq s$ and $y_3' + x_3' = s - y_2 = y_1 - y_2 > 0$.

(d) The conditions for this case, namely $y_2 < y_1, x_2 < y_1$, and $x_2 \geq y_2$ reduce to $y_2 \leq x_2 < y_1$. We distinguish between two main cases, namely whether $x_3 + y_3 \leq \min\{x_1, y_1\}$ or not. We first consider the case $x_3 + y_3 \leq \min\{x_1, y_1\}$.

   - If $y_2 < s = x_3 + y_3 \leq \min\{x_1, y_1\}$, then we can move to $\boldsymbol{p'} = (0, s - y_2, y_2, y_3, x_3, 0, s) \in S_4$. Since $\min\{s - y_2, y_2, x_3\} > 0$, the conditions of S_4 are satisfied.
   - If $s = x_3 + y_3 \leq y_2 \leq x_2$, then $x_2x_3y_3y_2y_1$ is either a shallow valley or a deep valley, depending on whether $x_3 + y_3 + y_2 > x_2$ or $x_3 + y_3 + y_2 \leq x_2$, and there is a move to S_1.

   Now we look at the second case, $x_3 + y_3 > x_1$ or $x_3 + y_3 > y_1$. We show that with this condition alone (disregarding the overall conditions of subcase d), we can show that there is a move to $S_4 \cup S_1$. We can therefore assume, without loss of generality, that $x_1 \geq y_1$, and consider two subcases, namely $x_1 \geq x_3 + y_3 > y_1$ and $x_3 + y_3 > x_1$.

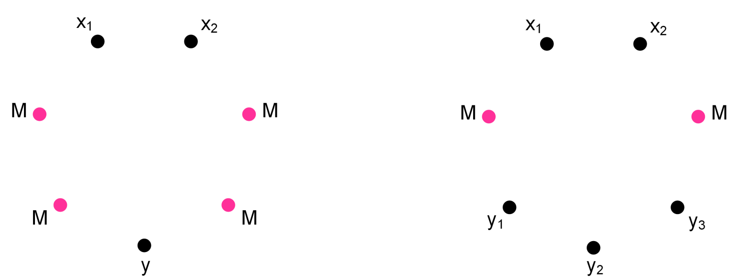
   - If $x_1 \geq x_3 + y_3 > y_1$ and $x_3 + y_3 > y_1 + y_2$, then we let $s = y_1 + y_2$ and can move to $\boldsymbol{p'} = (0, y_1, y_2, y_3', x_3', 0, s) \in S_4$ with $y_3' + x_3' = s$. We can adjust the sum $y_3' + x_3'$ such that $x_3' > 0$. Also, $\min\{y_1, y_2\} > 0$, so the S_4 conditions are satisfied. If, on the other hand, $x_3 + y_3 \leq y_1 + y_2$, then we can move to $\boldsymbol{p'} = (0, y_1', y_2, y_3, x_3, 0, s) \in S_4$ with $s = x_3 + y_3$ and $y_1' = s - y_2 > 0$ and the S_4 conditions are satisfied.
   - If $x_3 + y_3 > \max\{x_1, y_1\}$ and $x_1 \geq y_1 + y_2 = s$, then we can move to $\boldsymbol{p'} = (0, y_1, y_2, y_3', x_3', 0, s) \in S_4$ with $s = y_1 + y_2 = y_3' + x_3'$. Note that once more, $\min\{x_1, x_3 + y_3\} \geq y_1 + y_2 = s$, so the move is legal. Finally, assume that $y_1 + y_2 > x_1 = s$. Now we have a move to $\boldsymbol{p'} = (0, y_1, y_2', y_3', x_3', 0, x_1) \in S_4 \cup S_1$ with $y_1 + y_2' = x_3' + y_3' = s$. Since $y_1 \leq s$, we can make the sum $y_1 + y_2' = s$, and we can also adjust the sum $x_3' + y_3'$ while keeping $x_3' > 0$. If $y_3' = y_2' = 0$, then $\boldsymbol{p'} \in S_1$, otherwise $\boldsymbol{p'} \in S_4$.

This completes the proof in the case of exactly one zero. □

Finally, we deal with the case when the position $\boldsymbol{p}$ does not have a zero. In this case, we divide the positions according to where the maximum is located in relation to other maxima (if any). Note that when $\min(\boldsymbol{p}) > 0$, there is a close relation between positions in S_3 and S_4. A position $\boldsymbol{p} = (m, M, m, p_4, p_5, p_6, p_7)$ with $p_4 + p_5 = p_6 + p_7 = M + m$ and $\min\{p_4, p_7\} > m$ is in S_4 if $\max\{p_5, p_6\} > m$ and is in S_3 if $p_5 = p_6 = m$. Therefore, we will state that there is a move to $S_3 \cup S_4$ and need only check on the sum conditions and the minimum condition. This property will be used repeatedly in the Maximum Lemma.

Lemma 4 (Maximum Lemma)**.** *Let* $\boldsymbol{p} \in S^c$ *with* $\min(\boldsymbol{p}) > 0$*. Then there is a move from* $\boldsymbol{p}$ *to* $\boldsymbol{p}' \in S$*.*

Proof. Let $M = \max(\boldsymbol{p})$. We will first look at the antipodal case, where we have two maxima "opposite" (at distance two) of each other. The generic position is $\boldsymbol{p} = (x_1, x_2, M, y_3, y_2, y_1, M)$, shown in Figure 7b.



(a) (b)

Figure 7: Generic positions for antipodal maxima. (a) $y_3 = M$ and (b) $y_3 < M$.

Table 2 shows the subcases we will consider for antipodal maxima. Without loss of generality, we may assume that $y_3 \le y_1$.

<table>
<tr><td>$y_3 = M$</td><td colspan="2"></td><td>(a)</td><td>$\boldsymbol{p}' \in S_1 \cup S_3 \cup S_4$</td></tr>
<tr><td rowspan="3">$y_3 < M$</td><td>$y_2 + y_3 \le M$</td><td></td><td>(b1)</td><td>$\boldsymbol{p}' \in S_1$</td></tr>
<tr><td rowspan="2">$y_2 + y_3 > M$</td><td>$x_1 \ge x_2$</td><td>(b2)</td><td>$\boldsymbol{p}' \in S_3$</td></tr>
<tr><td>$x_1 < x_2$</td><td>(b3)</td><td>$\boldsymbol{p}' \in S_3 \cup S_4$</td></tr>
</table>

Table 2: Subcases for antipodal maxima.

(a) We start with the case $M = y_3$ shown in Figure 7a. Note that since $y_3 \le y_1$, we also have $y_1 = M$. In this case, the generic position becomes $\boldsymbol{p} = (x_1, x_2, M, M, y, M, M)$, where we have dropped the y subscript for ease of notation. We may also assume in this case that without loss of generality, $x_1 \le x_2$. If $x_1 + x_2 < M$, then Mx_1x_2MM forms a shallow valley and there is a move to S_1. Now assume that $M \le x_1 + x_2 \le M + y$. In this case, there is a move to $\boldsymbol{p}' = (x_1, x_2, x_1, M, x_1 + x_2 - M, x_1 + x_2 - M, M) \in S_3$. We can make the necessary adjustments since $x_1 \le M = \max(\boldsymbol{p})$, and $M \ge y \ge x_1 + x_2 - M \ge 0$ by assumption. Finally, when $M + y < x_1 + x_2$, then we can move to $\boldsymbol{p}' = (x_1', x_2', y, M, y, M, y) \in S_4$, with $x_1' + x_2' = M + y$. Note that $M + y < x_1 + x_2$ implies that $M > y$. We need to show that we can adjust the x_1 and x_2 stacks such that $x_1' > y$ and $x_2' > y$ to satisfy the S_4 conditions. This is possible since $x_1 + x_2 > M + y \ge y + 1 + y = 2y + 1$.

We now assume that $M > y_3$ (see Figure 7b) and consider the various subcases listed in Table 2.

(b1) Since $M \ge y_2 + y_3$, position $\boldsymbol{p}$ is either shallow valley (if $y_1 + y_2 + y_3 > M$) or deep valley (if $y_1 + y_2 + y_3 \le M$), so there is a move to $\boldsymbol{p}' \in S_1$.

Now let $s = \min\{y_2 + y_3, M + x_1, M + x_2\}$.

(b2) If $s = y_2 + y_3$ or $s = M + x_2$, then there is a move to $\boldsymbol{p}' = (s - M, s - M, M, y_3, y_2', y_3, M) \in S_3$ with $y_2' = s - y_3$. Note that in either case, we only play on four stacks. If $s = y_2 + y_3$, then $s \le \min\{M + x_1, M + x_2\}$, so $s - M \le \min\{x_1, x_2\}$, and $y_3 \le y_1$ by assumption. Also, $y_2' = s - y_3 = y_2$, so play is on the x_2, x_1, M, and y_3 stacks. If $s = M + x_1$, Since $M > y_3$, we have that $s - M = y_2 - (M - y_3) < y_2$ as needed for positions in S_3. On the other hand, if $s = M + x_2$, then $s - M = x_2$, so play is on the x_1, M, y_1, and y_2 stacks. By assumption, $x_1 < x_2$, $y_1 \le y_3$. Because $M + x_2 < y_2 + y_3$, we have that $y_2' \le y_2$, and $M > y_3$ implies that $y_2' = M + x_2 - y_3 > x_2$, so all conditions of S_3 are satiesfied in this case also.

(b3) If $s = M + x_1$, then $M + x_1 \le y_2 + y_3$. We move to $\boldsymbol{p}' = (x_1, x_2, s - x_2, y_3', y_2', x_1, M) \in S_3 \cup S_4$ with $y_2' + y_3' = M + x_1 = s$, playing on the one of the M stacks and the y_i stacks. This move is legal because $s - x_2 = M + x_1 - x_2 < M$ and $y_1 \ge y_3 \ge M + x_1 - y_2 \ge x_1$. Left to show is that $\min\{x_2, y_2'\} > x_1$. By assumption of this case, $x_2 > x_1$, and $0 < M - y_3 \le y_2 - x_1$ shows that we can satisfy the sum condition with $y_2' > x_1$.

This completes the case of antipodal maxima. We now consider the case when $M > \max\{x_3, y_3\}$, so the stacks that are opposite of M have strictly smaller height. Our generic position is shown in Figure 8. Without loss of generality, we may assume that $x_1 \le y_1$. Once more we move to either $\boldsymbol{p}' \in S_1$ or $\boldsymbol{p}' \in S_3 \cup S_4$. Now let $s = \min\{M + x_1, x_2 + x_3, y_2 + y_3\}$.

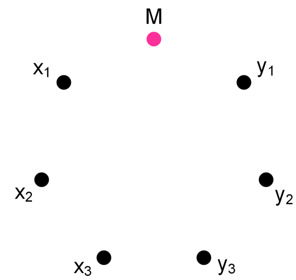


Figure 8: Generic position when $M > \max\{x_3, y_3\}$.

- If $s = M + x_1$, then we can move to $\boldsymbol{p}' = (M, x_1, y_2, y_3', x_3', x_2, x_1) \in S_3 \cup S_4$, with $x_2 + x_3' = y_2 + y_3' = M + x_1$. Play is on the y_i stacks and x_3; the move is legal because $x_1 \leq y_1$ by assumption, $x_3' = M + x_1 - x_2 \leq x_3$, and $x_3' > 0$ since $M = \max(\boldsymbol{p})$ and all stack heights are positive. Likewise, $0 < y_3' \leq y_3$. Left to show is that $\min\{x_2, y_2\} > x_1$. By assumption, $M > \max\{x_3, y_3\}$ which implies both $0 < M - x_3 \leq x_2 - x_1$ and $0 < M - y_3 \leq y_2 - x_1$, so the move is legal.

- If $s = x_2 + x_3$ and $y_3 \geq s = x_2 + x_3$, then $M > y_3$ implies that $\boldsymbol{p}$ is either shallow valley (if $y_3 < x_1 + x_2 + x_3$) or deep valley (if $y_3 \geq x_1 + x_2 + x_3$) and there is a move to S_1. If $y_3 < s = x_2 + x_3 < M + x_1$, then we move to $\boldsymbol{p}' = (M', m', y_2', y_3, x_3, x_2, m') \in S_4$ with $M' = \min\{s, M\}$, $m' = s - M' = \max\{0, s - M\} \geq 0$, and $y_2' + y_3 = s$. Let us check that this move is legal. If $M \geq s$, then we can clearly create the M' and m' stacks. If $M < s$, then $m' = s - M > 0$ and $s - M < x_1 \leq y_1$, so the adjustment on the M' and m' stacks is legal. Next we consider the y_2 stack. Because $y_3 < s$ and $y_3 < M$, we have $y_2' = s - y_3 > \max\{0, s - M\} = m' \geq 0$. Finally, we have that $x_2 > 0$ (by assumption of no zero stacks) and also $x_2 > x_2 + x_3 - M = s - M$ since $x_3 < M$, so in either case, $x_2 > m'$ and the conditions of S_4 are satisfied.

- If $s = y_2 + y_3$, then the same arguments apply as in the case $s = x_2 + x_3$, with the roles of x and y interchanged except for the inequality that $s < M + x_1$.

This completes the proof of the Maximum Lemma. □

These three lemmas together prove Proposition 2, because each position either has multiple zeros, a unique zero, or no zero. Together with Proposition 1 and Theorem 1, we have shown that the set S of Theorem 2 is the set of $\mathcal{P}$-positions of CN$(7, 4)$.

3. Discussion

Our goal in the investigations of $\mathrm{CN}(n,k)$ has always been to find a general structure of the $\mathcal{P}$-positions for families of games. So far we have found such results for $\mathrm{CN}(n,1)$, $\mathrm{CN}(n,n)$, and $\mathrm{CN}(n,n-1)$ (see [4]). In addition, in all previous results for $\mathrm{CN}(n,k)$, we have been able to find a single description of the $\mathcal{P}$-positions. The case of $\mathrm{CN}(7,4)$ is seemingly an anomaly in that four different sets make up the $\mathcal{P}$-positions. However, a careful look at the $\mathcal{P}$-positions of $\mathrm{CN}(3,2)$, $\mathrm{CN}(5,3)$, and $\mathrm{CN}(7,4)$, which are all examples of $\mathrm{CN}(2\ell+1,\ell+1)$, reveals a common structure. Recall that the $\mathcal{P}$-positions of $\mathrm{CN}(3,2)$ are given by $\{a,a,a\}$ for $a \geq 0$, and the $\mathcal{P}$-positions of $\mathrm{CN}(5,3)$ are given by $\{(x,0,x,a,b)|x = a+b\}$. This leads to the following result.

Lemma 5. *In the game* $\mathrm{CN}(2\ell+1,\ell+1)$*, the set of* $\mathcal{P}$*-positions contains the set* S_1*, where*

$$S_1 = \{\boldsymbol{p} = (x, \underbrace{0,\ldots,0}_{\ell-1}, x, a_1,\ldots,a_\ell)| \sum_{i=1}^{\ell} a_i = x\}.$$

Proof. Note that all positions in $\mathrm{CN}(2\ell+1,\ell+1)$ that have $\ell-1$ consecutive zeros are $\mathrm{CN}(3,2)$-equivalent with sets $\{p_1\}$,$\{p_{\ell+1}\}$ and $\{p_{\ell+2},\ldots,p_{2\ell+1}\}$. Those in S_1 are precisely the $\mathrm{CN}(3,2)$-equivalent $\mathcal{P}$-positions. Therefore, we cannot make a move from S_1 to S_1 because this would amount to a move from a $\mathcal{P}$-position in $\mathrm{CN}(3,2)$ to another $\mathcal{P}$-position in $\mathrm{CN}(3,2)$. On the other hand, we can make a $\mathrm{CN}(3,2)$ winning move into S_1 from any position in $\mathrm{CN}(2\ell+1,\ell+1)$ that has $\ell-1$ consecutive zeros. Therefore, S_1 must be a subset of the $\mathcal{P}$-positions of $\mathrm{CN}(2\ell+1,\ell+1)$. $\square$

While Lemma 5 does not settle the question regarding the set of $\mathcal{P}$-positions of the family of games $\mathrm{CN}(2\ell+1,\ell+1)$, the result shows that the set S_1 for $\mathrm{CN}(7,4)$, which has the requirement of the zero minima, is not an anomaly, but a fixture among the $\mathcal{P}$-positions of this family of games. Note that for $\mathrm{CN}(3,2)$ and $\mathrm{CN}(5,3)$, the set of $\mathcal{P}$-positions equals S_1. These two games are too small to show the more general structure of the $\mathcal{P}$-positions of this family. The question arises whether there are generalizations of the other components of the $\mathcal{P}$-positions of $\mathrm{CN}(7,4)$ that play a part of the $\mathcal{P}$-positions in this family. The obvious candidate would be S_2, with all equal stack heights. Interestingly enough, this set is NOT a part of the $\mathcal{P}$-positions (except for the terminal position) of $\mathrm{CN}(9,5)$. For example, the position $(2,2,2,2,2,2,2,2,2)$ is an $\mathcal{N}$-position of $\mathrm{CN}(9,5)$.

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