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MULTIPLICATIVE INEQUALITIES FOR PRIMES AND THE PRIME COUNTING FUNCTION

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Abstract

Multiplicative inequalities for primes and the prime counting function are proved. For any positive integers m and n, if p_n is the nth prime, then $p_{m\cdot n} < p_m p_n$, with only two exceptions. If $\pi(n)$ is the number of primes less than or equal to n, then $\pi(mn) \geq \pi(m)\pi(n)$, again with only two exceptions. These inequalities sharpen additive inequalities of Ishikawa. In addition, the nth primorial $p_n\# := p_1 \times \cdots \times p_n$ satisfies $p_{n!} < p_n\#$ if and only if $n > 1$. Similarly, $\pi(n!) > \pi(2) \times \cdots \times \pi(n)$ if and only if $n > 2$.

1. The Result

For any $n \in \mathbb{N} := \{1, 2, \ldots\}$, let p_n be the nth prime starting from $p_1 = 2$, and let $\pi(n)$ be the number of primes that do not exceed n (so $\pi(1) = 0$). We observe that

$$p_6 = p_{2\cdot 3} = 13 < p_2 p_3 = 15, \tag{1}$$

$$p_{12} = p_{3\cdot 4} = 37 > p_3 p_4 = 35, \tag{2}$$

$$\pi(p_1 \cdot p_2) = 3 > \pi(p_1) \cdot \pi(p_2) = 2, \tag{3}$$

$$\pi(p_2 \cdot p_3) = \pi(15) = 6 = \pi(p_2)\pi(p_3) = 2 \cdot 3, \tag{4}$$

$$\pi(p_3 \cdot p_4) = \pi(35) = 11 < \pi(p_3)\pi(p_4) = 3 \cdot 4 = 12. \tag{5}$$

We ask: which (if any) of Inequality (1)–Inequality (5) illustrates a rule, and which (if any) is exceptional?

Because the products mn, $p_m p_n$, and $\pi(m)\pi(n)$ commute for all $m, n \in \mathbb{N}$, it suffices to consider only $m \leq n$. As a partial answer to this question, we compared $p_{m\cdot n}$ with $p_m p_n$ for $m = 1, \ldots, 2.5 \times 10^4$, $n = m, m+1, \ldots, 2.5 \times 10^4$. We found $p_{m\cdot n} < p_m p_n$ (which is illustrated by Inequality (1) above) in all but two cases: $(m, n) = (3, 4)$ (which is Inequality (2) above) and $(m, n) = (4, 4)$ (which is $p_{16} =$

$53 > p_4^2 = 49$). (Bonus question: Why did we never find $p_{m\cdot n} = p_m p_n$? Hint: the left side is a prime.)

For $m = 1, \ldots, 10^3,\ n = m, \ldots, 10^3$, we also found $\pi(mn) \geq \pi(m)\pi(n)$ (which is illustrated by Inequality (3) and Equation (4) above, again with only two exceptions: $(m, n) = (5, 7) = (p_3, p_4)$ (which is Inequality (5) above) and $(m, n) = (7, 7) = (p_4, p_4)$ (which is $\pi(p_4^2) = \pi(49) = 15 < (\pi(p_4))^2 = 16$).

These numerical examples suggest and will help prove Theorem 1.

Theorem 1. *Let $m, n \in \mathbb{N},\ m \leq n$. Then*

$$p_{m\cdot n} < p_m p_n \tag{6}$$

unless $(m, n) = (3, 4)$ or $(m, n) = (4, 4)$, in which cases $p_{m\cdot n} > p_m p_n$. Also

$$\pi(mn) \geq \pi(m)\pi(n) \tag{7}$$

unless $(m, n) = (5, 7) = (p_3, p_4)$ or $(m, n) = (7, 7) = (p_4, p_4)$, in which cases $\pi(mn) < \pi(m)\pi(n)$.

Proof. To prove Inequality (6), we seek a lower bound on $p_m p_n$ and an upper bound on $p_{m\cdot n}$. Rosser [6, Theorem 1] proved that

$$k \log k < p_k \quad \text{for all } k \in \mathbb{N}.$$

Here and throughout, log is the natural logarithm. This lower bound has been greatly sharpened, especially by Dusart [1], but the simplicity of Rosser's result makes it convenient to use here. Therefore

$$(m \log m)(n \log n) < p_m p_n \quad \text{for all } m \leq n \in \mathbb{N}. \tag{8}$$

Dusart [1, Lemma 1] quotes earlier results of others that

$$\begin{aligned} p_k &\leq k(\log k + \log\log k) && \text{for all } k \in \mathbb{N},\ k \geq 6, \\ p_k &\leq k \log p_k && \text{for all } k \in \mathbb{N},\ k \geq 4. \end{aligned} \tag{9}$$

Therefore

$$p_{m\cdot n} < mn(\log(mn) + \log\log(mn)) \quad \text{for all } m, n \in \mathbb{N},\ m \leq n,\ 6 \leq mn. \tag{10}$$

Now compare Inequality (8) and Inequality (10). Then Inequality (6) holds under the conditions $m, n \in \mathbb{N},\ m \leq n,\ 6 \leq mn$ if we can show that

$$mn(\log(mn) + \log\log(mn)) < mn(\log m)(\log n). \tag{11}$$

Cancel mn from both sides and let $M := \log m,\ N := \log n$. Then Inequality (11) holds if and only if

$$M + N + \log(M + N) < MN. \tag{12}$$

For $M = N = 3$, the left side of Inequality (12) is $6 + \log 6 \approx 7.7918$ while the right side is 9, so Inequality (12) holds. Moreover, for $M \geq 3,\ N \geq 3$, the difference $f(M, N) := MN - (M + N + \log(M + N))$ increases with both M and N because $\partial f/\partial M = N - (1 + 1/(M + N)) > 0$ and by symmetry $\partial f/\partial N > 0$. Now $M, N \geq 3$ means $m, n \geq e^3 \approx 20.0855$. So the condition in Inequality (9) that $k \geq 6$ will be satisfied if $m, n \geq e^3$. We have now proved Inequality (6) when $m, n \in \mathbb{N},\ 6 \leq m \leq n$. Our initial numerical calculations proved Inequality (6) for the remaining smaller values of m, n, apart from the two exceptions.

We prove Inequality (7) similarly. Rosser and Schoenfeld [8, Corollary 1] proved that

$$\frac{x}{\log x} < \pi(x) \qquad \text{for all } x \in \mathbb{R},\ x \geq 17,$$

$$\pi(x) < \frac{1.25506x}{\log x} \qquad \text{for all } x \in \mathbb{R},\ x > 1.$$

Therefore for all $m, n \in \mathbb{N},\ mn \geq 17$, with $C := 1.25506$,

$$\pi(mn) > \frac{mn}{\log(mn)}, \quad \frac{Cm}{\log m} \cdot \frac{Cn}{\log n} > \pi(m)\pi(n).$$

Then Inequality (7) will be proved for all $m, n \in \mathbb{N},\ mn \geq 17$ if we show that

$$\frac{mn}{\log(mn)} > \frac{Cm}{(\log m)} \cdot \frac{Cn}{(\log n)}. \tag{13}$$

Cancel mn from both sides of Inequality (13) and let $M := \log m,\ N := \log n$. Then Inequality (13) is equivalent to

$$C^2(M + N) < MN. \tag{14}$$

When $M = N$, Inequality (14) holds if and only if $2C^2M < M^2$ or $2C^2 \approx 3.15035 < M$ or $\exp(2C^2) \approx 23.34426 < m$. Thus Inequality (14) and therefore Inequality (7) hold for any $m, n \geq 24$. Our initial numerical calculations proved Inequality (7) for the remaining smaller values of m, n apart from the two exceptions. □

Because $p_n \sim n \log n$ for large n, Inequality (6) would be expected to hold for large n. The point of Theorem 1 is that Inequality (6) holds with only two exceptions. Likewise, because $\pi(x) \sim x/\log x$ for large x, Inequality (7) would be expected to hold for large x. Theorem 1 shows that Inequality (7) holds with only two exceptions.

Does Inequality (6) governing primes imply Inequality (7) governing the prime counting function, or conversely? I do not know.

For $n \in \mathbb{N}$, define the nth *primorial* to be $p_n\# := p_1 \times \cdots \times p_n$. Define $p_{n!}$, as usual, to be the $n!$th prime. For example, $p_4\# := 2 \times 3 \times 5 \times 7 = 210,\ p_{4!} = p_{24} = 89$.

n	$p_n\#$	$p_{n!}$	$\pi(n!)$	$\pi(n)\#$
1	2	2	0	0
2	6	3	1	1
3	30	13	3	2
4	210	89	9	4
5	2310	659	30	12
6	30030	5443	128	36
7	510510	49033	675	144
8	9699690	484037	4231	576
9	223092870	5222429	30969	2304
10	6469693230	61194647	258689	9216

Table 1: First ten values of primorial, $p_n\#$, the $n!$th prime number, $p_{n!}$, the number $\pi(n!)$ of primes not exceeding $n!$, and the nth piorial.

Similarly, define the nth *piorial* to be $\pi(n)\# := \prod_{j=2}^{n} \pi(j)$ when $n \geq 2$ and define $\pi(1)\# := \pi(1) = 0$. To make the values of $\pi(n)\#,\ n > 1$, nonzero, $\pi(1)$ is excluded from the product. Table 1 reports the first 10 values of $p_n\#$, $p_{n!}$, $\pi(n!)$, and $\pi(n)\#$. In this table, $p_n\# \geq p_{n!}$ and $\pi(n!) \geq \pi(n)\#$, with strict inequality for $n > 2$.

Corollary 1. *For any $t \in \mathbb{N}$, let $n_1, \ldots, n_t \in \mathbb{N}$. Let $N := \prod_{j=1}^{t} n_j$. Then*

$$p_N < \prod_{j=1}^{t} p_{n_j} \qquad \text{if } \min_j n_j \geq 5, \tag{15}$$

$$\pi(N) \geq \prod_{j=1}^{t} \pi(n_j) \qquad \text{if } \min_j n_j \geq 8. \tag{16}$$

Consequently, for all $n \in \mathbb{N}$,

$$p_{n!} < p_n\# \qquad \text{if and only if } n > 1, \tag{17}$$

$$\pi(n!) > \prod_{j=2}^{n} \pi(j) \qquad \text{if and only if } n > 2. \tag{18}$$

Proof. Theorem 1 establishes Inequality (15) for $t = 2$. For $t = 3$ and $5 \leq k, m, n \in \mathbb{N}$, stepwise application of Theorem 1 gives $p_{kmn} < p_{km}p_n < p_k p_m p_n$. This stepwise use of Theorem 1 extends to any larger $t \in \mathbb{N}$, proving Inequality (15). Likewise, for $t = 3$ and $8 \leq k, m, n \in \mathbb{N}$, stepwise application of Theorem 1 gives $\pi(kmn) \geq \pi(km)\pi(n) \geq \pi(k)\pi(m)\pi(n)$. This stepwise use of Theorem 1 extends to any larger $t \in \mathbb{N}$, proving Inequality (16).

Table 1 establishes Inequality (17) for $n = 1, \ldots, 10$. For larger n, we see that $p_{n!} = p_{5!6\cdots n} < p_{5!}p_{6\cdots n}$ and $p_n\# = p_5\# \times p_6 \cdots p_n$. We have $p_{5!} < p_5\#$ and $p_{6\cdots n} <$

$p_6 \cdots p_n$. Multiplying these two inequalities proves Inequality (17). Inequality (18) is proved similarly. □

Ishikawa [4, Theorems 2, 3] proved that

$$p_{m+n} < p_m p_n \qquad \text{for all } m, n \in \mathbb{N},\ m \leq n,$$
$$\pi(xy) > \pi(x) + \pi(y) \qquad \text{for all } x, y \in \mathbb{R},\ x, y \geq 5. \tag{19}$$

Ribenboim [5] presents these and many related results. We combine Ishikawa's results with Theorem 1.

Theorem 2. *Let $m, n \in \mathbb{N},\ m \leq n$. Then*

$$p_{m+n} < p_{m \cdot n} \tag{20}$$

unless $m = 1$ (in which case $p_{m+n} > p_{m \cdot n}$) or $m = n = 2$ (in which case $p_{m+n} = p_{m \cdot n}$), and

$$p_{m \cdot n} < p_m p_n$$

unless $(m, n) = (3, 4)$ or $(m, n) = (4, 4)$ (in which cases $p_{m \cdot n} > p_m p_n$). Thus, avoiding the exceptional cases,

$$p_{m+n} < p_{mn} < p_m p_n \qquad \text{for all } m, n \in \mathbb{N},\ 4 < m \leq n. \tag{21}$$

Also, still assuming $m, n \in \mathbb{N},\ m \leq n$,

$$\pi(m)\pi(n) > \pi(m) + \pi(n) \tag{22}$$

unless $m = n = 1$ (in which case both sides of Inequality (22) are 0) or $m = 1$ or 2 and $n \geq 2$ (in which cases $\pi(m)\pi(n) < \pi(m) + \pi(n)$), and

$$\pi(mn) \geq \pi(m)\pi(n)$$

unless $(m, n) = (5, 7) = (p_3, p_4)$ or $(m, n) = (7, 7) = (p_4, p_4)$ (in which cases $\pi(mn) < \pi(m)\pi(n)$). Thus, avoiding the exceptional cases,

$$\pi(mn) \geq \pi(m)\pi(n) > \pi(m) + \pi(n) \qquad \text{for all } m, n \in \mathbb{N},\ 7 < m \leq n. \tag{23}$$

Proof. In the first exception to Inequality (20), if $m = 1$, then $mn < m + n$, so $p_{1+n} > p_{1 \cdot n}$. If $1 < m \leq n$, then $mn \geq m+n$ with equality if and only if $m = n = 2$. When $mn > m + n$, Inequality (20) is strict.

The second exceptions to Inequality (22) are $m = 1$ or 2 and $n \geq 2$. If $m = 1$ and $n \geq 2$, then $\pi(m)\pi(n) = 0$ while $\pi(m) + \pi(n) > 0$, so the direction of the inequality is reversed. If $m = 2$ and $n \geq 2$, then $\pi(m)\pi(n) = \pi(n)$ while $\pi(m) + \pi(n) = 1 + \pi(n)$, so again the direction of the inequality is reversed. Ishikawa's requirement in Inequality (19) that $x, y \geq 5$ excludes these exceptional cases. □

For each inequality in (21) and (23), the larger quantity asymptotically grows faster than the smaller. Theorem 3 gives details.

Theorem 3. *Let $m, n \in \mathbb{N}$. Then*

$$\lim_{m,n\to\infty} \frac{p_{m\cdot n}}{p_m p_n} = 0, \tag{24}$$

$$\lim_{m,n\to\infty} \frac{p_{m+n}}{p_{m\cdot n}} = 0, \tag{25}$$

$$\lim_{m,n\to\infty} \frac{\pi(m)\pi(n)}{\pi(mn)} = 0, \tag{26}$$

$$\lim_{m,n\to\infty} \frac{\pi(m)+\pi(n)}{\pi(m)\pi(n)} = 0. \tag{27}$$

Moreover, $p_{m+n}/p_{m\cdot n} \to 0$ faster than $p_{m\cdot n}/(p_m p_n) \to 0$, and $(\pi(m)+\pi(n))/(\pi(m)\pi(n)) \to 0$ faster than $\pi(m)\pi(n)/\pi(mn) \to 0$. Thus

$$\lim_{m,n\to\infty} \frac{p_{m+n}}{p_{m\cdot n}} \bigg/ \frac{p_{m\cdot n}}{p_m p_n} = 0, \tag{28}$$

$$\lim_{m,n\to\infty} \frac{\pi(m)+\pi(n)}{\pi(m)\pi(n)} \bigg/ \frac{\pi(m)\pi(n)}{\pi(mn)} = 0. \tag{29}$$

Proof. Using $p_n \sim n\log n$, $M := \log m$, $N := \log n$ gives, as $m, n, M, N \to \infty$,

$$\frac{p_{m\cdot n}}{p_m p_n} \sim \frac{mn\log(mn)}{(m\log m)(n\log n)} = \frac{M+N}{MN} = \frac{1}{M} + \frac{1}{N} \to 0.$$

This proves Inequality (24). Next,

$$\frac{p_{m+n}}{p_{m\cdot n}} \sim \frac{m+n}{mn} \cdot \frac{\log(m+n)}{\log(mn)}. \tag{30}$$

Then $(m+n)/(mn) = 1/m + 1/n \to 0$ as $m, n \to \infty$. Also $m+n > mn$ if and only if $m = 1$ or $n = 1$ or both, and $m+n = mn$ if and only if $m = n = 2$. In all other positive integer cases, $m+n < mn$ so $\log(m+n)/\log(mn) < 1$. Therefore the right side of Inequality (30) converges to 0. This proves Inequality (25).

Using $\pi(n) \sim n/\log n$, $M := \log m$, $N := \log n$ gives, as $m, n, M, N \to \infty$,

$$\frac{\pi(m)\pi(n)}{\pi(mn)} \sim \frac{m}{M}\frac{n}{N}\frac{M+N}{mn} = \frac{M+N}{MN} = \frac{1}{M} + \frac{1}{N} \to 0.$$

This proves Inequality (26). Similarly,

$$\frac{\pi(m)+\pi(n)}{\pi(m)\pi(n)} \sim \left(\frac{m}{M} + \frac{n}{N}\right)\frac{MN}{mn} = \frac{M}{m} + \frac{N}{n} \to 0.$$

This proves Inequality (27). Using the rightmost asymptotic approximations above then gives

$$\frac{p_{m+n}}{p_{m\cdot n}} \Big/ \frac{p_{m\cdot n}}{p_m p_n} \sim \frac{MN(m+n)\log(m+n)}{mn(M+N)^2}, \tag{31}$$

$$\frac{\pi(m)+\pi(n)}{\pi(m)\pi(n)} \Big/ \frac{\pi(m)\pi(n)}{\pi(mn)} \sim \frac{MN(mN+nM)}{mn(M+N)}. \tag{32}$$

The right sides of Inequality (31) and Inequality (32) converge to 0 by inspection or routine calculation. This proves Inequality (28) and Inequality (29). □

In 1923, Hardy and Littlewood [2] conjectured that

$$\pi(x)+\pi(y) \geq \pi(x+y) \quad \text{for all } x, y \in \mathbb{R},\ 2 \leq x \leq y. \tag{33}$$

It would be esthetically appealing to add this inequality to the right side of Inequality (23) (after replacing $x,\ y \in \mathbb{R}$ by $m,\ n \in \mathbb{N}$). Extensive numerical evidence supports Inequality (33). Dusart [2, Theorem 1] proved Inequality (33) for all $x,\ y \in \mathbb{R}$ such that $2 \leq x \leq y \leq \frac{7}{5}x \log x \log\log x$. However, Hensley and Richards [3] showed that this inequality is incompatible with another famous, well-supported conjecture, the "prime k-tuples conjecture," which they judge to be more likely to be true.

Similarly, it would be esthetically appealing to add to the left side of Inequality (21) the conjecture that, for every $m, n \in \mathbb{N},\ m \leq n$,

$$p_m + p_n \leq p_{m+n} \tag{34}$$

unless $m = n = 1$ (in which case $p_1 + p_1 = p_2 + 1$). Whenever the weak inequality (34) holds for $1 < m \leq n$, it must be strict because then the left side is the even sum of two odd numbers, while the right side is odd. We confirmed Inequality (34) numerically for $m = 1, \ldots, 2.5 \times 10^4,\ n = m, m+1, \ldots, 2.5 \times 10^4$. The proof of Inequality (34) is simple when $1 = m < n$. Then $p_m = 2$ and $2 + p_n \leq p_{1+n}$ because the gap between every two successive primes after $p_2 - p_1$ is at least 2. We have no proof in general. A helpful referee asks: does the observation [3] that the prime k-tuples conjecture is incompatible with Inequality (33) imply also that the prime k-tuples conjecture is incompatible with Inequality (34)? We do not know.

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