#A36 INTEGERS 22 (2022)

THE $\{1, s\}$-WEIGHTED DAVENPORT CONSTANT IN C_n^k

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Received: 7/6/21, Accepted: 3/2/22, Published: 3/14/22

Abstract

Let G be a finite abelian group and let $\varnothing \neq A \subset \mathbb{Z}$. The A-weighted Davenport constant of G is the smallest positive integer $\mathsf{D}_A(G)$ such that every sequence $x_1 \cdot x_2 \cdot \ldots \cdot x_{\mathsf{D}_A(G)}$ over G has a non-empty subsequence $(x_{j_i})_i$ such that $\varepsilon_1 x_{j_1} + \varepsilon_2 x_{j_2} + \ldots + \varepsilon_t x_{j_t} = 0$ for some $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_t \in A$. In this paper, we obtain both upper and lower bounds for $\mathsf{D}_{\{1,s\}}(C_n^k)$, where C_n denotes the cyclic group of order n, $s^2 \equiv 1 \pmod n$ and $s \not\equiv \pm 1 \pmod n$. These bounds become sharp in some "small" cases.

1. Introduction

Given a finite abelian group G written additively, the *zero-sum problems* study conditions to ensure that a given sequence over G has a non-empty subsequence with prescribed properties (for instance length, repetitions, and weights) such that the sum of their terms equals 0, the identity of G. This kind of problem dates back to the 1960s, with the works of Erdős, Ginzburg and Ziv [7] and Olson [13, 14]. It has been extensively studied for abelian groups, but in recent times several results over non-abelian groups have emerged. This paper deals with a weighted problem over C_n^k, where C_n denotes the cyclic group of order n. First of all, some definitions and notations are required.

1.1. Definitions and Notations

By a *sequence* S over a finite group G we mean a finite and unordered element of the free abelian monoid $\mathcal{F}(G)$ equipped with the sequence concatenation product

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denoted by $\boldsymbol{\cdot}$. A sequence $S \in \mathcal{F}(G)$ has the form

$$S = \prod_{1\le i\le k}^{\bullet} g_i = g_1 \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} g_k \in \mathcal{F}(G),$$

where $g_1, \ldots, g_k \in G$ are the *terms* of S and $k = |S| \geq 0$ is the *length* of S. Since the sequences are unordered, $S = \prod^{\bullet}_{1\le i\le k} g_{\tau(i)} \in \mathcal{F}(G)$ for any permutation $\tau : \{1, 2, \ldots, k\} \to \{1, 2, \ldots, k\}$. For a given $S \in \mathcal{F}(G)$ and for $t \geq 0$, we abbreviate $S^{[t]} = \underbrace{S \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} S}_{t \text{ times}}$. For $g \in G$, the *multiplicity* of the term g in S is denoted by $v_g(S) = \#\{i \in \{1, 2, \ldots, k\} : g_i = g\}$, and therefore our sequence S may also be written as $S = \prod^{\bullet}_{g\in G} g^{[v_g(S)]}$. A sequence T is a *subsequence* of S if $v_g(T) \leq v_g(S)$ for all $g \in G$; for this we use the notation $T \mid S$. In this case, we write $S \boldsymbol{\cdot} T^{[-1]} = \prod^{\bullet}_{g\in G} g^{[v_g(S)-v_g(T)]}$. We also define:

$$\sigma(S) = g_1 + \ldots + g_k \in G, \quad \text{the } \textit{sum} \text{ of } S;$$

$$\Sigma(S) = \bigcup_{\substack{T\mid S\\ |T|\ge 1}} \{\sigma(T)\} \subset G, \quad \text{the } \textit{set of subsums} \text{ of } S;$$

$$S \cap K = \prod_{\substack{g\mid G\\ g\in K}}^{\bullet} g^{[v_g(S)]}, \quad \text{the subsequence of } S \text{ that lie in a subset } K \text{ of } G.$$

The sequence S is called *zero-sum free* if $0 \notin \Sigma(S)$, and S is *zero-sum sequence* if $\sigma(S) = 0$.

1.2. The Davenport Constant

One of the most important types of zero-sum problems involves the *small Davenport constant* $\mathsf{d}(G)$ of a finite abelian group G. Namely, let $\mathsf{d}(G)$ be the maximal integer such that there exists a sequence over G (repetition allowed) of length $\mathsf{d}(G)$ which is zero-sum free, that is,

$$\mathsf{d}(G) = \sup\{|S| > 0 : S \in \mathcal{F}(G) \text{ is zero-sum free}\}.$$

It is well-known that $\mathsf{d}(C_n) = n - 1$. The inequality

$$\mathsf{d}(C_{n_1} \oplus \ldots \oplus C_{n_k}) \geq \sum_{i=1}^{k} (n_i - 1) \tag{1.1}$$

holds true, considering the zero-sum free sequence formed by the concatenation of $n_i - 1$ copies of a generator of C_{n_i} for each $1 \leq i \leq k$. Olson proved that the equality holds true for abelian p-groups [13] and for $k \leq 2$ [14].

Many variations and generalizations of zero-sum problems have been considered over the years. In this paper, we consider the following weighted problem. Let

$A \subset \mathbb{Z}$ be a *set of weights*. In order to avoid trivial cases, we assume that $A \neq \varnothing$ and A does not contain any multiple of $\exp(G)$, the exponent of G. Let

$$\sigma_A(S) = \{a_1 g_1 + \ldots + a_k g_k \in G : \ a_i \in A\} \quad \text{the } \textit{set of } A\textit{-weighted sums} \text{ of } S,$$
$$\Sigma_A(S) = \bigcup_{\substack{T \mid S \\ |T| \geq 1}} \sigma_A(T) \subset G \quad \text{the } \textit{set of } A\textit{-weighted subsequence sums} \text{ of } S.$$

Moreover, S is *A-zero-sum free* if $0 \notin \Sigma_A(S)$, and an *A-zero-sum sequence* if $0 \in \sigma_A(S)$.

The *A-weighted Davenport constant* of an abelian group G is defined by

$$\mathsf{D}_A(G) = \inf\{k > 0 : \ \text{every } S \in \mathcal{F}(G) \text{ with } |S| \geq k \text{ satisfies } 0 \in \Sigma_A(S)\}.$$

For instance, every abelian group G yields $\mathsf{D}_{\{1\}}(G) = \mathsf{d}(G) + 1$, where $\mathsf{d}(G)$ is the (unweighted) small Davenport constant of G. Using the Pigeonhole Principle, Adhikari et al. [1] proved that $\mathsf{D}_{\{\pm 1\}}(C_n) = \lfloor \log_2(n) \rfloor + 1$. More generally, Adhikari, Grynkiewicz and Sun [3] proved that if $n_1 \mid n_2 \mid \ldots \mid n_k$ and $G = C_{n_1} \oplus \ldots \oplus C_{n_k}$, then

$$\sum_{i=1}^{k} \lfloor \log_2(n_i) \rfloor + 1 \leq \mathsf{D}_{\{\pm 1\}}(G) \leq \lfloor \log_2(|G|) \rfloor + 1, \tag{1.2}$$

and Marchan, Ordaz and Schmid [12] removed the hypothesis $n_1 \mid n_2 \mid \ldots \mid n_k$.

In addition, Adhikari, David and Urroz [2] showed that if $A = \{1, 2, 3, \ldots, r\}$, then $\mathsf{D}_A(C_n) = \lceil \frac{n}{r} \rceil$ (it was previously proved for n prime by Adhikari and Rath [4]). Also in [2], it is proved that if A is the set of all quadratic residues modulo a squarefree number n and $\omega(n)$ is the number of distinct prime factors of n, then $\mathsf{D}_A(C_n) = 2\omega(n) + 1$.

Although very little is known about the A-weighted Davenport constant for a general set A, Halter-Koch [11] provided an arithmetical interpretation of certain types of weighted Davenport constants in terms of algebraic integers and of binary quadratic forms.

In this paper, we fix an integer $k \geq 1$, and establish some upper and lower bounds for $\mathsf{D}_{\{1,s\}}(C_n^k)$, where

$$s^2 \equiv 1 \pmod{n}, \qquad \text{but} \qquad s \not\equiv \pm 1 \pmod{n}. \tag{1.3}$$

In fact, for $s \equiv 1 \pmod{n}$ we have $\mathsf{D}_{\{1,s\}}(C_n^k) = \mathsf{d}(C_n^k) + 1$, and for $s \equiv -1 \pmod{n}$ we have $\mathsf{D}_{\{1,s\}}(C_n^k) = \mathsf{D}_{\{\pm 1\}}(C_n^k)$.

The main motivation for considering this set of weights was the inverse zero-sum problem related to the small Davenport constant over the non-abelian group $C_n \rtimes_s C_2$. Indeed, in [5] the authors considered the inverse problem over D_{2n}, the dihedral group of order $2n$. The main argument used the $\{\pm 1\}$-weighted Davenport constant over C_n in order to obtain the structure of the product-one free sequences of maximum length. The idea for $C_n \rtimes_s C_2$ would be similar, using the $\{1, s\}$-weighted Davenport constant over C_n. However, the upper bounds provided in this paper were only able to solve the inverse problem for values of n with sufficiently large factors, namely, factors n_1 and n_2 as in Lemma 1 (in particular, the proof would not work for $n = 2^t$, $t \geq 3$). Nevertheless, in [6] the authors completely solved the inverse problem over $C_n \rtimes_s C_2$, using the set of weights but without using directly the bounds on $\mathsf{D}_{\{1,s\}}(C_n^k)$ presented here.

This paper is organized as follows. In Section 2, we present a useful factorization of n in order to set a crucial projection, yielding an isomorphism in some cases. In Section 3, we prove lower bounds for $\mathsf{D}_{\{1,s\}}(C_n^k)$. In Sections 4 and 5, we prove upper bounds for $\mathsf{D}_{\{1,s\}}(C_n^k)$ when the factorization of n generates an isomorphism and a projection, respectively; in particular, we prove a relation among these two cases. In Section 6, we discuss the tightness of these bounds, and conclude the exact value in two families of "small" cases.

2. The Natural Projection/Isomorphism

We note that conditions (1.3) guarantee that n can be neither an odd prime power nor twice an odd prime power, for otherwise we would have $s \equiv \pm 1 \pmod{n}$. The following lemma, which is also [6, Lemma 2.2], and which we reproduce its proof here for convenience, ensures that these conditions suffice to factor n nicely.

Lemma 1 ([6, Lemma 2.2]). *Let $n \geq 8$ and s be positive integers satisfying the conditions (1.3).*

(i) *If both $n \neq p^t$ and $n \neq 2p^t$ for every prime p and every integer $t \geq 1$, then there exist coprime integers $n_1, n_2 \geq 3$ such that $s \equiv -1 \pmod{n_1}$, $s \equiv 1 \pmod{n_2}$, and either (A) $n = n_1 n_2$ or (B) $n = 2n_1 n_2$.*

(ii) *If $n = 2^t$ for some $t \geq 3$, then (B) $n = 2n_1n_2$, where either $(n_1, n_2) = (1, 2^{t-1})$ satisfies $s \equiv 1 \pmod{n_2}$ or $(n_1, n_2) = (2^{t-1}, 1)$ satisfies $s \equiv -1 \pmod{n_1}$.*

Proof. Let $n = 2^t m$, where m is odd and $t \geq 0$ is an integer. Since m divides $s^2 - 1$ and $\gcd(s-1, s+1) \in \{1, 2\}$ (depending on whether s is even or odd), each prime power factor of m divides either $s + 1$ or $s - 1$. Let $m_1 = \gcd(m, s + 1)$ and $m_2 = \gcd(m, s - 1)$, so that $m = m_1 m_2$. In addition, $s^2 \equiv 1 \pmod{2^t}$ implies that either $s \equiv \pm 1 \pmod{2^t}$ or $s \equiv 2^{t-1} \pm 1 \pmod{2^t}$ (depending on whether $t \geq 3$). We consider some cases:

(i) `CASE` $n \neq 2^t$, that is, $m \neq 1$.

(i.1) `SUBCASE` $t = 0$. In this case, $n = m = m_1 m_2$. We set $n_1 = m_1$ and $n_2 = m_2$, hence $n = n_1 n_2$ is the desired factorization, as in (A). This is the only case where s can be even; in the following, s must be odd.

(i.2) `SUBCASE` $t = 1$. It is possible to set either $n_1 = 2m_1$ and $n_2 = m_2$ or $n_1 = m_1$ and $n_2 = 2m_2$, hence $n = n_1n_2$ is the desired factorization, as in (A).

(i.3) `SUBCASE` $t \geq 2$ and $m \geq 3$. If $s \equiv -1 \pmod{2^t}$, then we set $n_1 = 2^t m_1$ and $n_2 = m_2$. If $s \equiv 1 \pmod{2^t}$, then we set $n_1 = m_1$ and $n_2 = 2^t m_2$. Therefore, $n = n_1n_2$ is the desired factorization, as in (A). In the case that $t \geq 3$, it is possible that $s \equiv 2^{t-1} \pm 1 \pmod{2^t}$. If $s \equiv 2^{t-1} - 1 \pmod{2^t}$, then we set $n_1 = 2^{t-1}m_1$ and $n_2 = m_2$. If $s \equiv 2^{t-1} + 1 \pmod{2^t}$, then we set $n_1 = m_1$ and $n_2 = 2^{t-1}m_2$. Hence, $n = 2n_1n_2$ is the desired factorization, as in (B).

(ii) `CASE` $n = 2^t$ with $t \geq 3$, that is, $m = 1$. In this case, $s \equiv 2^{t-1} \pm 1 \pmod{2^t}$, which implies that $s \equiv \pm 1 \pmod{2^{t-1}}$. For the negative sign, it follows that $(n_1, n_2) = (2^{t-1}, 1)$, and for the positive sign, it follows that $(n_1, n_2) = (1, 2^{t-1})$. Therefore, $n = 2n_1n_2$ is the factorization, as in (B).

□

By the previous lemma and Chinese Remainder Theorem, there exists a natural projection

$$\Psi_0 : C_n \to C_{n_1} \oplus C_{n_2}$$

satisfying

$$\Psi_0(e) = (e_1, e_2) \quad \text{and} \quad \Psi_0(s \cdot e) = (-e_1, e_2),$$

where $C_{n_1} = \langle e_1 \rangle$, $C_{n_2} = \langle e_2 \rangle$ and $C_n = \langle e \rangle$. In Case (A) of previous lemma, Ψ_0 is an isomorphism.

Let $\vec{0}_r$ be the identity of C_r^k, $\vec{a} = (a_1, \ldots, a_k) \in C_r^k$, and write $t \cdot \vec{a} = (ta_1, \ldots, ta_k)$. Denote by

$$\Psi : C_n^k \to C_{n_1}^k \oplus C_{n_2}^k \tag{2.1}$$

the similar projection satisfying

$$\Psi(\vec{a}) = (\vec{a}, \vec{a}) \quad \text{and} \quad \Psi(s \cdot \vec{a}) = (-\vec{a}, \vec{a}),$$

where the first coordinate is the restriction to $C_{n_1}^k$ and the second coordinate is the restriction to $C_{n_2}^k$. In Case (A) of Lemma 1, Ψ is an isomorphism. Moreover, for $g = (u, v) \in C_{n_1}^k \oplus C_{n_2}^k$, we write $sg = (-u, v)$.

When we look at the sequences over $C_{n_1}^k \oplus C_{n_2}^k$, since $s \equiv -1 \pmod{n_1}$ and $s \equiv 1 \pmod{n_2}$, in the second coordinate we have an unweighted problem, while in the first coordinate we have a $\{\pm 1\}$-weighted problem. In the case that $n = n_1n_2$, finding zero-sum sequences over C_n^k is equivalent to finding zero-sum sequences over $C_{n_1}^k \oplus C_{n_2}^k$. In the case that $n = 2n_1n_2$, zero-sums over C_n^k imply zero-sums over $C_{n_1}^k \oplus C_{n_2}^k$, and $k+1$ disjoint zero-sums over $C_{n_1}^k \oplus C_{n_2}^k$ generate a zero-sum over C_n^k, since $C_n^k/(C_{n_1}^k \oplus C_{n_2}^k) \simeq C_2^k$ and $\mathsf{d}(C_2^k) = k$.

It is worth mentioning that the bounds presented here refer to the parities of n_1 and n_2 (see [8, Conjecture 1.10] for a conjecture regarding the Erdős-Ginzburg-Ziv constant over C_n^3 that is concerned with the parity of n). For the upper bound, our method consists basically of two steps: using the set of weights $\{\pm 1\}$ in $C_{n_1}^k$ to produce more than $\mathsf{d}(C_{n_2}^k)$ disjoint sums in $\{\vec{0}_{n_1}\} \oplus C_{n_2}^k$, hence obtaining a zero-sum over C_n^k. However, the method can be improved, provided n_2 is even in the following way: in the first step and with a little more effort, it is possible to obtain sums in $\{\vec{0}_{n_1}\} \oplus C_{\frac{n_2}{2}}^k$, where $C_{\frac{n_2}{2}} = \{2g : g \in C_{n_2}\}$. This effort is rewarded, since now it is only required $\mathsf{d}(C_{\frac{n_2}{2}}^k) + 1$ sums for the second step. This method will be further applied for other zero-sum problems in future papers.

3. The Lower Bounds

In this section, we provide good although not always tight lower bounds. The first one deals with the case $n = n_1 n_2$.

Theorem 1. *Let n and s be as in (1.3), and write $n = n_1 n_2$ as in Lemma 1. Then:*

(i) *for any n_1 and n_2, we have*

$$\mathsf{D}_{\{1,s\}}(C_n^k) \geq \mathsf{D}_{\{\pm 1\}}(C_{n_1}^k) + \mathsf{d}(C_{n_2}^k) \geq k\left(\lfloor \log_2(n_1) \rfloor + n_2 - 1\right) + 1;$$

(ii) *if n_2 is odd and either $n_1 > n_2$ or n_1 is even, then*

$$\mathsf{D}_{\{1,s\}}(C_n^k) \geq k(2n_2 - 1) + 1.$$

Proof. First of all, the projection defined in (2.1) ensures that we may consider S over $C_{n_1}^k \oplus C_{n_2}^k$ instead of over C_n^k. For each case, we exhibit $\{1, s\}$-zero-sum free sequences S.

(i) Let S_1' be a $\{\pm 1\}$-zero-sum free sequence over $C_{n_1}^k$ with $|S_1'| = \mathsf{D}_{\{\pm 1\}}(C_{n_1}^k) - 1$ and S_2' be a zero-sum free sequence over $C_{n_2}^k$ with $|S_2'| = \mathsf{d}(C_{n_2}^k)$. Set

$$S_1 = \prod_{g \in S_1'}^{\bullet} (g, \vec{0}_{n_2}) \in \mathcal{F}(C_{n_1}^k \oplus C_{n_2}^k),$$

$$S_2 = \prod_{h \in S_2'}^{\bullet} (\vec{0}_{n_1}, h) \in \mathcal{F}(C_{n_1}^k \oplus C_{n_2}^k),$$

and $S = S_1 \cdot S_2$. Then $S \in \mathcal{F}(C_{n_1}^k \oplus C_{n_2}^k)$ is $\{1, s\}$-zero-sum free and $|S| = |S_1| + |S_2| = \mathsf{D}_{\{\pm 1\}}(C_{n_1}^k) + \mathsf{d}(C_{n_2}^k) - 1$, which proves (i).

(ii) Let $C_{n_1} = \langle e_1 \rangle$ and $C_{n_2} = \langle e_2 \rangle$. Set

$$g_i = \Big(\underbrace{0, \ldots, 0, e_1, 0, \ldots, 0}_{e_1 \text{ in the } i^{\text{th}} \text{ position}}, \underbrace{0, \ldots, 0, e_2, 0, \ldots, 0}_{e_2 \text{ in the } (k+i)^{\text{th}} \text{ position}} \Big) \in C_{n_1}^k \oplus C_{n_2}^k$$

for $1 \le i \le k$, and

$$S = \prod_{1 \le i \le k}^{\bullet} g_i^{[2n_2-1]} \in \mathcal{F}(C_{n_1}^k \oplus C_{n_2}^k).$$

Then $|S| = k(2n_2 - 1)$ and we claim that S is $\{1, s\}$-zero-sum free. Suppose otherwise, then there exists a non-empty subsequence $T \mid S$ such that $0 \in \sigma_{\{1,s\}}(T)$. Assume that $g_i \mid T$. By the $(k+i)^{th}$ position, we have $v_{g_i}(T) = n_2$. These g_i's are the only possible terms of T that modify the i^{th} position. If n_1 is even then the i^{th} position of the $\{1, s\}$-weighted sum is odd, hence it can not be 0. If n_1 is odd and $n_1 > n_2$, then the $\{1, s\}$-weighted sum in the i^{th} position is an odd element of $[-n_2, n_2]$, therefore it can not be 0 modulo n_1.

□

The second result deals with the case $n = 2n_1n_2$.

Theorem 2. *Let n and s be as in (1.3), and write $n = 2n_1n_2$ as in Lemma 1. Then*

$$\mathsf{D}_{\{1,s\}}(C_n^k) \ge k \cdot \mathsf{D}_{\{1,s\}}(C_{n_1n_2}^k) + 1.$$

Proof. Let $T_1 \in \mathcal{F}(C_{n_1n_2})$ be a $\{1, s\}$-zero-sum free sequence of length $|T_1| = \mathsf{D}_{\{1,s\}}(C_{n_1n_2}) - 1$. The sequence $T_2 = T_1 \boldsymbol{\cdot} ((n_1n_2)e)$, where $C_{2n_1n_2} = \langle e \rangle$, is $\{1, s\}$-zero-sum free over $C_{2n_1n_2}$, and $|T_2| = \mathsf{D}_{\{1,s\}}(C_{2n_1n_2})$. Let $S \in \mathcal{F}(C_{2n_1n_2}^k)$ be the sequence formed by concatenation of the sequences T_2 in each coordinate, that is,

$$S = \prod_{1 \le j \le k}^{\bullet} \prod_{g \mid T_2}^{\bullet} \underbrace{(0, \ldots, 0, g, 0, \ldots, 0)}_{g \text{ in the } j^{th} \text{ coordinate}}.$$

Hence, $|S| = k \cdot |T_2| = k \cdot \mathsf{D}_{\{1,s\}}(C_{n_1n_2})$ and S is $\{1, s\}$-zero-sum free. □

4. The Upper Bounds for $n = n_1n_2$

The projection $\Psi : C_n^k \to C_{n_1}^k \oplus C_{n_2}^k$ is actually an isomorphism in this case. Hence, finding zero-sum sequences over C_n^k is equivalent to finding zero-sum sequences over $C_{n_1}^k \oplus C_{n_2}^k$. The goal in most cases in this section is to find many subsequences with a $\{\pm 1\}$-zero-sum in the first coordinate, in order to obtain a zero-sum in the second coordinate.

Before claiming the main results of this section, we prove two simpler upper bounds.

Proposition 1. *Let n and s be as in (1.3), and write $n = n_1 n_2$ as in Lemma 1. The following hold.*

(i) *For every n,*

$$\mathsf{D}_{\{1,s\}}(C_n^k) \leq \left(\mathsf{d}(C_{n_2}^k) + 1\right) \mathsf{D}_{\{\pm 1\}}(C_{n_1}^k) \leq \left(\mathsf{d}(C_{n_2}^k) + 1\right) \Big(\lfloor k \log_2(n_1) \rfloor + 1 \Big).$$

(ii) *For n_2 even,*

$$\mathsf{D}_{\{1,s\}}(C_n^k) \leq \Big(\mathsf{d}(C_{\frac{n_2}{2}}^k) + 1\Big) \Big(\lfloor k \log_2(n_1) \rfloor + k + 1 \Big).$$

Proof. As in the proof of Theorem 1, we consider $S \in \mathcal{F}(C_{n_1} \oplus C_{n_2})$ instead of $S \in \mathcal{F}(C_n)$.

(i) Suppose that $|S| = \left(\mathsf{d}(C_{n_2}^k) + 1\right) \mathsf{D}_{\{\pm 1\}}(C_{n_1}^k)$. Choose a minimal subsequence $A_1 \mid S$ with $|A_1| \leq \mathsf{D}_{\{1,s\}}(C_{n_1}^k)$ such that $(\vec{0}_{n_1}, b_1) \in \sigma_{\{1,s\}}(A_1)$, where $b_1 \in C_{n_2}^k$. Inductively, we may construct $A_{j+1} \mid S \cdot (A_1 \cdot \ldots \cdot A_j)^{[-1]}$ for $j = 1, \ldots, \mathsf{d}(C_{n_2}^k)$ such that $|A_{j+1}| \leq \mathsf{D}_{\{1,s\}}(C_{n_1})$ and $(\vec{0}_{n_1}, b_{j+1}) \in \sigma_{\{1,s\}}(A_{j+1})$, where $b_{j+1} \in C_{n_2}^k$. On the other hand, it follows that we have $\mathsf{d}(C_{n_2}^k) + 1$ disjoint $\{1,s\}$-weighted sums in $\{\vec{0}_{n_1}\} \oplus C_{n_2}^k$, therefore we are done.

(ii) The proof of this inequality is similar to the previous one. The only difference is that each $\lfloor k \log_2(n_1) \rfloor + k + 1$ terms yield a $\{1,s\}$-weighted sum into $\{\vec{0}_{n_1}\} \oplus C_{\frac{n_2}{2}}^k$. In fact, let $T \mid S$ with $|T| = \lfloor k \log_2(n_1) \rfloor + k + 1$. We may construct $2^{|T|} > 2^k n_1^k$ sums using the terms of T, therefore the Pigeonhole Principle ensures that there exist distinct subsequences (but not necessarily pairwise disjoint) $T_1, T_2, \ldots, T_{2^k+1} \mid T$, such that the first coordinates of $\sigma(T_1), \sigma(T_2), \ldots, \sigma(T_{2^k+1})$ are all the same in $C_{n_1}^k$. Again by the Pigeonhole Principle, there exist two of them, say T_1 and T_2, such that $\sigma(T_1)$ and $\sigma(T_2)$ have the same parity in all their last k coordinates. We may transform T_1 and T_2 into disjoint subsequences. Indeed, if T_1 and T_2 share the term (a,b), then the first coordinate of $\sigma(T_1)$ equals the first coordinate of $\sigma(T_2)$ if and only if the first coordinate of $\sigma(T_1 \cdot (a,b)^{[-1]})$ equals the first coordinate of $\sigma(T_2 \cdot (a,b)^{[-1]})$, as well as all the last k coordinates of $\sigma(T_1)$ have the same parity as those of $\sigma(T_2)$ if and only if all the last k coordinates of $\sigma(T_1 \cdot (a,b)^{[-1]})$ have the same parity as those of $\sigma(T_2 \cdot (a,b)^{[-1]})$, where "last k coordinates" denotes the restriction of the elements to $C_{n_2}^k$. Therefore, assuming that T_1 and T_2 are disjoint, we obtain

$$\sum_{g \in T_1} sg + \sum_{h \in T_2} h \in \{\vec{0}_{n_1}\} \oplus C_{\frac{n_2}{2}}^k.$$

It is possible to construct $\mathsf{d}(C_{\frac{n_2}{2}}^k) + 1$ disjoint $\{1,s\}$-weighted sums as the previous one, therefore we are done.

$\square$

Our goal can be achieved faster once the subsequences with an $\{1,s\}$-zero-sum in the first coordinate have "small" length. For example, if we have two terms $(a_1,b_1)\cdot(a_2,b_2)\mid S$ with $a_1=\pm a_2$, then these two terms generate a $\{1,s\}$-zero-sum in the first coordinate. For three or four terms, we need the following:

Lemma 2. *Let $m\geq 2$ be an integer, and let $S\in\mathcal{F}(C_m^k)$ with $|S|\geq\sqrt{2m^k}$. Suppose that every term of S has multiplicity one, and the equation $g_1+g_2=0$ has no solution with $g_1\cdot g_2\mid S$. If $\vec{0}_m\nmid S$, then there exists $T\mid S$ with $3\leq|T|\leq 4$ such that $\sum_{g|T}\varepsilon_g\cdot g=\vec{0}_m$ with $\varepsilon_g\in\{\pm1\}$.*

Proof. Since $|S|\geq\sqrt{2m^k}>\sqrt{2m^k+1/4}-1/2$, we have that $\binom{|S|}{2}+\binom{|S|}{1}>m^k$. Therefore, the Pigeonhole Principle implies that S contains distinct subsequences T_1 and T_2 of lengths one or two such that $\sigma(T_1)=\sigma(T_2)$. Notice that T_1 and T_2 have no common terms, otherwise either $T_1=T_2$ or $\vec{0}_m\mid S$. Therefore, $\sigma(T_1)-\sigma(T_2)=\vec{0}_m$ and the latter has three or four terms. □

Theorem 3. *Let n and s be as in (1.3), and write $n=n_1n_2$ as in Lemma 1. The following hold.*

(i) *For n odd,*

$$\mathsf{D}_{\{1,s\}}(C_n^k)\leq 2\mathsf{d}(C_{n_2}^k)+\frac{n_1^k-1}{4}+\frac{\sqrt{2n_1^k}}{2}.$$

(ii) *For n_1 even,*

$$\mathsf{D}_{\{1,s\}}(C_n^k)\leq 2\mathsf{d}(C_{n_2}^k)+\frac{n_1^k+2^k-2}{4}+\frac{\sqrt{2n_1^k}}{2}.$$

(iii) *For n_2 even,*

$$\mathsf{D}_{\{1,s\}}(C_n^k)\leq 2\mathsf{d}(C_{\frac{n_2}{2}}^k)+\frac{n_1^k(k\cdot 2^{k+1}+1)+2\sqrt{2n_1^k}+k(2^{k+1}+4)+7}{4(k+1)}.$$

Proof. As in the proof of Theorem 1, we consider $S\in\mathcal{F}(C_{n_1}^k\oplus C_{n_2}^k)$ instead of $S\in\mathcal{F}(C_n^k)$.

(i) In this case, n_1 and n_2 are both odd. Suppose that $|S|\geq 2\mathsf{d}(C_{n_2}^k)+\frac{n_1^k-1}{4}+\frac{\sqrt{2n_1^k}}{2}$.
For $1\leq i\leq u$, let $A_i=(\vec{0}_{n_1},b_i)\mid S$ be the terms of S with $\vec{0}_{n_1}$ in the first coordinate. Furthermore, for $1\leq i\leq v$, let $A_{u+i}=(a_{u+i},b_{u+i})\cdot(a_{u+i}^{\pm1},b'_{u+i})\mid S\cdot\left(\prod^\bullet_{1\leq i\leq u}A_i\right)^{[-1]}$ denote the subsequences of S formed by two terms such that their first coordinates are distinct than $\vec{0}_{n_1}$ and are equals or inverses in $C_{n_1}^k$.
From S, if we remove the terms of each A_i with $1\leq i\leq u+v$, there will

remain at most $\frac{n_1^k-1}{2}$ terms (otherwise there would exist two equal or inverse terms in the first coordinate, that could be included in the previous A_{u+i}, $1 \le i \le v$).

Let $T = S \cdot \left(\prod^{\bullet}_{1\le i\le u+v} A_i\right)^{[-1]}$, so that

$$|T| = |S| - (u+2v) \le \frac{n_1^k - 1}{2}.$$

While $|T| \ge \sqrt{2n_1^k}$, Lemma 2 ensures that there exists $T_1 \mid T$ with $|T_1| \in \{3,4\}$ such that $\vec{0}_{n_1} \in \sigma_{\{1,s\}}(T_1)$. Inductively, whenever possible, we construct $T_2, \ldots, T_w$ satisfying the same conditions than T_1, hence

$$|T| - 4w = |S| - (u + 2v + 4w) < \sqrt{2n_1^k}.$$

It follows that there exist $u+v+w$ disjoint subsequences of S whose $\{1,s\}$-weighted sum belong to $\{\vec{0}_{n_1}\} \oplus C_{n_2}^k$. Since

$$u+v+w = \frac{u+2v}{4} + \frac{u+2v+4w}{4} + \frac{u}{2} > \frac{|S| - \frac{n_1^k-1}{2}}{4} + \frac{|S| - \sqrt{2n_1^k}}{4} \ge \mathsf{d}(C_{n_2}^k),$$

we are done.

(ii) This case is almost completely similar to the previous one. The only difference is the following: after defining A_i for $1 \le i \le u+v$, the subsequence $T = S \cdot \left(\prod^{\bullet}_{1\le i\le u+v} A_i\right)^{[-1]}$ satisfies

$$|T| = |S| - (u+2v) \le \frac{n_1^k + 2^k - 2}{2},$$

since there are $2^k - 1$ non-zero elements of $C_{n_1}^k$ that are their own inverses. The remainder follows exactly the same steps as Case (i).

(iii) In this case, n_1 is odd and n_2 is even. This proof is also similar to Case (i). Suppose that $S \in \mathcal{F}(C_{n_1}^k \oplus C_{n_2}^k)$ with $|S| \ge 2\mathsf{d}(C_{\frac{n_2}{2}}^k) + \frac{n_1^k(k\cdot 2^{k+1}+1)+2\sqrt{2n_1^k}+k\cdot 2^{k+1}+3}{4(k+1)}$. For $1 \le i \le u$, let $A_i \mid S$ such that $|A_i| = 1$, and its only term lies in $\{\vec{0}_{n_1}\} \oplus C_{\frac{n_2}{2}}^k$.

For $1 \le i \le v$, let $A_{u+i} \mid S \cdot \left(\prod^{\bullet}_{1\le i\le u} A_i\right)^{[-1]}$ such that $|A_{u+i}| = 2$ and $\sigma_{\{1,s\}}(A_{u+i}) \cap \left[\{\vec{0}_{n_1}\} \oplus C_{\frac{n_2}{2}}^k\right] \ne \varnothing$. Let $T_1 = S \cdot \left(\prod^{\bullet}_{1\le i\le u+v} A_i\right)^{[-1]}$, so that

$$|T_1| = |S| - (u+2v) \le 2^k\left(\frac{n_1^k - 1}{2}\right) + (2^k - 1) = 2^{k-1}(n_1^k + 1) - 1.$$

For $1 \le i \le w$, let $A_{u+v+i} \mid T_1$ such that $|A_{u+v+i}| = 2$ and $\sigma_{\{1,s\}}(A_{u+v+i}) \cap \left[\{\vec{0}_{n_1}\} \oplus C_{n_2}^k\right] \neq \varnothing$. Let $T_2 = T_1 \cdot \left(\prod^{\bullet}_{1 \le i \le u+v+w} A_i\right)^{[-1]}$, so that

$$|T_2| = |S| - (u + 2v + 2w) \le \frac{n_1^k - 1}{2}.$$

For $1 \le i \le t$, let $A_{u+v+w+i} \mid T_2$ be as in Lemma 2 in such way that $\sigma_{\{1,s\}}(A_{u+v+w+i}) \cap \left[\{\vec{0}_{n_1}\} \oplus C_{n_2}^k\right] \neq \varnothing$. We have

$$|S| - (u + 2v + 2w + 4t) < \sqrt{2n_1^k}.$$

Since $\mathsf{d}(C_2^k) = k$, the disjoint subsequences A_i for $1 \le i \le u+v+w+t$ produce at least $u + v + \left\lfloor \frac{w+t}{k+1} \right\rfloor$ terms into $\{\vec{0}_{n_1}\} \oplus C^k_{\frac{n_2}{2}}$. Since

$$\begin{aligned} u + v + \frac{w+t}{k+1} &= \frac{u + 2v + 2w + 4t}{4(k+1)} + \frac{u + 2v + 2w}{4(k+1)} + \frac{k(u+2v)}{2(k+1)} + \frac{u}{2} \\ &> \frac{|S| - \sqrt{2n_1^k}}{4(k+1)} + \frac{|S| - \frac{n_1^k - 1}{2}}{4(k+1)} + \frac{k\left(|S| - 2^{k-1}(n_1^k + 1) + 1\right)}{2(k+1)} \\ &\ge \mathsf{d}(C^k_{\frac{n_2}{2}}) + 1, \end{aligned}$$

we are done.

□

Remark 1. Notice that the previous theorem only used subsequences of length at most four in order to obtain a zero-sum in the first coordinate. It unfolds that it is possible to go further with a similar of Lemma 2, considering subsequences of length at most six, eight, etc. However, the main terms involving $\mathsf{d}(C_{n_2}^k)$ and n_1^k keep the same, the only change is that we get a larger denominator involving $\sqrt{2n_1^k}$, and we get other smaller terms of the form $\sqrt[3]{2n_1^k}$, $\sqrt[4]{2n_1^k}$, etc. For instance, if n is odd and we consider subsequences up to six terms then we obtain the following inequality:

$$\mathsf{D}_{\{1,s\}}(C_n^k) \le 2\mathsf{d}(C_{n_2}^k) + \frac{n_1^k - 1}{4} + \frac{\sqrt{2n_1^k}}{6} + \frac{\sqrt[3]{6n_1^k}}{3}.$$

Alternatively, we could use the remainder of at most $\sqrt{2n_1^k}$ terms to form subsequences of length much larger whose $\{1,s\}$-weighted sum is zero in the first coordinate and a better term (in some sense) in the second coordinate, but it also does not change the main terms. For these reasons, we decide to make it explicit until the first "error term".

5. The Upper Bounds for $n = 2n_1n_2$

Due to the projection Ψ, we have $C_n^k/(C_{n_1}^k \oplus C_{n_2}^k) \simeq C_2^k$. Therefore, one can simply multiply each bound of Theorem 3 by $k+1$, obtaining upper bounds for $\mathsf{D}_{\{1,s\}}(C_n^k)$. Indeed, it is possible to find $k+1$ disjoint subsequences whose $\{1,s\}$-weighted sums are $\vec{0}_{n_1} \oplus \vec{0}_{n_2}$, that is, belong to C_2^k. Since $\mathsf{d}(C_2^k) = k$, there exists a $\{1,s\}$-zero-sum subsubsequence. This proves the following result.

Proposition 2. *Let n and s be as in (1.3), and write $n = 2n_1n_2$ as in Lemma 1. Then*

$$\mathsf{D}_{\{1,s\}}(C_n^k) \leq (k+1)\mathsf{D}_{\{1,s\}}(C_{n_1n_2}).$$

In the case that $k = 1$, n_2 is even and $s \equiv n_2 + 1 \pmod{2n_2}$, we are able to get a precise bound. For this, we need the following lemmas:

Lemma 3 (See [10, Theorem 11.1])**.** *Let $m \geq 3$ be an integer and let $S \in \mathcal{F}(C_m)$ be a zero-sum free sequence of length $|S| > m/2$. Then, there exists $g \mid S$ such that*

$$v_g(S) \geq \max\left\{m - 2|S| + 1,\ |S| - \left\lfloor \frac{m-1}{3} \right\rfloor\right\}.$$

In addition, for a residue class $r \pmod m$, denote by $\overline{r}$ the integer such that $0 \leq \overline{r} \leq m-1$ and $r \equiv \overline{r} \pmod m$. Then, there exists an integer t with $\gcd(t,m) = 1$ and $\sum_{g|S} \overline{gt} < m$. Moreover, for every $1 \leq k \leq \sum_{g|S} \overline{gt}$, there exists $T_k \mid S$ such that $\sum_{g|T_k} \overline{gt} = k$.

We also need the value of $\mathsf{D}_{\{1,m+1\}}(C_{2m})$, where m is even (so that $(m+1)^2 \equiv 1 \pmod{2m}$).

Theorem 4. *Let $m \geq 4$ be an even integer. Then*

$$\mathsf{D}_{\{1,m+1\}}(C_{2m}) = m + 1.$$

Moreover, each sequence of length $m+1$ has a proper subsequence whose $\{1,m+1\}$-weighted sum is $0 \in C_{2m}$.

Proof. The lower bound $\mathsf{D}_{\{1,m+1\}}(C_{2m}) \geq m+1$ follows the same steps than Theorem 2, hence it is only required to prove the upper bound. Let $C_{2m} = \langle e \rangle$, so that $C_m = \langle 2e \rangle$. Let

$$S = \prod^{\bullet}_{1 \leq i \leq m+1} (t_i e) \in \mathcal{F}(C_{2m})$$

of length $|S| = m+1$. We observe that $\mathsf{D}_{\{1,m+1\}}(C_m) = m = \mathsf{d}(C_m) + 1$, and that

$$\begin{cases} (m+1)a \equiv a \pmod{2m} & \text{if } a \text{ is even,} \\ (m+1)a \equiv m + a \pmod{2m} & \text{if } a \text{ is odd.} \end{cases}$$

Since $|S| > m$, the projection $\Psi_1 : C_{2m} \to C_m$ that maps $[(t \pmod{2m})e]$ into $[(t \pmod{m})e]$ implies that there exists $S_1 \mid S$ with $|S_1| \leq m$ such that either $0 \in \sigma_{\{1,m+1\}}(S_1)$ or $me \in \sigma_{\{1,m+1\}}(S_1)$. If $0 \in \sigma_{\{1,m+1\}}(S_1)$, then we are done. Therefore, we assume that $me \in \sigma_{\{1,m+1\}}(S_1)$.

Notice that if $me \mid S$, then $|S \cdot (me)^{[-1]}| = m$, thus there exists $S' \mid S \cdot (me)^{[-1]}$ with $|S'| \leq m$ such that either $0 \in \sigma_{\{1,m+1\}}(S')$ or $me \in \sigma_{\{1,m+1\}}(S')$. If $0 \in \sigma_{\{1,m+1\}}(S')$, then we are done. Otherwise, $me \in \sigma_{\{1,m+1\}}(S')$, which implies that $0 \in \sigma_{\{1,m+1\}}(S' \cdot me)$. If $|S'| < m$, then S has a proper $\{1, m+1\}$-weighted zero-sum subsequence over C_{2m}. Thus, we may assume that $|S'| = m$ and S' has no proper $\{1, m+1\}$-weighted zero-sum subsequence. It follows from Lemma 3 that $S' = (te)^{[m]}$ for some integer $1 \leq t \leq m$ such that $\gcd(t, m) = 1$ (in particular, t is odd). In this case, we have $0 = (m+1)te + \underbrace{te + \cdots + te}_{m-1 \text{ times}} \in \sigma_{\{1,m+1\}}(S')$, and again we are done.

Hence, we assume that $me \nmid S$. Furthermore, suppose that S is zero-sum free over C_{2m} (otherwise, we are done). Since $|S| > |C_{2m}|/2$, after a possible change of generators (as in Lemma 3), we assume that $t_1, \ldots, t_{m+1}$ are positive integers in $[1, 2m-1]$ such that $\sum_{i=1}^{m+1} t_i \leq 2m - 1$.

In this way, S_1 can be taken as an unweighted sum, that is, $\sigma(S_1) = me$. If S_1 has some odd term, say $ae \mid S_1$, then

$$(m+1)a + \sum_{t_ie \mid (S_1 \cdot (ae)^{[-1]})} t_i \equiv 0 \pmod{2m},$$

hence, we are done. Otherwise, we assume that every term of any sequence S_1 with $\sigma(S_1) = me$ is even.

CLAIM: $(2e) \mid S_1$ and $e^{[2]} \mid S \cdot S_1^{[-1]}$.
If this is the case, we just consider $S_2 = (S_1 \cdot e^{[2]}) \cdot (2e)^{[-1]}$, therefore, we are done.

Proof of the claim. If $e^{[2]} \nmid S$, then

$$t_1 + \cdots + t_{m+1} \geq 1 + 2(m-1) = 1 + 2m > 2m,$$

a contradiction. If $2e \nmid S_1$ then $t_ie \mid S_1$ implies $t_i \geq 4$ is even. We have that $4|S_1| \leq \sum_{t_ie|S_1} t_i = m$, hence $|S_1| \leq \frac{m}{4}$. Therefore

$$|S \cdot S_1^{[-1]}| = |S| - |S_1| \geq m + 1 - \frac{m}{4} = \frac{3m}{4} + 1,$$

which implies

$$\sum_{t_ie \mid S \cdot S_1^{[-1]}} t_i \geq \frac{3m}{4} + 1.$$

Let $v_1 = v_e(S) = v_e(S \cdot S_1^{[-1]})$. Since $\sum_{t_i e | S \cdot S_1^{[-1]}} t_i \leq m-1$ and $|S_1| \leq \frac{m}{4}$, the average term of the subsequence $S \cdot (S_1 \cdot e^{[v_1]})^{[-1]}$ is at least 2. Therefore

$$\frac{m-1-v_1}{\frac{3m}{4}+1-v_1} \geq \frac{\sum_{t_i e | S \cdot S_1^{[-1]}} t_i - v_1}{|S \cdot S_1^{[-1]}| - v_1} \geq 2,$$

which implies $v_1 \geq \frac{m}{2}+3$. If $|S_1| = 1$, then $S_1 = me \mid S$, a contradiction. Therefore, $|S_1| \geq 2$. The Pigeonhole Principle ensures that there exists $t_i e \mid S_1$ such that $1 \leq t_i \leq \frac{m}{2}$. In this case, we replace the term $t_i e$ by t_i copies of e, which is possible since $v_1 > \frac{m}{2} \geq t_i$. This leads us to a contradiction, hence, we are done. □

Corollary 1. *Let $n = 2n_1n_2$ and s be as in Lemma 1 such that n_2 is even and $s \equiv n_2 + 1 \pmod{2n_2}$. Then*

$$\mathsf{D}_{\{1,s\}}(C_n) = n_1n_2 + 1.$$

Proof. In view of Theorem 2, it is only required to prove the upper bound. Let $S \in \mathcal{F}(C_n)$ such that $|S| = \mathsf{D}_{\{1,s\}}(C_{n_1n_2})+1 = n_1n_2+1$. Since $|S| \geq n_2+1$, Theorem 4 ensures that there exists a subsequence $T_1 \mid S$ with $|T_1| \leq n_2$ whose $\{1,s\}$-weighted sum belongs to the subgroup C_{n_1}. Now construct $T_2 \mid S \cdot T_1^{[-1]}$ with the same property. Inductively, for $j \leq n_1$, we construct $T_j \mid S \cdot (T_1 \cdot \dots \cdot T_{j-1})^{[-1]}$ with $|T_j| \leq n_2$ such that an $\{1,s\}$-weighted sum belongs to C_{n_1}. Since $\mathsf{d}(C_{n_1}) = n_1 - 1$, this corollary follows. □

6. The Tightness of the Bounds

For $n = n_1n_2$, if n_2 is odd and either $n_1 > n_2$ or n_1 is even, then the second bound of Theorem 1 is useful only when $n_2 \geq \lfloor \log_2(n_1) \rfloor$. Otherwise, the first bound is more useful. The worst bounds are those where n_2 is odd and $n_1 < n_2$, in which case we were unable to find a good lower bound. On the other hand, Proposition 1 is more useful than Theorem 3 only when n_1 is very small compared to n_2; otherwise, Theorem 3 turns out to be of better use.

For $n = 2n_1n_2$, we provided a good lower bound in terms of the previous case. In the particular case $k = 1$, the problem splits into two subcases: n_2 even with $s \equiv n_2 + 1 \pmod{2n_2}$ (whose exact value has been established in Corollary 1) and n_1 even with $s \equiv n_1 - 1 \pmod{2n_1}$. In both cases, the values of $\mathsf{D}_{\{1,s\}}(C_n)$ depend on $\mathsf{D}_{\{1,s\}}(C_{n_1n_2})$. On the other hand, the upper bound is asymptotically sharp for large values of k.

Notation. Let f, g be positive functions. As usual, we use the asymptotic Bachmann-Landau notations:

- $f(n) \sim g(n)$ means that $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 1$,
- $f(n) = O(g(n))$ means that there exists $c > 0$ such that $f(n) \le c \cdot g(n)$ for every positive integer n, and
- $f(n) = o(g(n))$ means that $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$.

Conjecture 1. Let n and s be as in (1.3), and write $n = n_1 n_2$ as in Lemma 1. We expect that

$$\mathsf{D}_{\{1,s\}}(C_n^k) = \begin{cases} 2kn_2 + O(k \cdot \log_2(n_1)) & \text{if } n_2 \text{ is odd,} \\ kn_2 + O(k \cdot \log_2(n_1)) & \text{if } n_2 \text{ is even,} \end{cases}$$

$$\mathsf{D}_{\{1,s\}}(C_n) = \begin{cases} 2n_2 + \lfloor \log_2(n_1) \rfloor & \text{if } n_2 \text{ is odd,} \\ n_2 + \lfloor \log_2(n_1) \rfloor & \text{if } n_2 \text{ is even.} \end{cases}$$

In the case that $n = 2n_1 n_2$, we expect that

$$\mathsf{D}_{\{1,s\}}(C_n^k) = k \cdot \mathsf{D}_{\{1,s\}}(C_{n_1 n_2}^k) + 1.$$

Remember that $\mathsf{d}(C_n) = n - 1$ and $\mathsf{d}(C_n^2) = 2n - 2$. It is conjectured that if $k \ge 3$ then equality in (1.1) holds for C_n^k, that is, $\mathsf{d}(C_n^k) = k(n-1)$. Girard [9] showed that $\mathsf{d}(C_n^k) \sim kn$ as $n \to \infty$. As a consequence of the upper bounds and Girard's result, we have the following.

Corollary 2. *Let n and s be as in (1.3), and write $n = n_1 n_2$ as in Lemma 1.*

(i) *With n_1 fixed, it follows that*

$$\mathsf{D}_{\{1,s\}}(C_n^k) = \begin{cases} 2kn_2(1 + o(1)) & \textit{if } n_2 \textit{ is odd,} \\ kn_2(1 + o(1)) & \textit{if } n_2 \textit{ is even.} \end{cases}$$

(ii) *With n_2 fixed, it follows that*

$$\mathsf{D}_{\{1,s\}}(C_n^k) = O(k \cdot \log_2(n_1)).$$

The next proposition and corollary yield the exact values of $\mathsf{D}_{\{1,s\}}(C_n)$ when $n_1 \in \{3, 5\}$ and n_2 is even for $n = n_1 n_2$ and for $n = 2n_1 n_2$, respectively, proving particular cases of Conjecture 1. It is worth mentioning that Theorems 1(i) and 3(iii) yield

$$n_2 + \lfloor \log_2(n_1) \rfloor \le \mathsf{D}_{\{1,s\}}(C_n) \le n_2 + \lfloor \log_2(n_1) \rfloor + 1$$

provided $n = n_1 n_2$, n_2 is even and $n_1 \in \{3, 5\}$, and similar intervals for the cases where $n = 2n_1 n_2$.

Proposition 3. *Let n and s be as in (1.3), and write $n = n_1 n_2$ as in Lemma 1, where $n_1 \in \{3, 5\}$ and n_2 even. Then*

$$\mathsf{D}_{\{1,s\}}(C_n) = n_2 + \lfloor \log_2(n_1) \rfloor.$$

Proof. The lower bound $\mathsf{D}_{\{1,s\}}(C_n) \geq n_2 + \lfloor \log_2(n_1) \rfloor$ is given by item (i) of Theorem 1. Therefore we just need to prove that if $S \in \mathcal{F}(C_{n_1} \oplus C_{n_2})$ and $|S| = n_2 + \lfloor \log_2(n_1) \rfloor$, then S is not $\{1, s\}$-zero-sum free. Let $C_{n_1} = \langle e_1 \rangle$, $C_{n_2} = \langle e_2 \rangle$, $C_{\frac{n_2}{2}} = \langle 2e_2 \rangle$, and $e_2 + C_{\frac{n_2}{2}} = \{te_2 \in C_{n_2} :\ t \text{ is odd}\}$.

Let $k \in \mathbb{Z}$ such that $|S \cap (\{0\} \oplus C_{n_2})| = n_2 - k$. Since $\mathsf{d}(C_{n_2}) = n_2 - 1$, if $k \leq 0$, then, we are done. Hence, assume $k \geq 1$.

(i) CASE $n_1 = 3$, $|S| = n_2 + 1$: It holds $|S \cap (\{e_1, 2e_1\} \oplus C_{n_2})| = k + 1 \geq 2$. From these, it is possible to obtain a zero-sum in the first coordinate using any subsequence formed by two terms $(g_1, h_1) \boldsymbol{\cdot} (g_2, h_2) \mid S \cap (\{e_1, 2e_1\} \oplus C_{n_2})$. In fact, if $g_1 = g_2$, then $(g_1 - g_2, h_1 + h_2) = (0, h_1 + h_2)$, and if $g_1 = e_1$ and $g_2 = 2e_1$ then $(g_1 + g_2, h_1 + h_2) = (0, h_1 + h_2)$.

Let $S = T_1 \boldsymbol{\cdot} T_2 \boldsymbol{\cdot} T_3$, where

$$\begin{aligned} &T_1 \mid S \cap (\{0\} \oplus C_{n_2}), \\ &T_2 \mid S \cap (\{e_1, 2e_1\} \oplus C_{\frac{n_2}{2}}), \\ &T_3 \mid S \cap (\{e_1, 2e_1\} \oplus (e_2 + C_{\frac{n_2}{2}})). \end{aligned}$$

Then it is possible to obtain at least

$$\begin{aligned} \left\lfloor \frac{|T_1|}{2} \right\rfloor + \left\lfloor \frac{|T_2|}{2} \right\rfloor + \left\lfloor \frac{|T_3|}{2} \right\rfloor &= \left\lfloor \frac{n_2 - k}{2} \right\rfloor + \left\lfloor \frac{|T_2|}{2} \right\rfloor + \left\lfloor \frac{k + 1 - |T_2|}{2} \right\rfloor \\ &\geq \frac{n_2 - k - 1}{2} + \frac{|T_2| - 1}{2} + \frac{k + 1 - |T_2| - 1}{2} \\ &\geq \frac{n_2}{2} - 1 \end{aligned}$$

elements in $\{0\} \oplus C_{\frac{n_2}{2}}$. Since $\mathsf{d}(C_{\frac{n_2}{2}}) = \frac{n_2}{2} - 1$, if the latter inequality is strict, then we are done. Otherwise, $n_2 - k$, $|T_2|$ and $k + 1 - |T_2|$ are all odd, hence k is odd. Therefore, after removing the sums of those pairs, it remains one term from each subsequence T_1, T_2 and T_3. Let $(0, t_1 e_2) \mid T_1$, $(ue_1, 2t_2 e_2) \mid T_2$ and $(ve_1, (2t_3 - 1)e_2) \mid T_3$ these remainder terms. If t_1 is even, then we have one more term in $\{0\} \oplus C_{\frac{n_2}{2}}$, and we are done. Hence, we suppose that t_1 is odd. In this case,

$$\big(0, (t_1 + 2t_2 + 2t_3 - 1)e_2\big) \ \in\ \sigma_{\{1,s\}}\big((0, t_1 e_2) \boldsymbol{\cdot} (ue_1, 2t_2 e_2) \boldsymbol{\cdot} (ve_1, (2t_3 - 1)e_2)\big),$$

and again we have one more element in $\{0\} \oplus C_{\frac{n_2}{2}}$. Therefore, we are done.

(ii) CASE $n_1 = 5$, $|S| = n_2 + 2$: As in the previous case, we have

$$|S \cap (\{e_1, 2e_1, 3e_1, 4e_1\} \oplus C_{n_2})| = k + 2 \geq 3.$$

Define $S = T_1 \cdot T_2 \cdot T_3 \cdot T_4 \cdot T_5$, where

$$\begin{aligned} &T_1 \mid S \cap (\{0\} \oplus C_{n_2}),\\ &T_2 \mid S \cap (\{e_1, 4e_1\} \oplus C_{e_2+\frac{n_2}{2}}),\\ &T_3 \mid S \cap (\{e_1, 4e_1\} \oplus C_{\frac{n_2}{2}}),\\ &T_4 \mid S \cap (\{2e_1, 3e_1\} \oplus C_{e_2+\frac{n_2}{2}}),\\ &T_5 \mid S \cap (\{2e_1, 3e_1\} \oplus C_{\frac{n_2}{2}}). \end{aligned}$$

Notice that the terms in each T_i can be grouped in pairs in order to obtain products in $\{0\} \oplus C_{\frac{n_2}{2}}$. Since n_2 and $|S|$ are even, $|T_2|+|T_3|+|T_4|+|T_5| = k+2$ and

$$\sum_{i=1}^{5} \left\lfloor \frac{|T_i|}{2} \right\rfloor \geq \frac{n_2 - k - 1}{2} + \sum_{i=2}^{5} \frac{|T_i| - 1}{2} + \frac{1}{2} = \frac{n_2}{2} - 1,$$

it is possible to obtain at least $\frac{n_2}{2} - 1$ disjoint subsums in $\{0\} \oplus C_{\frac{n_2}{2}}$. If the previous inequality is strict, then we are done. Otherwise, it remains at least one term in four of the subsequences T_i for $1 \leq i \leq 5$. We observe that one term from each T_2 and T_3 generates a subsum in $\{0\} \oplus C_{n_2}$, as well as one term from each T_4 and T_5 generates a subsum in $\{0\} \oplus C_{n_2}$. If $|T_1|$ is even, then it remains the terms from T_2, T_3, T_4 and T_5, and we obtain two subsums in $\{0\} \oplus C_{n_2}$, which produce one more subsum in $\{0\} \oplus C_{\frac{n_2}{2}}$. Otherwise, either T_2 and T_3 generate one more subsum in $\{0\} \oplus C_{n_2}$ or T_4 and T_5 generate one more subsum in $\{0\} \oplus C_{n_2}$, and adding the remainder term from T_1 led us to one more subsum in $\{0\} \oplus C_{\frac{n_2}{2}}$. Hence, we are done.

□

Remark 2. The previous argument can be adapted to higher values of n_1, but the inequality obtained will be weaker. Thereby, it will be required to produce more subsums in $\{0\} \oplus C_{\frac{n_2}{2}}$ using at most one remainder term from each subsequence $T_1, \ldots, T_{n_1}$. This creates several cases as n_1 grows.

From the previous proposition and Corollary 1, it follows that:

Corollary 3. *Let n and s be as in (1.3), and write $n = 2n_1n_2$ as in Lemma 1, where $n_1 \in \{3, 5\}$ and n_2 even. Then*

$$\mathsf{D}_{\{1,s\}}(C_n) = n_2 + \lfloor \log_2(n_1) \rfloor + 1.$$

Acknowledgement. The authors would like to thank the reviewer for the careful reading and valuable comments. The authors are partially supported by FAPEMIG APQ-02546-21, Brazil.

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