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TWO IDENTITIES FOR COMPLEX NUMBERS WITH APPLICATIONS TO SPECIAL SEQUENCES AND FUNCTIONS

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Abstract

We present two identities for n complex numbers.

1. Let $P(z)=\alpha z^m+\beta z^{m-1}+\cdots$ be a polynomial of degree m with $m\le n$, and let $a_1,\cdots,a_n$ be pairwise distinct complex numbers. Then,

$$\sum_{k=1}^{n} P(a_k)\prod_{\substack{j=1\\ j\neq k}}^{n}\frac{1}{a_k-a_j}=\begin{cases}0, & \text{if } m<n-1,\\ \alpha, & \text{if } m=n-1,\\ \alpha(a_1+\cdots+a_n)+\beta, & \text{if } m=n.\end{cases}$$

2. For all pairwise distinct complex numbers $z_1,\cdots,z_n$ $(n\ge 2)$,

$$1+\sum_{k=1}^{n} z_k-\prod_{k=1}^{n} z_k=\sum_{k=1}^{n}\prod_{\substack{j=1\\ j\neq k}}^{n}\left(1+\frac{z_j(z_j-1)(z_k-1)}{z_k-z_j}\right).$$

Applications lead to formulas involving special sequences and functions like, for example,

$$F_n=n-\sum_{k=1}^{n-1}\prod_{\substack{j=1\\ j\neq k}}^{n-1}\left(1+\frac{2(2\cos\frac{j\pi}{n}+i)\cos\frac{j\pi}{n}\cos\frac{k\pi}{n}}{\cos\frac{j\pi}{n}-\cos\frac{k\pi}{n}}\right),$$

where F_n denotes the n-th Fibonacci number.

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1. Introduction and Statement of the Main Results

The work on this paper has been inspired by an interesting article published by J.L. Díaz-Barrero [1] in 2003. He applied methods from complex analysis to obtain identities involving the classical Fibonacci and Lucas numbers. One of his results states that for $n \geq 0$,

$$\frac{L_nL_{n+1}}{(L_n-L_{n+2})(L_{n+1}-L_{n+2})}+\frac{L_{n+1}L_{n+2}}{(L_{n+1}-L_n)(L_{n+2}-L_n)} + \frac{L_{n+2}L_n}{(L_{n+2}-L_{n+1})(L_n-L_{n+1})}=1, \tag{1.1}$$

where L_k denotes the k-th Lucas number. The following companion to (1.1) is valid,

$$\frac{(L_nL_{n+1})^{-1}}{(L_n-L_{n+2})(L_{n+1}-L_{n+2})}+\frac{(L_{n+1}L_{n+2})^{-1}}{(L_{n+1}-L_n)(L_{n+2}-L_n)} + \frac{(L_{n+2}L_n)^{-1}}{(L_{n+2}-L_{n+1})(L_n-L_{n+1})}=0. \tag{1.2}$$

However, formulas (1.1) and (1.2) do not characterize the Lucas numbers. Indeed, both are, respectively, special cases of

$$\frac{ab}{(a-c)(b-c)}+\frac{bc}{(b-a)(c-a)}+\frac{ca}{(c-b)(a-b)}=1$$

and

$$\frac{(ab)^{-1}}{(a-c)(b-c)}+\frac{(bc)^{-1}}{(b-a)(c-a)}+\frac{(ca)^{-1}}{(c-b)(a-b)}=0 \tag{1.3}$$

which hold for all pairwise distinct complex numbers a, b, c. In (1.3), additionally, $abc \neq 0$.

The aim of this paper is to present related identities with n complex numbers. Our first theorem offers an extension of (1.3).

Theorem 1. *Let $P(z) = \alpha z^m + \beta z^{m-1} + \cdots$ be a polynomial of degree m with $m \leq n$, and let $a_1, \cdots, a_n$ be pairwise distinct complex numbers. Then,*

$$\sum_{k=1}^{n} P(a_k) \prod_{\substack{j=1\\ j\neq k}}^{n} \frac{1}{a_k-a_j} = \begin{cases} 0, & \text{if } m < n-1,\\ \alpha, & \text{if } m = n-1,\\ \alpha(a_1+\cdots+a_n)+\beta, & \text{if } m = n. \end{cases} \tag{1.4}$$

Remark 1. Let $a_1, \cdots, a_n$ be pairwise distinct complex numbers. From Theorem 1 with $P(z) = z$, we obtain for $n \geq 3$,

$$\sum_{k=1}^{n} a_k \prod_{\substack{j=1\\ j\neq k}}^{n} \frac{1}{a_k-a_j} = 0. \tag{1.5}$$

If we set $n = 3$ and multiply by $(a_1a_2a_3)^{-1}$, then (1.5) leads to (1.3).

Remark 2. Let $a_1, \cdots, a_n$ be pairwise distinct complex numbers which are different from 0. We apply (1.4) with $P(z) = 1$ and $n+1$ instead of n. Then, we set $a_{n+1} = 0$ and multiply both sides by $(-1)^n a_1 \cdots a_n$. This yields for $n \geq 2$,

$$\sum_{k=1}^{n} \prod_{\substack{j=1 \\ j\neq k}}^{n} \frac{a_j}{a_j - a_k} = 1. \tag{1.6}$$

Remark 3. An application of (1.6) leads to the following counterpart of (1.5). Let

$$\prod_{j=1}^{n} a_j = Q \neq 0,$$

where $a_1, \cdots, a_n$ are pairwise distinct complex numbers. Then,

$$\sum_{k=1}^{n} \frac{1}{a_k} \prod_{\substack{j=1 \\ j\neq k}}^{n} \frac{1}{a_j - a_k} = \frac{1}{Q}. \tag{1.7}$$

Our second theorem presents an identity which involves the sum and the product of n complex numbers.

Theorem 2. *Let $z_1, \cdots, z_n$ $(n \geq 2)$ be pairwise distinct complex numbers. Then,*

$$1 + \sum_{k=1}^{n} z_k - \prod_{k=1}^{n} z_k = \sum_{k=1}^{n} \prod_{\substack{j=1 \\ j\neq k}}^{n} \left(1 + \frac{z_j(z_j-1)(z_k-1)}{z_k - z_j}\right). \tag{1.8}$$

Remark 4. An application of the arithmetic mean - geometric mean inequality and Theorem 2 shows that the inequality

$$\sum_{k=1}^{n} \prod_{\substack{j=1 \\ j\neq k}}^{n} \left(1 + \frac{z_j(z_j-1)(z_k-1)}{z_k - z_j}\right) > 1 + nQ^{1/n} - Q$$

is valid for all positive real numbers $z_1, \cdots, z_n$ $(n \geq 2)$ which are pairwise distinct and satisfy $\prod_{k=1}^{n} z_k = Q$. Let $\epsilon > 0$. If we use (1.8) with

$$z_k = (1 + k\epsilon)Q^{1/n} \qquad \text{for } k = 1, \cdots, [n/2],$$

$$z_k = \frac{Q^{1/n}}{1 + (k - [n/2])\epsilon} \qquad \text{for } k = [n/2]+1, \cdots, 2[n/2],$$

$$z_n = Q^{1/n} \qquad \text{if } n \text{ is odd},$$

and let $\epsilon \to 0$, then we conclude that the constant lower bound $1 + nQ^{1/n} - Q$ is sharp.

The proofs of the two theorems are given in the next section. In Section 3, we apply Theorem 1, Theorem 2 and Remark 3 to obtain various identities involving Lucas and Fibonacci numbers, harmonic numbers, the central binomial coefficient, Chebyshev polynomials of the second kind and an n-th root of unity.

2. Proofs

Proof of Theorem 1. We consider two cases.

`Case 1.` $m \le n-1$. Using the partial fraction decomposition yields

$$\frac{P(z)}{(z-a_1)\cdots(z-a_n)} = \frac{A_1}{z-a_1} + \cdots + \frac{A_n}{z-a_n},$$

where

$$A_k = P(a_k)\prod_{\substack{j=1\\ j\neq k}}^{n}\frac{1}{a_k-a_j}.$$

This leads to

$$P(z) = \alpha z^m + \beta z^{m-1} + \cdots = \prod_{j=1}^{n}(z-a_j)\sum_{j=1}^{n}\frac{A_j}{z-a_j} = c_{n-1}z^{n-1} + \cdots \tag{2.1}$$

with

$$c_{n-1} = \sum_{k=1}^{n} A_k.$$

Comparing the coefficients gives that if $m < n-1$, then (2.1) leads to $c_{n-1} = 0$. If $m = n-1$, then we obtain from (2.1) that $c_{n-1} = \alpha$.

`Case 2.` $m = n$. We have $P(z) = \alpha z^n + P_1(z)$, where P_1 is a polynomial of degree $\le n-1$. It follows that

$$\sum_{k=1}^{n} P(a_k)\prod_{\substack{j=1\\ j\neq k}}^{n}\frac{1}{a_k-a_j} = \alpha S_1 + S_2 \tag{2.2}$$

with

$$S_1 = \sum_{k=1}^{n} a_k^n \prod_{\substack{j=1\\ j\neq k}}^{n}\frac{1}{a_k-a_j} \quad \text{and} \quad S_2 = \sum_{k=1}^{n} P_1(a_k)\prod_{\substack{j=1\\ j\neq k}}^{n}\frac{1}{a_k-a_j}.$$

From

$$\frac{z^{n-1}}{(z-a_1)\cdots(z-a_n)} = \frac{A_1^*}{z-a_1} + \cdots + \frac{A_n^*}{z-a_n}$$

with

$$A_k^* = a_k^{n-1} \prod_{\substack{j=1 \\ j \neq k}}^{n} \frac{1}{a_k - a_j}$$

we obtain

$$z^n = \prod_{j=1}^{n} (z - a_j) \sum_{k=1}^{n} A_k^* + \prod_{j=1}^{n} (z - a_j) \sum_{k=1}^{n} \frac{a_k A_k^*}{z - a_k}.$$

Since

$$\prod_{j=1}^{n} (z - a_j) = z^n - \sum_{k=1}^{n} a_k \cdot z^{n-1} + \cdots ,$$

we get

$$z^n = \left(z^n - \sum_{k=1}^{n} a_k \cdot z^{n-1} + \cdots \right) \sum_{k=1}^{n} A_k^* + \sum_{k=1}^{n} a_k A_k^* \cdot z^{n-1} + \cdots .$$

Next, we compare the coefficients. This gives

$$\sum_{k=1}^{n} A_k^* = 1 \quad \text{and} \quad \sum_{k=1}^{n} a_k \sum_{k=1}^{n} A_k^* = \sum_{k=1}^{n} a_k A_k^*.$$

It follows that

$$\sum_{k=1}^{n} a_k = \sum_{k=1}^{n} a_k A_k^* = S_1. \tag{2.3}$$

Since $\deg P_1 \leq n - 1$, we obtain (as in Case 1 with P_1 instead of P) $\beta = S_2$. From (2.2) and (2.3) we conclude that

$$\sum_{k=1}^{n} P(a_k) \prod_{\substack{j=1 \\ j \neq k}}^{n} \frac{1}{a_k - a_j} = \alpha \sum_{k=1}^{n} a_k + \beta.$$

The proof of Theorem 1 is complete. □

Proof of Theorem 2. We use induction on n. Let

$$F_n(z_1, \cdots , z_n) = 1 + \sum_{k=1}^{n} z_k - \prod_{k=1}^{n} z_k - \sum_{k=1}^{n} \prod_{\substack{j=1 \\ j \neq k}}^{n} \left(1 + \frac{z_j(z_j - 1)(z_k - 1)}{z_k - z_j} \right).$$

We have for $z_1 \neq z_2$,

$$F_2(z_1, z_2) = 1 + z_1 + z_2 - z_1 z_2 - \left(1 + \frac{z_2(z_2-1)(z_1-1)}{z_1 - z_2}\right) - \left(1 + \frac{z_1(z_1-1)(z_2-1)}{z_2 - z_1}\right) = 0.$$

Now, we assume that $F_n(w_1, \cdots, w_n) = 0$ for all pairwise distinct complex numbers $w_1, \cdots, w_n$. Let $z_1, \cdots, z_{n+1}$ be complex numbers which are pairwise distinct. We consider two cases.

`Case 1`. There exists a number z_r $(1 \leq r \leq n+1)$ such that $z_r = 0$ or $z_r = 1$. Since

$$F_{n+1}(z_1, \cdots, z_{r-1}, 0, z_{r+1}, \cdots, z_{n+1}) = F_n(z_1, \cdots, z_{r-1}, z_{r+1}, \cdots, z_{n+1})$$

and

$$F_{n+1}(z_1, \cdots, z_{r-1}, 1, z_{r+1}, \cdots, z_{n+1}) = F_n(z_1, \cdots, z_{r-1}, z_{r+1}, \cdots, z_{n+1}),$$

we conclude from the induction hypothesis that

$$F_{n+1}(z_1, \cdots, z_r, \cdots, z_{n+1}) = 0.$$

`Case 2`. Let $z_k \neq 0$ and $z_k \neq 1$ for $k = 1, \cdots, n+1$. We obtain for $z \in \mathbb{C} \setminus \{z_1, \cdots, z_n\}$,

$$F_{n+1}(z_1, \cdots, z_n, z) = 1 + \sum_{k=1}^{n} z_k + z - \left(\prod_{k=1}^{n} z_k\right) z - \sum_{k=1}^{n} \left[\left(1 + \frac{z(z-1)(z_k-1)}{z_k - z}\right) \prod_{\substack{j=1 \\ j \neq k}}^{n} \left(1 + \frac{z_j(z_j-1)(z_k-1)}{z_k - z_j}\right)\right] - \prod_{j=1}^{n} \left(1 + \frac{z_j(z_j-1)(z-1)}{z - z_j}\right).$$

Let

$$G(z) = F_{n+1}(z_1, \cdots, z_n, z) \prod_{j=1}^{n} (z - z_j).$$

Here, G is a polynomial of degree at most $n+1$. Since

$$G(0) = F_n(z_1, \cdots, z_n) \prod_{j=1}^{n} (-z_j) \quad \text{and} \quad G(1) = F_n(z_1, \cdots, z_n) \prod_{j=1}^{n} (1 - z_j),$$

we conclude from the induction hypothesis that $G(0) = G(1) = 0$. Next, we prove that G has additional zeros. Let $r \in \{1, \cdots, n\}$. Then,

$$\lim_{z \to z_r} (z - z_r) \left(1 + \sum_{k=1}^{n} z_k + z - \left(\prod_{k=1}^{n} z_k\right) z\right) = 0,$$

$$\lim_{z \to z_r} (z - z_r) \sum_{k=1}^{n} \left[\left(1 + \frac{z(z-1)(z_k - 1)}{z_k - z}\right) \prod_{\substack{j=1 \\ j \neq k}}^{n} \left(1 + \frac{z_j(z_j - 1)(z_k - 1)}{z_k - z_j}\right) \right]$$
$$= -z_r(z_r - 1)^2 \prod_{\substack{j=1 \\ j \neq r}}^{n} \left(1 + \frac{z_j(z_j - 1)(z_r - 1)}{z_r - z_j}\right)$$

and

$$\lim_{z \to z_r} (z - z_r) \prod_{j=1}^{n} \left(1 + \frac{z_j(z_j - 1)(z - 1)}{z - z_j}\right)$$
$$= z_r(z_r - 1)^2 \prod_{\substack{j=1 \\ j \neq r}}^{n} \left(1 + \frac{z_j(z_j - 1)(z_r - 1)}{z_r - z_j}\right).$$

This leads to

$$\lim_{z \to z_r} (z - z_r) F_{n+1}(z_1, \cdots, z_n, z) = 0.$$

Thus,

$$G(z_1) = \cdots = G(z_n) = 0.$$

It follows that G has $n+2$ distinct zeros. This can only happen if $G(z) \equiv 0$. Hence,

$$F_{n+1}(z_1, \cdots, z_{n+1}) = G(z_{n+1}) \prod_{j=1}^{n} \frac{1}{z_{n+1} - z_j} = 0.$$

This completes the proof of (1.8). □

3. Applications

We apply both theorems and Remark 3 to provide various identities involving special sequences and functions.

I. Let L_n and F_n be the n-th Lucas and Fibonacci numbers, respectively.
(i) Using the representation

$$L_{n+2} = 1 + \sum_{k=0}^{n} L_k$$

we conclude from (1.8) with $z_k = L_{k-1}$ and $n+1$ instead of n that

$$L_{n+2} = \prod_{k=0}^{n} L_k + \sum_{k=0}^{n} \prod_{\substack{j=0 \\ j\neq k}}^{n} \left(1 + \frac{L_j(L_j-1)(L_k-1)}{L_k - L_j}\right).$$

(ii) We have the product formula

$$F_n = \prod_{k=1}^{n-1} \left(1 - 2i\cos\frac{k\pi}{n}\right),$$

see Rudolph-Lilith [2]. Applying (1.8) gives

$$F_n = n - \sum_{k=1}^{n-1} \prod_{\substack{j=1 \\ j\neq k}}^{n-1} \left(1 + \frac{2(2\cos\frac{j\pi}{n} + i)\cos\frac{j\pi}{n}\cos\frac{k\pi}{n}}{\cos\frac{j\pi}{n} - \cos\frac{k\pi}{n}}\right)$$

and (1.7) yields

$$\frac{1}{F_n} = \left(\frac{i}{2}\right)^{n-2} \sum_{k=1}^{n-1} \frac{1}{1 - 2i\cos\frac{k\pi}{n}} \prod_{\substack{j=1 \\ j\neq k}}^{n-1} \frac{1}{\cos\frac{j\pi}{n} - \cos\frac{k\pi}{n}}.$$

II. Let $H_n = 1 + 1/2 + \cdots + 1/n$ be the n-th harmonic number.
(i) From (1.8) with $z_k = 1 + 1/k$ we conclude that

$$H_n = \sum_{k=1}^{n} \prod_{\substack{j=1 \\ j\neq k}}^{r} \left(1 + \frac{j+1}{j(j-k)}\right).$$

(ii) Let B_n be the Bernoulli polynomial of degree $n \geq 2$. Then,

$$B_n(z) = z^n - \frac{n}{2} z^{n-1} + \cdots .$$

We use (1.4) with $P = B_n$ and $a_k = 1/k$. This leads to

$$H_n = \frac{n}{2} + \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} k^{n-1} B_n\left(\frac{1}{k}\right).$$

(iii) We set $z_k = (-1)^{k+1}/k$ and apply (1.8) with $2n$ instead of n. Since

$$H_{2n} - H_n = \sum_{k=1}^{2n} \frac{(-1)^{k+1}}{k},$$

we get

$$1 + H_{2n} - H_n = \frac{(-1)^n}{(2n)!} + \sum_{k=1}^{2n} \prod_{\substack{j=1 \\ j\neq k}}^{2n} \left(1 + \frac{\left(1 + (-1)^j j\right)\left(1 + (-1)^k k\right)}{j\left(j - (-1)^{j+k} k\right)}\right).$$

III. We have

$$\prod_{k=1}^{n-1}\left(1-\frac{n^2}{k^2}\right)=\frac{(-1)^{n-1}}{2}\binom{2n}{n}\quad\text{and}\quad\sum_{k=1}^{n-1}\left(1-\frac{n^2}{k^2}\right)=n-1-n^2H_{n-1}^{(2)},$$

where $H_n^{(2)}=1+1/2^2+\cdots+1/n^2$ denotes the n-th harmonic number of second order. Then, (1.8) with $z_k=1-n^2/k^2$ yields the following representation for the central binomial coefficient,

$$\frac{(-1)^n}{2}\binom{2n}{n}=n^2H_{n-1}^{(2)}-n+\sum_{k=1}^{n-1}\prod_{\substack{j=1\\j\neq k}}^{n-1}\left(1+\frac{n^2(n^2-j^2)}{j^2(j^2-k^2)}\right).$$

IV. Let U_n be the Chebyshev polynomial of the second kind of degree n. Applying (1.8) and

$$U_n(z)=2^n\prod_{k=1}^{n}\left(z-\cos\frac{k\pi}{n+1}\right)$$

leads to

$$\frac{1}{2^n}U_n(z)=1+nz-\sum_{k=1}^{n}\prod_{\substack{j=1\\j\neq k}}^{n}\left(1+\frac{(z-\cos\frac{j\pi}{n+1})(z-1-\cos\frac{j\pi}{n+1})(z-1-\cos\frac{k\pi}{n+1})}{\cos\frac{j\pi}{n+1}-\cos\frac{k\pi}{n+1}}\right).$$

Provided that $U_n(z)\neq 0$, we conclude from (1.7) that

$$\frac{2^n}{U_n(z)}=\sum_{k=1}^{n}\frac{1}{z-\cos\frac{k\pi}{n+1}}\prod_{\substack{j=1\\j\neq k}}^{n}\frac{1}{\cos\frac{k\pi}{n+1}-\cos\frac{j\pi}{n+1}}.$$

In particular, since $U_n(1)=n+1$, we obtain

$$(n+1)\left(1-\frac{1}{2^n}\right)=\sum_{k=1}^{n}\prod_{\substack{j=1\\j\neq k}}^{n}\left(1+\frac{(1-\cos\frac{j\pi}{n+1})\cos\frac{j\pi}{n+1}\cos\frac{k\pi}{n+1}}{\cos\frac{j\pi}{n+1}-\cos\frac{k\pi}{n+1}}\right)$$

and

$$\frac{2^n}{n+1}=\sum_{k=1}^{n}\frac{1}{1-\cos\frac{k\pi}{n+1}}\prod_{\substack{j=1\\j\neq k}}^{n}\frac{1}{\cos\frac{k\pi}{n+1}-\cos\frac{j\pi}{n+1}}.$$

V. Let $\omega_n=\exp(2\pi i/n)$ be an n-th root of unity. An application of

$$1-z^n=\prod_{k=0}^{n-1}(1-z\omega_n^k)$$

gives

$$z^n+n=\sum_{k=0}^{n-1}\prod_{\substack{j=0\\j\neq k}}^{n-1}\left(1+\frac{z\omega_n^{j+k}(1-z\omega_n^j)}{\omega_n^j-\omega_n^k}\right).$$

References

[1] J. L. Díaz-Barrero, Rational identities and inequalities involving Fibonacci and Lucas numbers, *J. Inequal. Pure Appl. Math.* **4**(5) (2003), Article 83.

[2] M. Rudolph-Lilith, On the product representation of number sequences, with application to the Fibonacci family, *J. Integer Seq.* **19**(3) (2016), Article 16.3.6.