THE DISTRIBUTIONS OF HIGH MOMENTS OF CUSP FORM COEFFICIENTS OVER ARITHMETIC PROGRESSIONS

Guodong Hua[1]
School of Mathematics and Statistics, Weinan Normal University, Shaanxi, Weinan, China
and
School of Mathematics, Shandong University, Shandong, Jinan, China
gdhua@mail.sdu.edu.cn

Bin Chen
School of Mathematics and Statistics, Weinan Normal University, Shaanxi, Weinan, China
ccbb3344@163.com

Rong Miao
School of Mathematics and Statistics, Weinan Normal University, Shaanxi, Weinan, China
251506277@qq.com

Received: 3/2/22, Accepted: 2/1/23, Published: 2/10/23

Abstract

Let f be a normalized primitive holomorphic cusp form of even integral weight k for the full modular group $\Gamma = SL(2,\mathbb{Z})$. Denote by $\lambda_f(n)$ the nth normalized Fourier coefficient of f. In this paper, we are interested in an asymptotic formula for the sum $\sum_{\substack{n\leq x\\ n\equiv l (\mathrm{mod}\ q)}} \lambda_f^{2j}(n)$, where $j \geq 5$ is any positive integer, and q is a prime with $(l,q) = 1$.

1. Introduction

The Fourier coefficients of modular forms are important and interesting objects in number theory. Let H_k^* be the set of all normalized primitive holomorphic cusp forms of even integral weight $k \geq 2$ for the full modular group $\Gamma = SL(2,\mathbb{Z})$. Then

DOI: 10.5281/zenodo.7625134

[1]Corresponding author

the Hecke eigenform $f(z) \in H_k^*$ has the following Fourier expansion at the cusp ∞:

$$f(z) = \sum_{n=1}^{\infty} \lambda_f(n) n^{\frac{k-1}{2}} e^{2\pi i n z}, \quad \Im(z) > 0,$$

where $\lambda_f(n)$ is the nth normalized Fourier coefficient (Hecke eigenvalue) such that $\lambda_f(1) = 1$. Then $\lambda_f(n)$ is real and satisfies the multiplicative property

$$\lambda_f(m)\lambda_f(n) = \sum_{d|(m,n)} \lambda_f\left(\frac{mn}{d^2}\right),$$

where $m \geq 1$ and $n \geq 1$ are positive integers. In 1974, P. Deligne [8] proved the Ramanujan–Petersson conjecture

$$|\lambda_f(n)| \leq d(n), \tag{1}$$

where $d(n)$ is the divisor function. By Inequality (1), Deligne's bound is equivalent to the fact that there exist $\alpha_f(p), \beta_f(p) \in \mathbb{C}$ satisfying

$$\alpha_f(p) + \beta_f(p) = \lambda_f(p), \quad \alpha_f(p)\beta_f(p) = |\alpha_f(p)| = |\beta_f(p)| = 1. \tag{2}$$

More generally, for all positive integers $l \geq 1$, one has

$$\lambda_f(p^l) = \alpha_f(p)^l + \alpha_f(p)^{l-1}\beta_f(p) + \cdots + \alpha_f(p)\beta_f(p)^{l-1} + \beta_f(p)^l.$$

In 1927, Hecke [11] proved that

$$\sum_{n \leq x} \lambda_f(n) \ll x^{\frac{1}{2}}. \tag{3}$$

Later, the upper bound in Inequality (3) was improved by several authors (see, for example, [8, 14, 37]). In particular, Wu [40] has shown that

$$\sum_{n \leq x} \lambda_f(n) \ll x^{\frac{1}{3}} \log^{\rho} x,$$

where

$$\rho = \frac{102 + 7\sqrt{21}}{210}\left(\frac{6 - \sqrt{21}}{5}\right)^{\frac{1}{2}} + \frac{102 - 7\sqrt{21}}{210}\left(\frac{6 + \sqrt{21}}{5}\right)^{\frac{1}{2}} - \frac{33}{35} = -0.118\cdots.$$

In the 1930s, Rankin [36] and Selberg [38] independently proved the following asymptotic formula:

$$\sum_{n \leq x} \lambda_f^2(n) = c_f x + O(x^{3/5}), \tag{4}$$

where $c_f > 0$ is a positive constant depending on f, and $\varepsilon > 0$ is an arbitrarily small positive number. Very recently, the exponent in Equation (4) was improved to $\frac{3}{5} - \delta$ in place of $\frac{3}{5}$ by Huang [16], where $\delta \leq 1/560$. This remains the best possible result.

Later, based on work about symmetric power L-functions, Moreno and Shahidi [31] were able to prove $\sum_{n\leq x} \tau_0^4(n) \sim c_1 x \log x$, as $x \to \infty$, where $\tau_0(n) = \tau(n)/n^{\frac{11}{2}}$ is the normalized Ramanujan tau-function, and $c_1 > 0$ is a positive constant. Obviously, Moreno and Shahidi's result also holds true if we replace $\tau_0(n)$ with the normalized Fourier coefficient $\lambda_f(n)$.

Let $f \in H_k^*$ be a Hecke eigenform and denote its nth normalized Fourier coefficient by $\lambda_f(n)$. Define

$$S_j(f;x) = \sum_{n\leq x} \lambda_f^j(n),$$

where $j \in \mathbb{Z}^+$ and $x \geq 1$.

Based on the work of Moreno and Shahidi concerning the symmetric power L-functions $L(\mathrm{sym}^j f, s)$ for $j = 1, 2, 3, 4$, Fomenko [9] established the following estimates

$$S_3(f;x) \ll_{f,\varepsilon} x^{5/6+\varepsilon}, \quad S_4(f;x) = c_f x \log x + d_f x + O_{f,\varepsilon}(x^{9/10+\varepsilon}),$$

where $c_f > 0$ and d_f are suitable constants depending on f, and ε is an arbitrarily small positive number. Later, Lü (see, for example, [26, 27, 28]) considered higher moments $S_j(f;x)$ for $3 \leq l \leq 8$, which improved and generalized the work of Fomenko. Later, Lau, Lü and Wu [29] proved that

$$S_j(f;x) = xP_j^*(\log x) + O_{f,\varepsilon}\big(x^{\theta_j+\varepsilon}\big), \quad 3 \leq j \leq 8,$$

where $P_j^*(t) \equiv 0$ are zero functions for $j = 3, 5, 7$, and $P_4^*(t), P_6^*(t), P_8^*(t)$ are polynomials of degree $1, 4, 13$, respectively, and

$$\begin{aligned} \theta_3 &= \frac{7}{10}, & \theta_5 &= \frac{40}{43}, & \theta_7 &= \frac{176}{179}, \\ \theta_4 &= \frac{151}{175}, & \theta_6 &= \frac{175}{181}, & \theta_8 &= \frac{2933}{2957}. \end{aligned}$$

Lau and Lü [30] derived the general results for $S_j(f;x)$ for all $j \geq 2$ under the assumption that $L(\mathrm{sym}^r f, s)$ is automorphic cuspidal for some positive r. Later, Zhai [42] investigated the higher power moments over the sum of two squares for

$$S_j^*(f;x) = \sum_{a^2+b^2\leq x} \lambda_f(a^2+b^2)^l$$

for $2 \leq l \leq 8$. Very recently, Xu [41] improved and extended the above work that the author established on general asymptotics for $S_j(f;x)$ and $S_j^*(f;x)$ for all large

j by utilizing the recent progress that $L(\mathrm{sym}^j f, s)$ is automorphic for all $j \geq 1$. This is due to the recent celebrated work of Newton and Thorne [33, 34] and some nice analytic properties of these automorphic L-functions.

Andrianov and Fomenko [1] first considered the second power moment of $\lambda_f(n)$ over arithmetic progressions for holomorphic cusp forms. Later, Andrianov [2] improved the error term. Ichihara [18, 19] investigated $\lambda_f^2(n)$ over arithmetic progressions for holomorphic cusp forms for $x \ll q^2$:

$$\begin{aligned}\sum_{\substack{n\leq x\\ n\equiv l(\mathrm{mod}\ q)}} \lambda_f^2(n) &= \frac{c}{\varphi(q)}\prod_{p|q}(1-\alpha_f(p)^2p^{-1})(1-p^{-1})(1-\beta_f(p)^2)(1+p^{-1})^{-1}x\\ &\quad +O_{f,\varepsilon}(x^{\frac{3}{5}}q^{\frac{4}{5}+\varepsilon}),\end{aligned}$$

where c is some suitable constant depending on f, and $\alpha_f(p), \beta_f(p)$ are the Satake parameters given by Equation (2). Later, Jiang and Lü [21] considered the sum of $\lambda_f^{2j}(n)$ over arithmetic progressions for $j = 2, 3, 4$, respectively. In a similar manner, they also established the corresponding results for the normalized Hecke-Maass cusp form with respect to $SL(2, \mathbb{Z})$ for $j = 2, 3, 4$, respectively.

Very recently, Zou et al. [43] by using the existence of automorphic cuspidal self-dual representation $\mathrm{sym}^j\pi_f$ for all $j \geq 1$ due to Newton and Thorne [33, 34], in combination with some nice properties of the corresponding automorphic L-functions, established the following result.

Theorem 1.1 ([43, Theorem 1]). *Let $f \in H_k^*$ be a Hecke eigenform and let q be a prime with $(q, l) = 1$. For $j \geq 2$ and $q \leq x^{\frac{3}{4}\delta_j}$, one has*

$$\sum_{\substack{n\leq x\\ n\equiv l(\bmod q)}} \lambda_f^2(n^j) = \frac{c_j x}{\varphi(q)} + O_{f,\varepsilon}\big(qx^{1-\frac{3}{2}\delta_j+\varepsilon}\big)$$

for any $\varepsilon > 0$, where $c_j > 0$ are some suitable constants, and $\delta_2 = \frac{92}{597}$ and $\delta_j = \frac{92}{69(j-1)(j+3)+247}$ for $j \geq 3$.

By adopting a similar approach to that given by Theorem 1.1, in this paper we consider the higher moments of cusp form coefficients over arithmetic progressions by adopting the similar approach given by Zou et al. [43]. More precisely, we establish the following result.

Theorem 1.2. *Let $f \in H_k^*$ be a Hecke eigenform and let q be a prime with $(q, l) = 1$. For $j \geq 5$ and $q \ll x^{\vartheta_j}$, one has*

$$\sum_{\substack{n\leq x\\ n\equiv l(\bmod q)}} \lambda_f^{2j}(n) = \frac{xP_{2j}(\log x)}{\varphi(q)} + O_{f,\varepsilon}\big(qx^{1-\frac{3}{2}\vartheta_j+\varepsilon}\big)$$

for any $\varepsilon > 0$, and $P_{2j}(t)$ denotes a polynomial of t with degree $\frac{(2j)!}{j!(j+1)!} - 1$, and $\vartheta_j = \frac{92}{69\cdot 2^{2j}-23A_j-6C_j(1)+4}$ for $j \geq 5$.

Throughout the paper, we always assume that $f \in H_k^*$ is a Hecke eigenform. Let $\varepsilon > 0$ be an arbitrarily small positive constant that may vary in different contexts. The symbol p always denote a prime number.

2. Preliminaries

In this section, we introduce some background on the analytic properties of automorphic L-functions and give some useful lemmas which play an important role in the proof of the main result in this paper.

Let $f \in H_k^*$ be a Hecke eigenform, and let $\lambda_f(n)$ denotes its nth normalized Fourier coefficient. It is natural to define the *Hecke L-function* $L(f,s)$ associated with f by

$$\begin{aligned} L(f,s) &= \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^s} = \prod_p (1-\lambda_f(p)p^{-s}+p^{-2s})^{-1} \\ &= \prod_p \left(1-\frac{\alpha_f(p)}{p^s}\right)^{-1}\left(1-\frac{\beta_f(p)}{p^s}\right)^{-1}, \ \Re(s)>1, \end{aligned}$$

where $\alpha_f(p)$ and $\beta_f(p)$ are the local parameters satisfying Equations (2). The *jth symmetric power L-function* associated with f is defined by

$$L(\mathrm{sym}^j f,s) = \prod_p \prod_{m=0}^{j} (1-\alpha_f(p)^{j-m}\beta_f(p)^m p^{-s})^{-1}, \ \Re(s)>1.$$

We may expand it into a Dirichlet series

$$\begin{aligned} L(\mathrm{sym}^j f,s) &= \sum_{n=1}^{\infty} \frac{\lambda_{\mathrm{sym}^j f}(n)}{n^s}, \\ &= \prod_p \left(1+\frac{\lambda_{\mathrm{sym}^j f}(p)}{p^s}+\cdots+\frac{\lambda_{\mathrm{sym}^j f}(p^k)}{p^{ks}}+\cdots\right), \ \Re(s)>1. \end{aligned}$$

We know that the coefficients $\lambda_{\mathrm{sym}^j f}(n)$ are real, multiplicative functions. For $j=1$, we have $L(\mathrm{sym}^1 f,s) = L(f,s)$.

It is standard to find that

$$\lambda_f(p^j) = \lambda_{\mathrm{sym}^j f}(p) = \frac{\alpha_f(p)^{j+1}-\beta_f(p)^{j+1}}{\alpha_f(p)-\beta_f(p)} = \sum_{m=0}^{j} \alpha_f(p)^{j-m}\beta_f(p)^m,$$

which can be written as $\lambda_f(p^j) = \lambda_{\mathrm{sym}^j f}(p) = U_j(\lambda_f(p)/2)$, where $U_j(x)$ is the jth Chebyshev polynomial of the second kind.

Let χ be a Dirichlet character modulo q. Then we can define the *twisted* j*th symmetric power* L*-function* by the Euler product representation with degree $j+1$,

$$\begin{aligned} L(\mathrm{sym}^j f \otimes \chi, s) &:= \prod_p \prod_{m=0}^{j} (1-\alpha_f(p)^{j-m}\beta_f(p)^m\chi(p)p^{-s})^{-1} \\ &:= \sum_{n=1}^{\infty} \frac{\lambda_{\mathrm{sym}^j f}(n)\chi(n)}{n^s} \end{aligned}$$

for $\Re(s) > 1$.

It is well known that a primitive form f is associated with an automorphic cuspidal representation π_f of $GL_2(\mathbb{A}_{\mathbb{Q}})$, and hence an automorphic L-function $L(\pi_f, s)$ which coincides with $L(f, s)$. It is predicted that π_f gives rise to a symmetric power lift, an automorphic representation whose L-function is the symmetric power L-function attached to f.

For $1 \le j \le 8$, the special Langlands functoriality conjecture states that $\mathrm{sym}^j f$ is automorphic cuspidal, as has been shown by a series of important work of Gelbart and Jacquet [10], Kim [25], Kim and Shahidi [24, 23], Shahidi [39], and Clozel and Thorne [5, 6, 7]. Very recently, Newton and Thorne [33, 34] proved that $\mathrm{sym}^j f$ corresponds with a cuspidal automorphic representation of $GL_{j+1}(\mathbb{A}_{\mathbb{Q}})$ for all $j \ge 1$ (with f being a holomorphic cusp form). Then we know that $L(\mathrm{sym}^j f, s)$, where $j \ge 1$, has analytic continuation to the whole complex plane as an entire function and satisfies certain Riemann-type functional equation. We refer the interested readers to [20, Chapter 5] for a more comprehensive treatment.

First, we give the functional equations of the L-functions $L(\mathrm{sym}^j f \otimes \chi, s)$ for any $j \ge 1$, which is a special case due to [43, Lemma 3].

Lemma 2.1. *Let $f \in H_k^*$ be a Hecke eigenform, and let χ be a primitive character modulo a prime q. Then the complete L-function*

$$\Lambda(sym^j f \otimes \chi, s) := q^{(j+1)s/2}\gamma(s)L(sym^j f \otimes \chi, s)$$

can be extended to the whole complex plane as an entire function and satisfies the functional equation

$$\Lambda(sym^j f \otimes \chi, s) = \varepsilon(f, \chi)\Lambda(sym^j f \otimes \bar{\chi}, 1-s),$$

where $j \ge 1$ and $|\varepsilon(f,\chi)| = 1$, and $\gamma(s)$ denotes the product of some gamma functions $\Gamma((s+\kappa_n))/2, n = 1, 2, \ldots, (j+1)$, with κ_n depending on the weight of f and the parity of the character χ and $\Re(\kappa_n) \ge 0$.

Lemma 2.2. *For any $\varepsilon > 0$, we have $\int_1^T \left|\zeta\left(\frac{1}{2}+it\right)\right|^{12} dt \ll T^{2+\varepsilon}$, uniformly for $T \ge 1$, and*

$$\zeta(\sigma+it) \ll \big(1+|t|\big)^{\max\{\frac{13}{42}(1-\sigma),0\}+\varepsilon} \tag{5}$$

uniformly for $\frac{1}{2} \leq \sigma \leq 1+\varepsilon$ *and* $|t| \geq 1$.

Proof. This first result is given by Heath-Brown [12], and the second result is the recent breakthrough due to Bourgain [4, Theorem 5]. □

Lemma 2.3. *For any* $\varepsilon > 0$, *we have*

$$L(sym^2 f, \sigma+it) \ll \big(1+|t|\big)^{\max\{\frac{27}{20}(1-\sigma),0\}+\varepsilon} \tag{6}$$

uniformly for $\frac{1}{2} \leq \sigma \leq 1+\varepsilon$ *and* $|t| \geq 1$.

Proof. From the result given by Aggarwal [3, Theorem 1.1], we can easily deduce that

$$L\bigg(\mathrm{sym}^2 f, \frac{1}{2}+it\bigg) \ll \big(1+|t|\big)^{\frac{27}{40}+\varepsilon}. \tag{7}$$

We can obtain the required result from the Phragmén–Lindelöf principle for a strip [20, Theorem 5.53] and the Inequality (7). □

Lemma 2.4. *Let* χ *be a primitive character modulo* q. *For* $T \geq 1$ *and* $q \ll T^2$,

$$L(\sigma+iT,\chi) \ll_\varepsilon (q(1+|T|))^{\max\{\frac{1}{3}(1-\sigma),0\}+\varepsilon} \tag{8}$$

$$L(\sigma+iT, sym^2 f\otimes\chi) \ll_\varepsilon (q(1+|T|))^{\max\{\frac{67}{46}(1-\sigma),0\}+\varepsilon}, \tag{9}$$

and if q *is a prime,* $\displaystyle\int_0^T |L(\sigma+it,\chi)|^{12} dt \ll_\varepsilon q^{4(1-\sigma)} T^{3-2\sigma+\varepsilon}$.

Proof. The results follows from work of Heath-Brown [13], Huang [15] and Motohashi [32], together with the Phragmén-Lindelöf principle for a strip, respectively. □

From above, we note that the L-functions $L(\mathrm{sym}^j f, s)$ and $L(\mathrm{sym}^j f\otimes\chi, s)$ are the general L-functions in the sense of Perelli [35]. For these general L-functions, we have the following individual or averaged convexity bounds.

Lemma 2.5. *Let* χ *be a primitive character modulo* q. *For the general* L*-functions* $\mathfrak{L}(s,\chi)$ *of degree* $2A$ *indicated above, we have*

$$\int_T^{2T} |\mathfrak{L}(\sigma+it,\chi)|^2 dt \ll (qT)^{2A(1-\sigma)+\varepsilon} \tag{10}$$

uniformly for $\frac{1}{2} \leq \sigma \leq 1$ *and* $T \geq 1$. *Furthermore,*

$$\mathfrak{L}(\sigma+it,\chi) \ll (q(|t|+1))^{\max\{A(1-\sigma),0\}+\varepsilon} \tag{11}$$

uniformly for $-\varepsilon \leq \sigma \leq 1+\varepsilon$.

Proof. This result can be derived by following the similar argument to that used in Zou et al. [43], which was originally deduced from Jiang and Lü [21]. □

Remark 2.6. For the automorphic L-functions $L(\mathrm{sym}^j f, s)$, where $j \geq 1$, we can regard the modulus q to be 1.

Let $f \in H_k^*$ be a Hecke eigenform. For $j \geq 2$, define

$$\mathfrak{R}_j(s) := \sum_{n=1}^{\infty} \frac{\lambda_f^j(n)}{n^s}, \quad \Re(s) > 1. \tag{12}$$

Lemma 2.7. *Let $\mathfrak{R}_j(s)$ be defined by Equation (12). For $j = 2n$ with $n \geq 5$, we have*

$$\mathfrak{R}_j(s) = \zeta(s)^{A_n} L(sym^{2n} f, s) \prod_{1 \leq r \leq n-1} L(sym^{2r} f, s)^{C_n(r)} H_j(s),$$

where $A_n, C_n(r), (1 \leq r \leq n-1)$ are given by

$$A_n = \frac{(2n)!}{n!(n+1)!} \qquad \text{and} \qquad C_n(1) = \frac{3 \cdot (2n)!}{(n-1)!(n+2)!}. \tag{13}$$

The L-function $\mathfrak{R}_j(s)$ is of degree 2^j, all coefficients of $\mathfrak{R}_j(s)$ are nonnegative, the function $H_j(s)$ admits the Dirichlet series which converges absolutely and uniformly in the half-plane $\Re(s) \geq \frac{1}{2} + \varepsilon$, and $H_j(s) \neq 0$ for $\Re(s) = 1$.

Proof. This result is given by Lau and Lü [30, Lemma 7.1]. □

Let $f \in H_k^*$ be a Hecke eigenform and let χ be a Dirichlet character modulo q. For $j \geq 2$, define

$$\mathfrak{F}_j(s, \chi) := \sum_{n=1}^{\infty} \frac{\lambda_f^j(n)\chi(n)}{n^s}, \quad \Re(s) > 1. \tag{14}$$

Lemma 2.8. *Let $\mathfrak{F}_j(s, \chi)$ be defined by Equation (14). And let χ be a primitive character modulo a prime q. For $j = 2n$ with $n \geq 5$, we have*

$$\mathfrak{F}_j(s, \chi) = L(s, \chi)^{A_n} L(sym^{2n} f \otimes \chi, s) \prod_{1 \leq r \leq n-1} L(sym^{2r} f \otimes \chi, s)^{C_n(r)} U_j(s, \chi),$$

where $A_n, C_n(r), (1 \leq r \leq n-1)$ are suitable constants, and $A_n, C_n(1)$ are the same as in Equation (13). The L-function $\mathfrak{F}_j(s, \chi)$ is of degree 2^j, the function $U_j(s, \chi)$ admits the Dirichlet series which converges absolutely and uniformly in the half-plane $\Re(s) \geq \frac{1}{2} + \varepsilon$, and $U_j(s, \chi) \neq 0$ for $\Re(s) = 1$.

Proof. This follows from an argument similar to that of Lau and Lü [30, Lemma 7.1]. □

3. Proof of Theorem 1.2

In this section, we give the proof of the main result. Firstly, we need the following two important propositions which play an essential role in the proof of the main theorem in this paper.

Let $f \in H_k^*$ be a Hecke eigenform, and let χ be a primitive character modulo a prime q. Define

$$S_j(\chi;x) = \sum_{n\leq x} \lambda_f^{2j}(n)\chi(n), \tag{16}$$

where $j \geq 5$ is any given positive integer.

Proposition 3.1. *Let $S_j(\chi;x)$ be defined as in Equation (16). Then, for $q \ll x^{\vartheta_j}$, we have*

$$S_j(\chi;x) = O_{f,\varepsilon}\big(qx^{1-\frac{3}{2}\vartheta_j+\varepsilon}\big), \tag{17}$$

where $\vartheta_j = \frac{92}{69\cdot 2^{2j}-23A_j-6C_j(1)+4}$ for $j \geq 5$, where A_j is given by (13).

Proof. Applying Perron's formula [20, Proposition 5.54] and Deligne's bound (2), we obtain

$$S_j(\chi;x) = \frac{1}{2\pi i}\int_{1+\varepsilon-iT}^{1+\varepsilon+iT} \mathfrak{F}_{2j}(s,\chi)\frac{x^s}{s}ds + O_{f,\varepsilon}\left(\frac{x^{1+\varepsilon}}{T}\right),$$

where $s = \sigma + it$ and $1 \leq T \leq x$ is some parameter to be specified later.

By shifting line of integration to the parallel line with $\Re(s) = \frac{1}{2}+\varepsilon$ and applying Cauchy's residue theorem, we obtain

$$\begin{aligned} S_j(\chi;x) &= \frac{1}{2\pi i}\left(\int_{\frac{1}{2}+\varepsilon-iT}^{\frac{1}{2}+\varepsilon+iT} + \int_{1+\varepsilon-iT}^{\frac{1}{2}+\varepsilon-iT} + \int_{\frac{1}{2}+\varepsilon+iT}^{1+\varepsilon+iT}\right)\mathfrak{F}_{2j}(s,\chi)\frac{x^s}{s}ds \\ &\quad + O_{f,\varepsilon}\left(\frac{x^{1+\varepsilon}}{T}\right) \\ &= I_{j,1} + I_{j,2} + I_{j,3} + O_{f,\varepsilon}\left(\frac{x^{1+\varepsilon}}{T}\right), \end{aligned} \tag{18}$$

where we have used the fact that $\mathfrak{F}_{2j}(s,\chi)$ has no poles in $\Re(s) > \frac{1}{2}$ and $U_j(s,\chi)$ converges absolutely and uniformly in q for the same region.

For the integrals over the horizontal segments $I_{j,2}$ and $I_{j,3}$, by Inequalities (8), (9) and (11), along with Lemma 2.8, for $j \geq 5$ and $q \ll T^2$, it follows that

$$\begin{aligned}
I_{j,2}+I_{j,3} &\ll \int_{\frac{1}{2}+\varepsilon}^{1+\varepsilon} x^{\sigma}|\mathfrak{F}_{2j}(s,\chi)|T^{-1}d\sigma \\
&\ll \int_{\frac{1}{2}+\varepsilon}^{1+\varepsilon} x^{\sigma}\Bigg|L(s,\chi)^{A_j}L(\mathrm{sym}^2 f\otimes\chi,s)^{C_j(1)}L(\mathrm{sym}^{2j} f\otimes\chi,s) \\
&\quad \times \prod_{2\leq r\leq j-1} L(\mathrm{sym}^{2r} f\otimes\chi,s)^{C_j(r)}\Bigg|T^{-1}d\sigma \\
&\ll \max_{\frac{1}{2}+\varepsilon\leq\sigma\leq 1+\varepsilon} x^{\sigma}(qT)^{(2^{2j-1}-\frac{1}{6}A_j-\frac{1}{23}C_j(1))(1-\sigma)+\varepsilon}T^{-1} \\
&\ll \frac{x^{1+\varepsilon}}{T}+x^{\frac{1}{2}+\varepsilon}q^{2^{2j-2}-\frac{1}{12}A_j-\frac{1}{46}C_j(1)+\varepsilon}T^{2^{2j-2}-\frac{1}{12}A_j-\frac{1}{46}C_j(1)-1+\varepsilon},
\end{aligned} \tag{19}$$

where $A_j, C_j(1)$ are defined as in Equation (13).

For the integral over a vertical segment $I_{j,1}$, by Lemma 2.4, Lemma 2.8 and Inequality (10), we have

$$\begin{aligned}
I_{j,1} &\ll x^{\frac{1}{2}+\varepsilon}\log T \sup_{1\leq T_1\leq T/2}\left\{\frac{1}{T_1}\int_{T_1}^{2T_1}\left|\mathfrak{F}_{2j}\left(\frac{1}{2}+\varepsilon+it,\chi\right)\right|dt\right\} \\
&\ll x^{\frac{1}{2}+\varepsilon}\log T \sup_{1\leq T_1\leq T/2}\left\{\frac{1}{T_1}\left(\int_{T_1}^{2T_1}\left|L\left(\frac{1}{2}+\varepsilon+it,\chi\right)^{A_j}\right|^6 dt\right)^{\frac{1}{6}}\right. \\
&\quad \times\left(\int_{T_1}^{2T_1}\left|L\left(\mathrm{sym}^2 f\otimes\chi,\frac{1}{2}+\varepsilon+it\right)^{C_j(1)}\right|^3 dt\right)^{\frac{1}{3}} \\
&\quad \times\left(\int_{T_1}^{2T_1}\left|\prod_{2\leq r\leq j-1} L\left(\mathrm{sym}^{2r} f\otimes\chi,\frac{1}{2}+\varepsilon+it\right)^{C_j(r)}\right.\right. \\
&\quad \left.\left.\times L\left(\mathrm{sym}^{2j} f\otimes\chi,\frac{1}{2}+\varepsilon+it\right)\right|^2 dt\right)^{\frac{1}{2}}\Bigg\} \\
&\ll x^{\frac{1}{2}+\varepsilon}\log T \sup_{1\leq T_1\leq T/2}\left\{\frac{1}{T_1}\left(\left|L\left(\frac{1}{2}+\varepsilon+iT_1,\chi\right)\right|^{6A_j-12}\right.\right. \\
&\quad \left.\times\int_{T_1}^{2T_1}\left|L\left(\frac{1}{2}+\varepsilon+it,\chi\right)\right|^{12}dt\right)^{\frac{1}{6}} \\
&\quad \times\left(\left|L\left(\mathrm{sym}^2 f\otimes\chi,\frac{1}{2}+\varepsilon+it\right)\right|^{3C_j(1)-2}\right. \\
&\quad \left.\times\int_{T_1}^{2T_1}\left|L\left(\mathrm{sym}^2 f\otimes\chi,\frac{1}{2}+\varepsilon+it\right)\right|^2 dt\right)^{\frac{1}{3}}
\end{aligned}$$

$$\times\left(\int_{T_1}^{2T_1}\left|\prod_{2\le r\le j-1}L\left(\mathrm{sym}^{2r}f\otimes\chi,\frac{1}{2}+\varepsilon+it\right)^{C_j(r)}\right.\right.$$
$$\left.\left.\times L\left(\mathrm{sym}^{2j}f\otimes\chi,\frac{1}{2}+\varepsilon+it\right)\right|^2dt\right)^{\frac{1}{2}}\Bigg\}$$
$$\ll\quad x^{\frac{1}{2}+\varepsilon}q^{2^{2j-2}-\frac{1}{12}A_j-\frac{1}{46}C_j(1)+\frac{1}{69}+\varepsilon}T^{2^{2j-2}-\frac{1}{12}A_j-\frac{1}{46}C_j(1)-\frac{68}{69}+\varepsilon}. \qquad (20)$$

By inserting the Equations (19) and (20) into Equation (18), we obtain Equation (17) by taking

$$T=x^{\frac{138}{69\cdot 2^{2j}-23A_j-6C_j(1)+4}}/q.$$

Since $q\ll T^2$, we have the required restriction that

$$q\ll x^{\frac{92}{69\cdot 2^{2j}-23A_j-6C_j(1)+4}}.$$

This completes the proof of the proposition. □

Let χ_0 be a principal character modulo a prime q. Define

$$S_j(\chi_0;x)=\sum_{n\le x}\lambda_f^{2j}(n)\chi_0(n), \qquad (21)$$

where $j\ge 5$ is any given positive integer.

Proposition 3.2. *Let $S_j(\chi_0;x)$ be defined as in Equation (21). Then for $q\ll x$,*

$$S_j(\chi_0;x)=xP_{2j}(\log x)+O\big(x^{\kappa_j+\varepsilon}\big),$$

where $P_{2j}(t)$ denotes a polynomial of t with degree $\frac{(2j)!}{j!(j+1)!}-1$, and $\kappa_j=1-\frac{420}{2^{2j+1}\cdot 105-80A_j-63C_j(1)+62}$.

Proof. Define

$$\mathfrak{R}_j(s,\chi_0)=\sum_{n=1}^{\infty}\frac{\lambda_f^{2j}(n)\chi_0(n)}{n^s}=\mathfrak{R}_{2j}(s)H(s,\chi_0), \qquad (22)$$

where $\mathfrak{R}_{2j}(s)$ is given by Lemma 2.7, and $H(s,\chi_0)$ is a finite Euler product which does not equal 0 for $\Re(s)>\frac{1}{2}$.

Applying Perron's formula [20, Proposition 5.54] and Deligne's bound (2), we obtain

$$S_j(\chi_0;x)=\frac{1}{2\pi i}\int_{1+\varepsilon-iT}^{1+\varepsilon+iT}\mathfrak{R}_j(s,\chi_0)\frac{x^s}{s}ds+O_{f,\varepsilon}\left(\frac{x^{1+\varepsilon}}{T}\right),$$

where $s=\sigma+it$ and $1\le T\le x$ is some parameter to be specified later.

By shifting the line of integration to the parallel line with $\Re(s)=\frac{1}{2}+\varepsilon$ and applying Cauchy's residue theorem, we obtain

$$\begin{aligned}
S_j(\chi_0;x) &= \frac{1}{2\pi i}\left(\int_{\frac{1}{2}+\varepsilon-iT}^{\frac{1}{2}+\varepsilon+iT}+\int_{1+\varepsilon-iT}^{\frac{1}{2}+\varepsilon-iT}+\int_{\frac{1}{2}+\varepsilon+iT}^{1+\varepsilon+iT}\right)\mathfrak{R}_j(s,\chi_0)\frac{x^s}{s}ds\\
&\quad+\mathrm{Res}_{s=1}\left\{\mathfrak{R}_j(s,\chi_0)\frac{x^s}{s}\right\}+O_{f,\varepsilon}\left(\frac{x^{1+\varepsilon}}{T}\right)\\
&:= J_{j,1}+J_{j,2}+J_{j,3}+xP_{2j}(\log x)+O_{f,\varepsilon}\left(\frac{x^{1+\varepsilon}}{T}\right), \qquad (23)
\end{aligned}$$

where $P_{2j}(t)$ denotes a polynomial of t with degree A_j-1, and A_j is given by (13).

For the integrals over the horizontal segments $J_{j,2}$ and $J_{j,3}$, by Equations (5), (6), and (11), (22), together with Lemma 2.7, for $j\ge 5$ and $q\ll T^2$, it follows that

$$\begin{aligned}
J_{j,2}+J_{j,3} &\ll \int_{\frac{1}{2}+\varepsilon}^{1+\varepsilon} x^\sigma|\mathfrak{R}_{2j}(s,\chi_0)|T^{-1}d\sigma\\
&\ll \int_{\frac{1}{2}+\varepsilon}^{1+\varepsilon} x^\sigma\left|\zeta(s)^{A_j}L(\mathrm{sym}^2 f,s)^{C_j(1)}L(\mathrm{sym}^{2j}f,s)\right.\\
&\qquad\times\left.\prod_{2\le r\le j-1}L(\mathrm{sym}^{2r}f,s)^{C_j(r)}\right|T^{-1}d\sigma\\
&\ll \max_{\frac{1}{2}+\varepsilon\le\sigma\le 1+\varepsilon} x^\sigma T^{(2^{2j-1}-\frac{4}{21}A_j-\frac{3}{20}C_j(1))(1-\sigma)+\varepsilon}T^{-1}\\
&\ll \frac{x^{1+\varepsilon}}{T}+x^{\frac{1}{2}+\varepsilon}T^{2^{2j-2}-\frac{2}{21}A_j-\frac{3}{40}C_j(1)-1+\varepsilon}, \qquad (24)
\end{aligned}$$

where $A_j, C_j(1)$ are defined as in Equation (13).

For the integral over a vertical segment $J_{j,1}$, by Lemma 2.2, Lemma 2.3 and Inequality (10), we have

$$\begin{aligned}
J_{j,1} &\ll x^{\frac{1}{2}+\varepsilon}\log T\sup_{1\le T_1\le T/2}\left\{\frac{1}{T_1}\int_{T_1}^{2T_1}\left|\mathfrak{R}_j\left(\frac{1}{2}+\varepsilon+it\right)\right|dt\right\}\\
&\ll x^{\frac{1}{2}+\varepsilon}\log T\sup_{1\le T_1\le T/2}\left\{\frac{1}{T_1}\left(\int_{T_1}^{2T_1}\left|\zeta\left(\frac{1}{2}+\varepsilon+it\right)^{A_j}\right|^6dt\right)^{\frac{1}{6}}\right.\\
&\quad\times\left(\int_{T_1}^{2T_1}\left|L\left(\mathrm{sym}^2 f,\frac{1}{2}+\varepsilon+it\right)^{C_j(1)}\right|^3dt\right)^{\frac{1}{3}}\\
&\quad\times\left(\int_{T_1}^{2T_1}\left|\prod_{2\le r\le j-1}L\left(\mathrm{sym}^{2r}f,\frac{1}{2}+\varepsilon+it\right)^{C_j(r)}\right.\right.\\
&\quad\times\left.\left.\left.L\left(\mathrm{sym}^{2j}f,\frac{1}{2}+\varepsilon+it\right)\right|^2dt\right)^{\frac{1}{2}}\right\}
\end{aligned}$$

$$\begin{aligned}
&\ll x^{\frac{1}{2}+\varepsilon}\log T \sup_{1\le T_1\le T/2}\Bigg\{\frac{1}{T_1}\Bigg(\bigg|\zeta\Big(\frac{1}{2}+\varepsilon+iT_1\Big)\bigg|^{6A_j-12}\\
&\quad\times\int_{T_1}^{2T_1}\bigg|\zeta\Big(\frac{1}{2}+\varepsilon+it\Big)\bigg|^{12}dt\Bigg)^{\frac{1}{6}}\\
&\quad\times\Bigg(\bigg|L\Big(\mathrm{sym}^2 f,\frac{1}{2}+\varepsilon+it\Big)\bigg|^{3C_j(1)-2}\int_{T_1}^{2T_1}\bigg|L\Big(\mathrm{sym}^2 f,\frac{1}{2}+\varepsilon+it\Big)\bigg|^{2}dt\Bigg)^{\frac{1}{3}}\\
&\quad\times\Bigg(\int_{T_1}^{2T_1}\bigg|\prod_{2\le r\le j-1}L\Big(\mathrm{sym}^{2r} f,\frac{1}{2}+\varepsilon+it\Big)^{C_j(r)}\\
&\quad\times L\Big(\mathrm{sym}^{2j} f,\frac{1}{2}+\varepsilon+it\Big)\bigg|^{2}dt\Bigg)^{\frac{1}{2}}\Bigg\}\\
&\ll x^{\frac{1}{2}+\varepsilon}T^{2^{2j-2}-\frac{2}{21}A_j-\frac{3}{40}C_j(1)-\frac{389}{420}+\varepsilon}.
\end{aligned}\tag{25}$$

By inserting the Equations (24), (25) into Equation (23), we obtain

$$S_j(\chi_0;x)=xP_{2j}(\log x)+O\Big(\frac{x^{1+\varepsilon}}{T}\Big)+O\big(x^{\frac{1}{2}+\varepsilon}T^{2^{2j-2}-\frac{2}{21}A_j-\frac{3}{40}C_j(1)-\frac{389}{420}+\varepsilon}\big).\tag{26}$$

On taking

$$T=x^{\frac{420}{2^{2j+1}\cdot 105-80A_j-63C_j(1)+62}}$$

in Equation (26), we obtain the desired result. □

Proof of Theorem 1.2 We are now able to prove Theorem 1.2. Let q be a Dirichlet character modulo a prime q. By the orthogonality relation of Dirichlet characters, we have

$$\begin{aligned}
\sum_{\substack{n\le x\\ n\equiv l(\bmod q)}}\lambda_f^{2j}(n) &= \frac{1}{\varphi(q)}\sum_{\chi(\bmod q)}\bar{\chi}(l)\sum_{n\le x}\lambda_f^{2j}(n)\chi(n)\\
&= \frac{1}{\varphi(q)}\sum_{n\le x}\lambda_f^{2j}(n)\chi_0(n)+O\Big(\sum_{n\le x}\lambda_f^{2j}(n)\chi(n)\Big).
\end{aligned}$$

It is obvious that $\varphi(q)=q-1$ since q is a prime.

From Propositions 3.1 and 3.2 and noting that $1-\frac{3}{2}\vartheta_j>\kappa_j$ for $j\ge 5$, we have

$$\sum_{\substack{n\le x\\ n\equiv l(\bmod q)}}\lambda_f^{2j}(n)=\frac{xP_{2j}(\log x)}{\varphi(q)}+O_{f,\varepsilon}\big(qx^{1-\frac{3}{2}\vartheta_j+\varepsilon}\big).$$

This completes the proof of Theorem 1.2. □

Acknowledgements. The first author would like to express his gratitude to Professor Guangshi Lü and Dr. Wei Zhang for their constant encouragement and valuable suggestions. The authors are extremely grateful to the anonymous referees for their meticulous checking, for thoroughly reporting countless typos and inaccuracies as well as for their valuable comments. These corrections and additions have made the manuscript clearer and more readable. This work was supported in part by The National Key Research and Development Program of China (Grant No. 2021YFA1000700) and Natural Science Basic Research Program of Shaanxi (Program No. 2023-JC-QN-0024, 2023-JC-YB-077).

References

[1] A. N. Andrianov and O.M. Fomenko, On square means of progressions of Fourier coefficients of parabolic forms, *Trudy Mat. Inst. Steklov.* **80** (1965), 5-15 (in Russian).

[2] M. K. Akbarov, On the quadratic mean over arithmetic progressions of the Fourier coefficients of cusp forms, *Math. USSR-Izv.* **22** (1983), 191-209.

[3] K. Aggarwal, A new subconvex bound for $GL(3)$ L-functions in the t-aspect, *Int. J. Number Theory* **17** (2021), 1111-1138.

[4] J. Bourgain, Decoupling, exponential sums and the Riemann zeta function, *J. Amer. Math. Soc.* **30** (2017), 205-224.

[5] L. Clozel and J. A. Thorne, Level-raising and symmetric power functoriality. I, *Compos. Math.* **150** (2014), 729-748.

[6] L. Clozel and J. A. Thorne, Level-raising and symmetric power functoriality. II, *Ann. of Math.* **181** (2015), 303-359.

[7] L. Clozel and J. A. Thorne, Level-raising and symmetric power functoriality. III, *Duke Math. J.* **166** (2017), 325-402.

[8] P. Deligne, La Conjecture de Weil. I, *Inst. Hautes Études Sci. Pull. Math.* **43** (1974), 273-307.

[9] O. M. Fomenko, Fourier coefficients of parabolic forms, and automorphic L-functions, *J. Math. Sci. (New York)* **95** (1999), 2295-2316.

[10] S. Gelbart and H. Jacquet, A relation between automorphic representations of $GL(2)$ and $GL(3)$, *Ann. Sci. École Norm. Sup.* **11** (1978), 471-542.

[11] E. Hecke, Theorie der Eisensteinsche Reihen höherer Stufe und ihre Anwendung auf Funktionentheorie und Arithmetik, *Monatshefte Math.* **5** (1927), 199-224.

[12] D. R. Heath-Brown, The twelfth power moment of Riemann-function, *Quart. J. Math. (Oxford)* **29** (1978), 443-462.

[13] D. R. Heath-Brown, *Hybrid bounds for L-functions: a q-analogue of Van der Corput's method and a t-analogue of Burgess' method,* Academic Press, London, 1981.

[14] J. L. Hafner and A. Ivić, On sums of Fourier coefficients of cusp forms, *L'Enseignement Math.* **35** (1989), 375-382.

[15] B. R. Huang, Hybrid subconvexity bounds for twisted L-functions on $GL(3)$, *Sci. China Math.* **64** (3) (2021), 443-478.

[16] B. R. Huang, On the Rankin-Selberg problem, *Math. Ann.* **381** (2021), 1217-1251.

[17] A. Ivić, On zeta-functions associated with Fourier coefficients of cusp forms, in: *Proc. Amal. Conf. Analytic Number Theory (Maiori, 1989)*, Univ. Saleron, Salerno, 1992, 231-246.

[18] Y. Ichihara, The evaluation of the sum over arithmetic progressions for the coefficients of the Rankin-Selberg series, *Japan. J. Math. (N.S.)* **29** (2003), 297-314.

[19] Y. Ichihara, *The evaluation of the sum over arithmetic progressions for the coefficients of the Rankin-Selberg series. II,* in: Analytic Number Theory (Beijing/Kyoto, 1999), Kluwer, Dordrecht, 2002, 173-182.

[20] H. Iwaniec and E. Kowalski, *Analytic Number Theory*, Amer. Math. Soc., Providence, Vol. 53, RI, 2004.

[21] Y. J. Jiang and G. S. Lü, On the higher mean over arithmetic progressions of Fourier coefficients of cusp form, *Acta Arith.* **166** (3) (2014), 231-252.

[22] Y. J. Jiang and G. S. Lü, Uniform estimates for sums of coefficients of symmetric square L-function, *J. Number Theory* **148** (2015), 220-234.

[23] H. Kim and F. Shahidi, Cuspidality of symmetric power with applications, *Duke Math. J.* **112** (2002), 177-197.

[24] H. Kim and F. Shahidi, Functorial products for $GL_2 \times GL_3$ and functorial symmetric cube for GL_2, with an appendix by C. J. Bushnell and G. Heniart, *Ann. of Math.* **155** (2002), 837-893.

[25] H. Kim, Functoriality for the exterior square of GL_4 and symmetric fourth of GL_2, Appendix 1 by D. Ramakrishan, Appendix 2 by H. Kim and P. Sarnak, *J. Amer. Math. Soc.* **16** (2003), 139-183.

[26] G. S. Lü, Average behavior of Fourier coefficients of cusp forms, *Proc. Amer. Soc.* **137** (2009), 1961-1969.

[27] G. S. Lü, The sixth and eighth moments of Fourier coefiicients of cusp forms, *J. Number Theory* **129** (2009), 2790-2800.

[28] G. S. Lü, On higher moments of Fourier coefficients of holomorphic cusp forms, *Canad. J. Math.* **63** (2011), 634-647.

[29] Y. K. Lau, G. S. Lü and J. Wu, Integral power sums of Hecke eigenvalues, *Acta Math.* **150** (2011), 193-207.

[30] Y.-K. Lau and G. S. Lü, Sums of Fourier coefficients of cusp forms, *Quart. J. Math.* **62** (2011), 687-716.

[31] C. J. Moreno and F. Shahidi, The fourth moment of Ramanujan τ-function, *Math. Ann.* **263** (1983), 233-239.

[32] Y. Motohashi, A note on the mean value of the zeta and L-functions. III, *Proc. Japan Acad. Ser. A. Math. Sci.* **62** (4) (1986), 152-154.

[33] J. Newton, J.A. Thorne, Symmetric power functoriality for holomorphic modular forms, *Publ. Math., Inst. Hautes Étud. Sci.* **134** (2021) 1-116.

[34] J. Newton, J.A. Thorne, Symmetric power functoriality for holomorphic modular forms. II, *Publ. Math., Inst. Hautes Étud. Sci.* **134** (2021) 117-152.

[35] A. Perelli, General L-functions, *Ann. Mat. Pura Appl* **130** (4) (1982), 287-306.

[36] R. A. Rankin, Contributions to the theory of Ramanujan's function $\tau(n)$ and similar arithmetical functions II. The order of the Fourier coefficients of the integral modular forms, *Proc. Cambridge Philos. Soc.* **35** (1939), 357-372.

[37] R. A. Rankin, Sums of cusp form coefficients, in: *Automorphic Forms and Analytic Number Theory*, (Montreal, PQ, 1989), University Montreal: Montreal, QC, Canada, 1990, 115-121.

[38] A. Selberg, Bemerkungen über eine Dirichletsche, die mit der Theorie der Modulformen nahe verbunden ist, *Arch. Math. Naturvid.* **43** (1940), 47-50.

[39] F. Shahidi, Third symmetric power L-functions for $GL(2)$, *Compos. Math.* **70** (1989), 245-273.

[40] J. Wu, Power sums of Hecke eigenvalues and applications, *Acta Arith.* **137** (2009), 333-344.

[41] C. R. Xu, General asymptotic formula of Fourier coefficients of cusp forms over sum of two squares, *J. Number Theory* **236** (2022), 214-229.

[42] S. Zhuai, Average behavior of Fourier coefficients of cusp forms over sum of two squares, *J. Number Theory* **133** (11) (2013), 3862-3876.

[43] A. Y. Zou, H. X. Lao and L. L. Wei, Distributions of Fourier coefficients of cusp forms over arithmetic progressions, *Integers* **22** (2022), #A10, 16pp.