ARITHMETIC PROPERTIES FOR OVERPARTITION TRIPLES WITH ODD PARTS

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Abstract

Let $\overline{PT}_o(n)$ denote the number of overpartition triples of a positive integer n into odd parts. In this paper, we prove some infinite families of congruences modulo small powers of 2 and 3 for $\overline{PT}_o(n)$. For example, for any $\alpha \geq 0$ and $1 \leq r \leq p-1$, we prove that

$$\overline{PT}_o\Big(8 \cdot p^{2\alpha+1}(pn+r) + 3 \cdot p^{2\alpha+2}\Big) \equiv 0 \pmod{16}.$$

1. Introduction

A partition of a positive integer n is a non-increasing sequence of positive integers $\zeta_1 \geq \zeta_2 \geq \zeta_3 \geq \cdots \geq \zeta_k$ such that $\sum_{i=1}^{k} \zeta_i = n$. The ζ_i are called the parts of the partition. If $p(n)$ denotes the number of partitions of a positive integer n, then the generating function of $p(n)$ is given by

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q;q)_\infty}, \qquad p(0) = 1,$$

where for any complex number a and $|q| < 1$,

$$(a;q)_\infty = \prod_{n=0}^{\infty}(1 - aq^n).$$

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For brevity, we often write $f_t := (q^t; q^t)_\infty$, for any integer $t \geq 1$.

Ramanujan [14] offered the following congruences satisfied by the partition function $p(n)$:

$$p(5n+4) \equiv 0 \pmod 5, \quad p(7n+5) \equiv 0 \pmod 7$$

and

$$p(11n+6) \equiv 0 \pmod{11}.$$

Inspired by Ramanujan's congruences for $p(n)$, many other partition functions have been investigated in recent years for their congruence properties. In this paper, we are interested in the overpartition triples of n with odd parts. An overpartition of a positive integer n is a partition of n in which the first occurrence of a number can be overlined. Example, for $n = 3$ has 8 overpartitions, namely:

$$3, \quad \overline{3}, \quad 2+1, \quad \overline{2}+1, \quad 2+\overline{1}, \quad \overline{2}+\overline{1}, \quad 1+1+1, \quad \overline{1}+1+1.$$

The number of overpartitions of n is denoted by $\bar{p}(n)$, so $\bar{p}(3) = 8$ and its generating function is given by

$$\sum_{n=0}^{\infty} \bar{p}(n) q^n = \frac{(-q;q)_\infty}{(q;q)_\infty}, \qquad p(0) = 1.$$

The arithmetic properties of the overpartition function was first studied by Fortin et al. [6] and Hirschhorn and Sellers [8]. More details on overpartitions can be found in [3, 4, 10, 11, 12, 13]. An *overpartition triple* ζ of n is a triplet of overpartitions $(\zeta_1, \zeta_2, \zeta_3)$ such that the sum of all the parts is n. An overpartition triple into odd parts is a triplet of overpartitions $(\zeta_1, \zeta_2, \zeta_3)$ such that the parts of all the overpartitions $\zeta_1, \zeta_2, \zeta_3$ are restricted to be odd integers. Let $\overline{PT}_o(n)$ denote the number of overpartition triplets of n into odd parts, then the generating function for $\overline{PT}_o(n)$ is given by

$$\sum_{n=0}^{\infty} \overline{PT}_o(n) q^n = \frac{f_2^9}{f_1^6 f_4^3}, \qquad \overline{PT}_o(0) = 1. \tag{1}$$

In Section 3 of this paper, we establish infinite families of congruences modulo small powers of 2 and 3 for the overpartition function $\overline{PT}_o(n)$ by employing q-series and Ramanujan's theta-function identities. Section 2 is devoted to collecting some q-series identities that are used in proving our results.

2. Preliminaries

Ramanujan's general theta-function $f(a, b)$ is defined by

$$f(a,b) = \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \ |ab| < 1.$$

Two special cases of $f(a,b)$ [2, Page 36, Entry 22 (i), (ii))] are the theta-functions $\phi(q)$ and $\psi(q)$ given by

$$\phi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = (-q;q^2)_\infty^2 (q^2;q^2)_\infty = \frac{f_2^5}{f_1^2 f_4^2} \tag{2}$$

and

$$\psi(q) := f(q,q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} = \frac{f_2^2}{f_1}. \tag{3}$$

Employing elementary q-operations, it is easily seen that

$$\phi(-q) = \frac{(q;q)_\infty^2}{(q^2;q^2)_\infty} = \frac{f_1^2}{f_2}. \tag{4}$$

Lemma 1 ([5, Theorem 2.1])**.** *If p is an odd prime, then*

$$\psi(q) = \sum_{j=0}^{(p-3)/2} q^{(j^2+j)/2} f\left(q^{(p^2+(2j+1)p)/2}, q^{(p^2-(2j+1)p)/2}\right) + q^{(p^2-1)/8}\psi(q^{p^2}). \tag{5}$$

Furthermore, $(j^2+j)/2 \not\equiv (p^2-1)/8 \pmod{p}$ *for* $0 \le j \le (p-3)/2$.

Lemma 2 ([5, Theorem 2.2])**.** *If $p \ge 5$ is a prime, then*

$$f_1 = \sum_{\substack{k=-(p-1)/2 \\ k \neq (\pm p-1)/6}}^{(p-1)/2} (-1)^k q^{\frac{3k^2+k}{2}} f\left(-q^{\frac{3p^2+(6k+1)p}{2}}, -q^{\frac{3p^2-(6k+1)p}{2}}\right)$$
$$+(-1)^{(\pm p-1)/6} q^{\frac{p^2-1}{24}} f_{p^2}, \tag{6}$$

where

$$\frac{\pm p - 1}{6} = \begin{cases} \dfrac{(p-1)}{6}, & if\ p \equiv 1 \pmod{6} \\ \dfrac{(-p-1)}{6}, & if\ p \equiv -1 \pmod{6}. \end{cases}$$

Furthermore, if $\dfrac{-(p-1)}{2} \le k \le \dfrac{(p-1)}{2}$ *and* $k \neq \dfrac{(\pm p-1)}{6}$, *then*

$$\frac{3k^2+k}{2} \not\equiv \frac{p^2-1}{24} \pmod{p}.$$

Lemma 3 ([2, Page 303, Entry 17(v)])**.** *We have*

$$f_1 = f_{49}\left(\frac{B(q^7)}{C(q^7)} - q\frac{A(q^7)}{B(q^7)} - q^2 + q^5 \frac{C(q^7)}{A(q^7)}\right), \tag{7}$$

where $A(q) = f(-q^3,-q^4), B(q) = f(-q^2,-q^5)$ *and* $C(q) = f(-q,-q^6)$.

Lemma 4 ([7])**.** *We have*

$$f_1 = f_{25}(R(q^5) - q - q^2 R(q^5)^{-1}), \tag{8}$$

where

$$R(q) = \frac{(q^2;q^5)_\infty (q^3;q^5)_\infty}{(q;q^5)_\infty (q^4;q^5)_\infty}.$$

Lemma 5 ([9])**.** *We have*

$$f_1^2 = \frac{f_2 f_8^5}{f_4^2 f_{16}^2} - 2q\frac{f_2 f_{16}^2}{f_8}, \tag{9}$$

$$\frac{1}{f_1^2} = \frac{f_8^5}{f_2^5 f_{16}^2} + 2q\frac{f_4^2 f_{16}^2}{f_2^5 f_8}, \tag{10}$$

$$\frac{1}{f_1^4} = \frac{f_4^{14}}{f_2^{14} f_8^4} + 4q\frac{f_4^2 f_8^4}{f_2^{10}} \tag{11}$$

and

$$f_1^4 = \frac{f_4^{10}}{f_2^2 f_8^4} - 4q\frac{f_2^2 f_8^4}{f_4^2}. \tag{12}$$

Lemma 6 ([8])**.** *We have*

$$\frac{1}{\phi(-q)} = \frac{1}{\phi(-q^4)^4}\Big(\phi(q^4)^3 + 2q\phi(q^4)^2\psi(q^8) + 4q^2\phi(q^4)\psi(q^8)^2 + 8q^3\psi(q^8)^3\Big). \tag{13}$$

Lemma 7 ([2, Page 345])**.** *We have*

$$f_1^3 = \frac{f_6 f_9^6}{f_3 f_{18}^3} - 3qf_9^3 + 4q^3\frac{f_3^2 f_{18}^6}{f_6^2 f_9^3}. \tag{14}$$

To end this section, we record the following congruences which can be easily proved using the binomial theorem.

Lemma 8 ([1, Lemma 1.4])**.** *If p is a prime, then*

$$f_p \equiv {f_1}^p \pmod{p}, \tag{15}$$

$$f_1^{p^2} \equiv {f_p}^p \pmod{p^2}, \tag{16}$$

$$f_1^{p^3} \equiv {f_p}^{p^2} \pmod{p^3}. \tag{17}$$

3. Congruences for $\overline{PT}_o(n)$ Modulo Powers of 2

Theorem 1. *Let p be any odd prime with $\left(\frac{-2}{p}\right) = -1$ and r be any integer with $1 \leq r \leq p-1$. Then for all integers $\alpha \geq 0$, we have*

$$\overline{PT}_o\Big(16n+14\Big) \equiv 0 \pmod{8}, \tag{18}$$

$$\sum_{n=0}^{\infty} \overline{PT}_o\Big(16 \cdot p^{2\alpha}n + 6 \cdot p^{2\alpha}\Big)q^n \equiv 4\psi(q)\psi(q^2) \pmod{8}, \tag{19}$$

$$\overline{PT}_o\Big(16 \cdot p^{2\alpha+1}(pn+r) + 6 \cdot p^{2\alpha+2}\Big) \equiv 0 \pmod{8}. \tag{20}$$

Proof. From Equation (1), we have

$$\sum_{n=0}^{\infty} \overline{PT}_o(n)q^n = \frac{f_2^9}{f_4^3}\Big(\frac{1}{f_1^4}\Big)\Big(\frac{1}{f_1^2}\Big). \tag{21}$$

Employing (10) and (11) in (21), we arrive at

$$\sum_{n=0}^{\infty} \overline{PT}_o(n)q^n = \frac{f_2^9}{f_4^3}\Big(\frac{f_4^{14}}{f_2^{14}f_8^4} + 4q\frac{f_4^2f_8^4}{f_2^{10}}\Big)\Big(\frac{f_8^5}{f_2^5f_{16}^2} + 2q\frac{f_4^2f_{16}^2}{f_2^5f_8}\Big). \tag{22}$$

Extracting the terms involving even powers of q from (22), we obtain

$$\sum_{n=0}^{\infty} \overline{PT}_o(2n)q^n = \frac{f_2^{11}f_4}{f_1^{10}f_8^2} + 8q\frac{f_2f_4^3f_8^2}{f_1^6}. \tag{23}$$

With the aid of (17), we can rewrite (23) as

$$\sum_{n=0}^{\infty} \overline{PT}_o(2n)q^n \equiv \frac{f_2^7}{f_4^3}\Big(\frac{1}{f_1^2}\Big) \pmod{8}. \tag{24}$$

Employing (10) in (24), we have

$$\sum_{n=0}^{\infty} \overline{PT}_o(2n)q^n \equiv \frac{f_2^7}{f_4^3}\Big(\frac{f_8^5}{f_2^5f_{16}^2} + 2q\frac{f_4^2f_{16}^2}{f_2^5f_8}\Big) \pmod{8}. \tag{25}$$

Extracting the terms involving odd powers of q from (25), we deduce that

$$\sum_{n=0}^{\infty} \overline{PT}_o(4n+2)q^n \equiv 2\frac{f_1^2f_8^2}{f_2f_4} \pmod{8}. \tag{26}$$

Using (9) in (26), we obtain

$$\sum_{n=0}^{\infty} \overline{PT}_o(4n+2)q^n \equiv 2\frac{f_8^2}{f_2f_4}\Big(\frac{f_2f_8^5}{f_4^2f_{16}^2} - 2q\frac{f_2f_{16}^2}{f_8}\Big) \pmod{8}. \tag{27}$$

Extracting the terms involving odd powers of q from (27), we arrive at

$$\sum_{n=0}^{\infty}\overline{PT}_o(8n+6)q^n \equiv 4\frac{f_4f_8^2}{f_2} \pmod 8. \tag{28}$$

Extracting the terms involving odd powers of q from (28), we arrive at the result (18). Again, extracting the terms involving even powers of q from (28), we have

$$\sum_{n=0}^{\infty}\overline{PT}_o(16n+6)q^n \equiv 4\frac{f_2f_4^2}{f_1} \pmod 8. \tag{29}$$

Employing (3) in (29), we obtain

$$\sum_{n=0}^{\infty}\overline{PT}_o(16n+6)q^n \equiv 4\psi(q)\psi(q^2) \pmod 8, \tag{30}$$

which is the $\alpha = 0$ case of (19). Assume that (19) is true for some integer $\alpha \geq 0$. Using (5) in (19), we establish that

$$\begin{aligned}
&\sum_{n=0}^{\infty}\overline{PT}_o\Big(16\cdot p^{2\alpha}n+6\cdot p^{2\alpha}\Big)q^n \\
&\equiv 4\left[\sum_{k=0}^{(p-3)/2} q^{(k^2+k)/2} f\left(q^{(p^2+(2k+1)p)/2}, q^{(p^2-(2k+1)p)/2}\right) + q^{(p^2-1)/8}\psi(q^{p^2})\right] \\
&\times\left[\sum_{j=0}^{(p-3)/2} q^{(j^2+j)} f\left(q^{(p^2+(2j+1)p)}, q^{(p^2-(2j+1)p)}\right) + q^{(p^2-1)/4}\psi(q^{2p^2})\right] \pmod 8.
\end{aligned} \tag{31}$$

Consider the congruence

$$\frac{(k^2+k)}{2} + (j^2+j) \equiv \frac{3(p^2-1)}{4} \pmod p,$$

which is equivalent to

$$(4j+2)^2 + 2(2k+1)^2 \equiv 0 \pmod p.$$

For $\left(\frac{-2}{p}\right) = -1$, the above congruence has only the solution $k = j = \frac{(p-1)}{2}$. Therefore, extracting the term involving $q^{pn+3(p^2-1)/8}$ from (31), dividing throughout by $q^{3(p^2-1)/8}$ and then replacing q^p by q, we have

$$\sum_{n=0}^{\infty}\overline{PT}_o\Big(16\cdot p^{2\alpha+1}n+6\cdot p^{2\alpha+2}\Big)q^n \equiv 4\psi(q^p)\psi(q^{2p}) \pmod 8. \tag{32}$$

Extracting the terms involving q^{pn} from (32) and replacing q^p by q, we have

$$\sum_{n=0}^{\infty}\overline{PT}_o\Big(16\cdot p^{2(\alpha+1)}n+6\cdot p^{2\alpha+2}\Big)q^n\equiv 4\psi(q)\psi(q^2)\pmod 8,$$

which is the $\alpha+1$ case of (19). Hence, by the method of induction, we complete the proof of (19). Lastly, extracting the terms involving q^{pn+r}, for $1\le r\le p-1$, from (32), we arrive at the result (20). □

Theorem 2. *Let $p\ge 5$ be a prime with $\left(\frac{-2}{p}\right)=-1$ and r be any integer with $1\le r\le p-1$. Then for all integers $\alpha\ge 0$, we have*

$$\overline{PT}_o\Big(16n+10\Big)\equiv 0\pmod 8, \tag{33}$$

$$\sum_{n=0}^{\infty}\overline{PT}_o\Big(16\cdot p^{2\alpha}n+2\cdot p^{2\alpha}\Big)q^n\equiv 2f_1f_2\pmod 8, \tag{34}$$

$$\overline{PT}_o\Big(16\cdot p^{2\alpha+1}(pn+r)+2\cdot p^{2\alpha+2}\Big)\equiv 0\pmod 8. \tag{35}$$

Proof. Extracting the terms involving even powers of q from (27) and using (16), we obtain

$$\sum_{n=0}^{\infty}\overline{PT}_o(8n+2)q^n\equiv 2\frac{f_4^7}{f_2^3f_8^2}\equiv 2f_2f_4\pmod 8. \tag{36}$$

Extracting the terms involving odd powers of q from (36), we arrive at the result (33). Next, extracting the terms involving even powers of q from (36), we get

$$\sum_{n=0}^{\infty}\overline{PT}_o(16n+2)q^n\equiv 2f_1f_2\pmod 8, \tag{37}$$

which is the $\alpha=0$ case of (34). Assume that (34) is true for some integer $\alpha\ge 0$. Using (6) in (34), we deduce that

$$\sum_{n=0}^{\infty}\overline{PT}_o\Big(16\cdot p^{2\alpha}n+2\cdot p^{2\alpha}\Big)q^n$$

$$\equiv 2\left[\sum_{\substack{k=-(p-1)/2\\ k\neq(\pm p-1)/6}}^{(p-1)/2}q^{(3k^2+k)/2}f\Big(q^{(3p^2+(6k+1)p)/2},q^{(3p^2-(6k+1)p)/2}\Big)+q^{(p^2-1)/24}f_{p^2}\right]$$

$$\times\left[\sum_{\substack{m=-(p-1)/2\\ m\neq(\pm p-1)/6}}^{(p-1)/2}q^{(3m^2+m)}f\left(q^{(3p^2+(6m+1)p)},q^{(3p^2-(6m+1)p)}\right)\right.$$

$$+q^{(p^2-1)/12}f_{2p^2}\Bigg] \pmod 8. \tag{38}$$

Consider the congruence

$$\frac{(3k^2+k)}{2}+(3m^2+m)\equiv\frac{(p^2-1)}{8} \pmod p,$$

which is equivalent to

$$(6k+1)^2+2(6m+1)^2\equiv 0 \pmod p.$$

Since $\left(\frac{-2}{p}\right)=-1$, the above congruence has only the solution $k=m=\frac{(\pm p-1)}{6}$. Therefore, extracting the terms involving $q^{pn+(p^2-1)/8}$ from both sides of (38), dividing throughout by $q^{(p^2-1)/8}$ and then replacing q^p by q, we establish that

$$\sum_{n=0}^{\infty}\overline{PT}_o\Big(16\cdot p^{2\alpha+1}n+2\cdot p^{2\alpha+2}\Big)q^n\equiv 2f_pf_{2p} \pmod 8. \tag{39}$$

Extracting the terms involving q^{pn} from (39) and replacing q^p by q, we obtain

$$\sum_{n=0}^{\infty}\overline{PT}_o\Big(16\cdot p^{2(\alpha+1)}n+2\cdot p^{2\alpha+2}\Big)q^n\equiv 2f_1f_2 \pmod 8,$$

which is the $\alpha+1$ case of (34). Thus, by the principle of mathematical induction, we complete the proof of (34).

Finally, extracting the coefficients of the terms involving q^{pn+r}, for $1\le r\le p-1$, from (39), we arrive at the result (35). □

Theorem 3. *Let p be any odd prime with $\left(\frac{-2}{p}\right)=-1$ and r be any integer with $1\le r\le p-1$. Then for all integers $\alpha\ge 0$, we have*

$$\overline{PT}_o\Big(8n+7\Big)\equiv 0 \pmod{16}, \tag{40}$$

$$\sum_{n=0}^{\infty}\overline{PT}_o\Big(8\cdot p^{2\alpha}n+3\cdot p^{2\alpha}\Big)q^n\equiv 12\psi(q)\psi(q^2) \pmod{16}, \tag{41}$$

$$\overline{PT}_o\Big(8\cdot p^{2\alpha+1}(pn+r)+3\cdot p^{2\alpha+2}\Big)\equiv 0 \pmod{16}. \tag{42}$$

Proof. Extracting the terms involving odd powers of q from (22), we obtain

$$\sum_{n=0}^{\infty}\overline{PT}_o\Big(2n+1\Big)q^n=2\frac{f_2^{13}f_8^2}{f_1^{10}f_4^5}+4\frac{f_4^9}{f_2f_1^6f_8^2}. \tag{43}$$

Employing (16) and (17) in (43), we have

$$\sum_{n=0}^{\infty} \overline{PT}_o\Big(2n+1\Big)q^n \equiv 2\frac{f_2^9}{f_1^2 f_4} + 4\frac{f_4^5}{f_2^3 f_1^2} \pmod{16}. \tag{44}$$

Employing (10) in (44) and then extracting the terms involving odd powers of q, we obtain

$$\sum_{n=0}^{\infty} \overline{PT}_o\Big(4n+3\Big)q^n \equiv 4\frac{f_1^4 f_2 f_8^2}{f_4} + 8\frac{f_2^7 f_8^2}{f_1^8 f_4} \pmod{16}. \tag{45}$$

Employing (12) in the first term of (45), we establish that

$$\sum_{n=0}^{\infty} \overline{PT}_o\Big(4n+3\Big)q^n \equiv 4\frac{f_2 f_8^2}{f_4}\Big(\frac{f_4^{10}}{f_2^2 f_8^4} - 4q\frac{f_2^2 f_8^4}{f_4^2}\Big) + 8\frac{f_2^7 f_8^2}{f_1^8 f_4} \pmod{16}. \tag{46}$$

Extracting the terms involving odd powers of q from (46), we arrive at the result (40). Next, extracting the terms involving even powers of q from (46), we have

$$\sum_{n=0}^{\infty} \overline{PT}_o\Big(8n+3\Big)q^n \equiv 4\frac{f_2^9}{f_1 f_4^2} + 8f_1^3 f_2^3 \pmod{16}. \tag{47}$$

With the help of (16) in (47), we obtain

$$\sum_{n=0}^{\infty} \overline{PT}_o\Big(8n+3\Big)q^n \equiv 12 f_1^3 f_2^3 \pmod{16}. \tag{48}$$

Using (3) in (48), we have

$$\sum_{n=0}^{\infty} \overline{PT}_o\Big(8n+3\Big)q^n \equiv 12\psi(q)\psi(q^2) \pmod{16}, \tag{49}$$

which is the $\alpha = 0$ case of (41). Assume that (41) is true for some integer $\alpha \geq 0$. Using (5) in (41) and proceeding as in the proof of (19), we arrive at

$$\sum_{n=0}^{\infty} \overline{PT}_o\Big(8\cdot p^{2\alpha+1}n + 3\cdot p^{2\alpha+2}\Big)q^n \equiv 12\psi(q^p)\psi(q^{2p}) \pmod{16}. \tag{50}$$

Extracting the terms involving q^{pn} from (50) and replacing q^p by q, we obtain

$$\sum_{n=0}^{\infty} \overline{PT}_o\Big(8\cdot p^{2(\alpha+1)}n + 3\cdot p^{2\alpha+2}\Big)q^n \equiv 12\psi(q)\psi(q^2) \pmod{16},$$

which is the $\alpha+1$ case of (41). Hence, by the method of induction, we complete the proof of (41).

Next, extracting the coefficients of the terms containing q^{pn+r}, for $1 \leq r \leq p-1$, from both sides of (50), we arrive at result (42). □

Theorem 4. *For all $n \geq 0$ and α, β, $\gamma \geq 0$ we have*

$$\sum_{n=0}^{\infty} \overline{PT}_o\Big(32 \cdot 3^{4\alpha} \cdot 5^{2\beta} \cdot 7^{2\gamma}(n) + 12 \cdot 3^{4\alpha} \cdot 5^{2\beta} \cdot 7^{2\gamma}\Big)q^n \equiv 4f_1^9 \pmod 8, \tag{51}$$

$$\sum_{n=0}^{\infty} \overline{PT}_o\Big(32 \cdot 3^{4\alpha} \cdot 5^{2\beta} \cdot 7^{2\gamma+1}(n) + 4 \cdot 3^{4\alpha} \cdot 5^{2\beta+1} \cdot 7^{2\gamma+1}\Big)q^n \equiv 4q^2 f_7^9 \pmod 8, \tag{52}$$

$$\sum_{n=0}^{\infty} \overline{PT}_o\Big(32 \cdot 3^{4\alpha} \cdot 5^{2\beta+1} \cdot 7^{2\gamma}(n) + 4 \cdot 3^{4\alpha} \cdot 5^{2\beta+1} \cdot 7^{2\gamma+1}\Big)q^n \equiv 4q f_5^9 \pmod 8, \tag{53}$$

$$\overline{PT}_o\Big(32 \cdot 3^{4\alpha} \cdot 5^{2\beta+1} \cdot 7^{2\gamma}(5n+i) + 4 \cdot 3^{4\alpha} \cdot 5^{2\beta+1} \cdot 7^{2\gamma+1}\Big) \equiv 0 \pmod 8, \tag{54}$$

$$\overline{PT}_o\Big(32 \cdot 3^{4\alpha} \cdot 5^{2\beta} \cdot 7^{2\gamma+1}(7n+j) + 4 \cdot 3^{4\alpha} \cdot 5^{2\beta+1} \cdot 7^{2\gamma+1}\Big) \equiv 0 \pmod 8, \tag{55}$$

$$\sum_{n=0}^{\infty} \overline{PT}_o\Big(32 \cdot 3^{4\alpha+1} \cdot 5^{2\beta} \cdot 7^{2\gamma}(n) + 44 \cdot 3^{4\alpha} \cdot 5^{2\beta} \cdot 7^{2\gamma}\Big)q^n \equiv 4f_2 f_3^3 \pmod 8, \tag{56}$$

$$\sum_{n=0}^{\infty} \overline{PT}_o\Big(32 \cdot 3^{4\alpha+1} \cdot 5^{2\beta} \cdot 7^{2\gamma}(n) + 76 \cdot 3^{2\alpha} \cdot 5^{2\beta} \cdot 7^{2\gamma}\Big)q^n \equiv 4f_1 f_6^3 \pmod 8, \tag{57}$$

$$\sum_{n=0}^{\infty} \overline{PT}_o\Big(32 \cdot 3^{4\alpha+1} \cdot 5^{2\beta+1} \cdot 7^{2\gamma}(n) + 28 \cdot 3^{4\alpha} \cdot 5^{2\beta+1} \cdot 7^{2\gamma}\Big)q^n \equiv 4q^2 f_{10} f_{15}^3 \pmod 8, \tag{58}$$

$$\sum_{n=0}^{\infty} \overline{PT}_o\Big(32 \cdot 3^{4\alpha+1} \cdot 5^{2\beta+1} \cdot 7^{2\gamma}(n) + 92 \cdot 3^{4\alpha} \cdot 5^{2\beta+1} \cdot 7^{2\gamma}\Big)q^n \equiv 4q^3 f_5 f_{30}^3 \pmod 8, \tag{59}$$

$$\overline{PT}_o\Big(32 \cdot 3^{4\alpha+1} \cdot 5^{2\beta+1} \cdot 7^{2\gamma}(5n+k) + 28 \cdot 3^{4\alpha} \cdot 5^{2\beta+1} \cdot 7^{2\gamma}\Big) \equiv 0 \pmod 8, \tag{60}$$

$$\overline{PT}_o\Big(32 \cdot 3^{4\alpha+1} \cdot 5^{2\beta+1} \cdot 7^{2\gamma}(5n+\ell) + 92 \cdot 3^{4\alpha} \cdot 5^{2\beta+1} \cdot 7^{2\gamma}\Big) \equiv 0 \pmod 8, \tag{61}$$

where $i = 0, 2, 3, 4$, $j = 0, 1, 3, 4, 5, 6$, $k = 0, 1, 3, 4$ and $\ell = 0, 1, 2, 4$.

Proof. Extracting the terms involving even powers of q from (25), we obtain

$$\sum_{n=0}^{\infty} \overline{PT}_o(4n)q^n \equiv \frac{f_1^2 f_4^5}{f_2^3 f_8^2} \pmod 8. \tag{62}$$

Employing (9) in (62) and then extracting the terms involving odd powers of q, we have

$$\sum_{n=0}^{\infty} \overline{PT}_o(8n+4)q^n \equiv 6\frac{f_2^5 f_8^2}{f_1^2 f_4^3} \pmod 8. \tag{63}$$

Employing (10) in (63) and then extracting the terms involving odd powers of q, we deduce that

$$\sum_{n=0}^{\infty}\overline{PT}_o(16n+12)q^n \equiv 4\frac{f_4f_8^2}{f_2} \pmod 8. \tag{64}$$

Extracting the terms involving even powers of q from (64) and using (15), we obtain

$$\sum_{n=0}^{\infty}\overline{PT}_o(32n+12)q^n \equiv 4f_1^9 \pmod 8,$$

which is $\alpha=\beta=\gamma=0$ case of (51). Assume that (51) is true for $\alpha\geq 0$ and $\beta=\gamma=0$. From (51) with $\beta=\gamma=0$, we arrive at

$$\sum_{n=0}^{\infty}\overline{PT}_o(32\cdot 3^{4\alpha}n+12\cdot 3^{4\alpha})q^n \equiv 4f_1^9 \pmod 8. \tag{65}$$

Employing (14) in (65) with $\beta=\gamma=0$ and then extracting the coefficients of q^{3n}, we obtain

$$\begin{aligned}\sum_{n=0}^{\infty}\overline{PT}_o(32\cdot 3^{4\alpha+1}n+12\cdot 3^{4\alpha})q^n &\equiv 4f_1^3+4qf_3^9\\ &\equiv 4f_3+4qf_3^9+4qf_9^3 \pmod 8. \end{aligned}\tag{66}$$

Substituting (14) in (66) and extracting the terms involving q^{3n+1}, we obtain

$$\begin{aligned}\sum_{n=0}^{\infty}\overline{PT}_o(32\cdot 3^{4\alpha+2}n+12\cdot 3^{4\alpha+2})q^n &\equiv 4f_1^9+4f_3^3\\ &\equiv 4qf_6f_9^3+4q^2f_3f_9^6+4q^3f_9^9 \pmod 8. \end{aligned}\tag{67}$$

Again, using (14) in (67) and extracting the terms involving q^{3n}, we arrive at

$$\sum_{n=0}^{\infty}\overline{PT}_o(32\cdot 3^{4\alpha+3}n+12\cdot 3^{4\alpha+2})q^n \equiv 4qf_3^9 \pmod 8. \tag{68}$$

Extracting the terms involving q^{3n+1} from (68), we arrive at

$$\sum_{n=0}^{\infty}\overline{PT}_o(32\cdot 3^{4\alpha+4}n+12\cdot 3^{4\alpha+4})q^n \equiv 4f_1^9 \pmod 8,$$

which proves that (51) is true for $\alpha+1$. By mathematical induction, (51) is true for all integers $\alpha\geq 0$ with $\beta=\gamma=0$. Suppose that (51) is true for $\alpha,\beta\geq 0$ with $\gamma=0$. From (51) with $\gamma=0$ and then employing (8), we get

$$\sum_{n=0}^{\infty}\overline{PT}_o(32\cdot 3^{4\alpha}\cdot 5^{2\beta+1}n+28\cdot 3^{4\alpha}\cdot 5^{2\beta+1})q^n \equiv 4qf_5^9 \pmod 8. \tag{69}$$

Extracting the terms involving q^{5n+1} from (69), we obtain

$$\sum_{n=0}^{\infty}\overline{PT}_o(32\cdot 3^{4\alpha}\cdot 5^{2\beta+2}n+12\cdot 3^{4\alpha}\cdot 5^{2\beta+2})q^n\equiv 4f_1^9 \pmod 8,$$

which proves that (51) is true for $\beta+1$ with $\gamma=0$. So, by the method of induction, (51) is true for all integers $\alpha,\beta\geq 0$ with $\gamma=0$. Suppose that (51) is true for $\alpha,\beta,\gamma\geq 0$ and then utilizing (7), we arrive at

$$\sum_{n=0}^{\infty}\overline{PT}_o(32\cdot 3^{4\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma+1}n+20\cdot 3^{4\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma+1})q^n\equiv 4q^2f_7^9 \pmod 8. \quad (70)$$

Extracting the terms involving q^{7n+2} from (70), we obtain

$$\sum_{n=0}^{\infty}\overline{PT}_o(32\cdot 3^{4\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma+2}n+12\cdot 3^{4\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma+2})q^n\equiv 4f_1^9 \pmod 8,$$

which implies that (51) is true for $\gamma+1$. By induction, (51) is true for all integers $\alpha,\beta,\gamma\geq 0$. Using (7) in (51), we obtain (52). Congruence (52) implies (55). Using (8) in (51), we obtain (53). Congruence (53) implies (54). Using (14) in (51) and then comparing the coefficients of q^{3n+1} and q^{3n+2} in the resultant equation, we obtain (56) and (57), respectively. Utilizing (8) in (56) and (57), we obtain (58) and (59), respectively. Congruence (58) implies congruence (60), and congruence (59) implies congruence (61). □

Theorem 5. *For all $n\geq 0$ and α, β, $\gamma\geq 0$ we have*

$$\sum_{n=0}^{\infty}\overline{PT}_o\Big(8\cdot 3^{2\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma}(n)+3^{2\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma}\Big)q^n\equiv 2f_1^3 \pmod 4, \quad (71)$$

$$\sum_{n=0}^{\infty}\overline{PT}_o\Big(8\cdot 3^{2\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma+1}(n)+3^{2\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma+2}\Big)q^n\equiv 2f_7^3 \pmod 4, \quad (72)$$

$$\overline{PT}_o\Big(8\cdot 3^{2\alpha+1}\cdot 5^{2\beta}\cdot 7^{2\gamma}(n)+3^{2\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma}\Big)\equiv\begin{cases}2 \pmod 4, & \textit{if } n \textit{ is a pentagonal number}\\ 0 \pmod 4, & \textit{otherwise},\end{cases} \quad (73)$$

$$\sum_{n=0}^{\infty}\overline{PT}_o\Big(8\cdot 3^{2\alpha+1}\cdot 5^{2\beta}\cdot 7^{2\gamma}(n)+3^{2\alpha+2}\cdot 5^{2\beta}\cdot 7^{2\gamma}\Big)q^n\equiv 2f_3^3 \pmod 4, \quad (74)$$

$$\overline{PT}_o\Big(8\cdot 3^{2\alpha+1}\cdot 5^{2\beta}\cdot 7^{2\gamma}(n)+17\cdot 3^{2\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma}\Big)\equiv 0 \pmod 4, \quad (75)$$

$$\overline{PT}_o\Big(8\cdot 3^{2\alpha+1}\cdot 5^{2\beta}\cdot 7^{2\gamma}(3n+i)+3^{2\alpha+2}\cdot 5^{2\beta}\cdot 7^{2\gamma}\Big)\equiv 0 \pmod 4, \quad (76)$$

$$\sum_{n=0}^{\infty}\overline{PT}_o\Big(8\cdot 3^{2\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}(n)+3^{2\alpha}\cdot 5^{2\beta+2}\cdot 7^{2\gamma}\Big)q^n\equiv 2f_5^3 \pmod 4, \quad (77)$$

$$\overline{PT}_o\Big(8\cdot 3^{2\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma}(5n+j)+3^{2\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma}\Big)\equiv 0 \pmod 4, \tag{78}$$

$$\overline{PT}_o\Big(8\cdot 3^{2\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}(5n+k)+3^{2\alpha}\cdot 5^{2\beta+2}\cdot 7^{2\gamma}\Big)\equiv 0 \pmod 4, \tag{79}$$

$$\overline{PT}_o\Big(8\cdot 3^{2\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma+1}(7n+\ell)+3^{2\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma+2}\Big)\equiv 0 \pmod 4, \tag{80}$$

where $i=1,2,\ j=2,4,\ k=1,2,3,4$ *and* $\ell=1,2,3,4,5,6$.

Proof. With the help of (16) and using (4), we can rewrite Equation (1) as

$$\sum_{n=0}^{\infty}\overline{PT}_o(n)q^n\equiv\frac{f_2}{f_1^2}\equiv\frac{1}{\phi(-q)} \pmod 4. \tag{81}$$

Employing (13) in (81) and then extracting the terms involving q^{4n+1} from the resultant equation, we obtain

$$\sum_{n=0}^{\infty}\overline{PT}_o(4n+1)q^n\equiv 2\frac{\phi(q)^2\psi(q^2)}{\phi(-q)^4} \pmod 4. \tag{82}$$

Using (2), (3), (4) and (15) in (82), we obtain

$$\sum_{n=0}^{\infty}\overline{PT}_o(4n+1)q^n\equiv 2f_2^3 \pmod 4. \tag{83}$$

Extracting the terms involving even powers of q from (83), we obtain

$$\sum_{n=0}^{\infty}\overline{PT}_o(8n+1)q^n\equiv 2f_1^3 \pmod 4. \tag{84}$$

Congruence (84) is the case $\alpha=\beta=\gamma=0$ of (71). Suppose that (71) is true for any integer $\alpha\geq 0$ with $\beta=\gamma=0$. Utilizing (14) in (71) with $\beta=\gamma=0$ and then extracting the coefficients of q^{3n+1}, we obtain

$$\sum_{n=0}^{\infty}\overline{PT}_o(8\cdot 3^{2\alpha+1}n+3^{2\alpha+2})q^n\equiv 2f_3^3 \pmod 4. \tag{85}$$

Extracting the coefficient of the terms involving q^{3n}, from (85), we obtain

$$\sum_{n=0}^{\infty}\overline{PT}_o(8\cdot 3^{2\alpha+2}n+3^{2\alpha+2})q^n\equiv 2f_1^3 \pmod 4,$$

which implies that (71) is true for $\alpha+1$ with $\beta=\gamma=0$. By mathematical induction, (71) is true for all α. Suppose that (71) holds for $\alpha,\beta\geq 0$ with $\gamma=0$. Utilising (8) in (71) and then extracting the terms involving q^{5n+3}, we obtain

$$\sum_{n=0}^{\infty}\overline{PT}_o(8\cdot 3^{2\alpha}\cdot 5^{2\beta+1}n+3^{2\alpha}\cdot 5^{2\beta+2})q^n\equiv 2f_5^3 \pmod 4. \tag{86}$$

Extracting the coefficient of the terms involving q^{5n} from (86), we obtain

$$\sum_{n=0}^{\infty} \overline{PT}_o(8 \cdot 3^{2\alpha} \cdot 5^{2\beta+2} n + 3^{2\alpha} \cdot 5^{2\beta+2}) q^n \equiv 2f_1^3 \pmod 4,$$

which implies that (71) is true for $\beta+1$ with $\gamma = 0$. By mathematical induction, (71) is true for all non-negative integers α, β with $\gamma = 0$. Suppose that (71) holds for $\alpha, \beta, \gamma \geq 0$. Utilising (7) in (71) and then extracting the terms involving q^{7n+6}, we obtain

$$\sum_{n=0}^{\infty} \overline{PT}_o(8 \cdot 3^{2\alpha} \cdot 5^{2\beta+2} \cdot 7^{2\gamma+1} n + 3^{2\alpha} \cdot 5^{2\beta} \cdot 7^{2\gamma+2}) q^n \equiv 2f_7^3 \pmod 4, \tag{87}$$

which proves (72). Extracting the coefficients of the terms involving q^{7n} from (87), we obtain

$$\sum_{n=0}^{\infty} \overline{PT}_o\Big(8 \cdot 3^{2\alpha} \cdot 5^{2\beta} \cdot 7^{2\gamma+2}(n) + 3 \cdot 3^{2\alpha} \cdot 5^{2\beta} \cdot 7^{2\gamma+2}\Big) q^n \equiv 2f_1^3 \pmod 4,$$

which implies that (71) is true for all $\gamma+1$. By mathematical induction, (71) is true for all non-negative integers α, β and γ. Employing (14) in (71) and then extracting the terms involving q^{3n}, q^{3n+1} and q^{3n+2}, we arrive at (73), (74) and (75), respectively. Extracting the coefficients of q^{3n+i} for $i = 1, 2$ on both sides of (74), we arrive at (76). Employing (8) in (71) and then extracting the terms involving q^{5n+3}, we arrive at (77). Again, employing (8) in (71) and then extracting the terms involving q^{5n+i} for $i = 2, 4$, we arrive at (78). Extracting the terms involving q^{5n+i} for $i = 1, 2, 3, 4$ from (77), yields (79). Finally, extracting the terms involving q^{7n+i} for $i = 1, 2, 3, 4, 5, 6$ from (72), yields (80). □

Theorem 6. *For any integer $n \geq 0$, we have*

$$\overline{PT}_o\Big(3n + i\Big) \equiv 0 \pmod 3, \tag{88}$$

$$\overline{PT}_o\Big(24(3n + j) + 3\Big) \equiv 0 \pmod 3, \tag{89}$$

where $i = j = 1, 2$.

Proof. With the help of (15), we can rewrite Equation (1) as

$$\sum_{n=0}^{\infty} \overline{PT}_o(n) q^n \equiv \frac{f_6^3}{f_3^2 f_{12}} \pmod 3. \tag{90}$$

Extracting the terms involving q^{3n+i} where $i \in \{1, 2\}$ from (90), we arrive at the desired result (88). Next, extracting the terms involving q^{3n} from (90), we obtain

$$\sum_{n=0}^{\infty} \overline{PT}_o(3n) q^n \equiv \frac{f_2^3}{f_1^2 f_4} \pmod 3. \tag{91}$$

Employing (10) in (91) and then extracting the terms involving odd powers of q, we obtain

$$\sum_{n=0}^{\infty} \overline{PT}_o(6n+3)q^n \equiv 2\frac{f_2 f_8^2}{f_1^2 f_4} \equiv \frac{\psi(q^4)}{\phi(-q)} \pmod 3. \tag{92}$$

Employing (13) in (92) and then extracting the terms involving q^{4n} from resultant equations, we obtain

$$\sum_{n=0}^{\infty} \overline{PT}_o(24n+3)q^n \equiv 2\frac{\psi(q)\phi^3(q)}{\phi^4(-q)} \pmod 3. \tag{93}$$

Employing (2), (3) and (4) in (93), we obtain

$$\sum_{n=0}^{\infty} \overline{PT}_o(24n+3)q^n \equiv 2\frac{f_6^7}{f_3^5 f_{12}^2} \pmod 3. \tag{94}$$

Extracting the terms involving q^{3n+j} where $j \in \{1, 2\}$ from (94), we arrive at the final result (89). □

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