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ON A RELATION BETWEEN SCHREIER-TYPE SETS AND A MODIFICATION OF TURÁN GRAPHS

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Abstract

Recently, a relation between Schreier-type sets and Turán graphs was discovered. In this note, we give a combinatorial proof and obtain a generalization of the relation. Specifically, for $p, q \geq 1$, let $\mathcal{A}_q := \{F \subset \mathbb{N} : |F| = 1 \text{ or } F \text{ is an arithmetic progression with difference } q\}$ and $Sr(n, p, q) := \#\{F \subset \{1, \ldots, n\} : p \min F \geq |F| \text{ and } F \in \mathcal{A}_q\}$. We show that $Sr(n, p, q) = T(n + 1, pq + 1, q)$, where $T(\cdot, \cdot, \cdot)$ is the number of edges of a modification of Turán graphs. We also prove that $Sr(n, p, q)$ is the partial sum of certain sequences.

1. Introduction

A set $F \subset \mathbb{N}$ is said to be *Schreier* if $\min F \geq |F|$. Schreier sets have appeared frequently in Banach space theory. For example, Schreier sets were used in the construction by Schreier to answer a question of Banach and Saks [8] and were the central concept in a celebrated theorem by Odell [7]. Besides, Schreier sets are of independent interests. A. Bird [2] showed a surprising connection between these sets and the Fibonacci sequence. Since then, there has been research on generalizing Bird's result to connect Schreier-type sets with other sequences (see [3, 4, 5, 6].) For instance, [4, Theorem 1.1] investigated the recurrence relation for the count of sets $F \subset \mathbb{N}$ satisfying $\min F \geq p|F|$, where $p \geq 1$. Recently, Beanland et al. [1] proved a recurrence for the more general form $q \min F \geq p|F|$, where $p, q \geq 1$. In the same note, the authors showed a connection between Schreier-type sets and Turán graphs, which we now describe.

For $n, p \geq 1$, let $[n] = \{1, 2, \ldots, n\}$ and

$$Sr(n, p) := \#\{F \subset [n] : p \min F \geq |F| \text{ and } F \text{ is an interval}\}.$$

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A *Turán graph* [10], denoted by $T(n,p)$, is the n-vertex complete p-partite graph whose classes differ in size by at most 1 vertex. In other words, to form $T(n,p)$, we divide the n vertices into p classes as equally as possible, and then connect all pairs of vertices that are not in the same class. We also use $T(n,p)$ to indicate the number of edges of the Turán graph $T(n,p)$. For each $p \geq 2$, the sequence $(T(n,p))_{n=1}^{\infty}$ is available in OEIS [9] (for example, see A002620, A000212, A033436, and A033437.)

By [1, Theorem 1.2], we have that

$$Sr(n,p) \;=\; T(n+1,p+1). \tag{1}$$

In the proof of [1, Theorem 1.2], the authors explicitly found the formulas for $Sr(n,p)$ and $T(n+1,p+1)$ and then, by algebraic manipulations, showed that the two formulas give the same number for all $n,p \geq 1$. The first goal of this note is to give a combinatorial proof of (1), which is concise and sheds a better light on why the equality holds. Our second goal is to generalize (1). To do so, we need to introduce some new notions.

1.1. Three Types of Graphs

We introduce several types of graphs that are recursively defined.

1.1.1. An $M(n,p)$-Graph

In a Turán graph, each vertex in a particular class shares an edge only with vertices in other classes. We shall present a slight modification that allows a vertex to share an edge with vertices in the same class. Let $M(n,p)$ denote the following modification of $T(n,p)$. Fix $p \geq 1$. Let $M(1,p)$ be the graph with p classes, exactly one of which contains a vertex. Suppose that $M(n,p)$ has been defined for some $n \geq 1$. We construct $M(n+1,p)$ by adding a vertex v to $M(n,p)$ under a certain rule. Write $n = p\ell + k$, where $\ell \geq 0$ and $1 \leq k \leq p$.

- `Case 1`: If $k = p$, then add v to one of the p classes. From the remaining $p-1$ classes, choose an arbitrary class P. Form edges between v and all vertices in the same class as v, and form edges between v and vertices in all other classes except P.
- `Case 2`: If $k < p$, then add v to a class P' with exactly ℓ vertices. Form an edge between v and every vertex in every class other than P'.

With an abuse of notation, let $M(n,p)$ also denote the number of edges of $M(n,p)$. Clearly, $M(n,p) = T(n,p)$ for all $n,p \geq 1$.

1.1.2. An $M(n,p,q)$-Graph

For fixed $p, q \geq 1$, we now define a graph $M(n,p,q)$, where the number of edges of $M(n,p,1)$ and $M(n,p)$ are equal. Let $M(1,p,q)$ be the graph with p classes, exactly one of which contains a vertex. Suppose that $M(n,p,q)$ has been defined for some $n \geq 1$. We construct $M(n+1,p,q)$ by adding a vertex v to $M(n,p,q)$ under the following rule. Write $n = p\ell + k$, where $\ell \geq 0$ and $1 \leq k \leq p$. There are k classes with $\ell + 1$ vertices and $p - k$ classes with ℓ vertices.

- `Case 1`: If $p = k$, then form $M(n+1,p,q)$ in the same way as we form $M(n+1,p)$ above.
- `Case 2`: If $p - q < k < p$, then add v to a class with exactly ℓ vertices. Let P be a class with $\ell + 1$ vertices. Form edges between v and all vertices in the same class as v, and form edges between v and vertices in all other classes except P.
- `Case 3`: If $k \leq p - q$, then add v to a class P' with exactly ℓ vertices. Form an edge between v and every vertex in every class other than P'.

By definition, $M(n,p,1) = M(n,p)$ for all $n, p \geq 1$.

1.1.3. A $T(n,p,q)$-Graph

A $T(n,p,q)$-graph is formed in almost the same way as an $M(n,p,q)$-graph. Fix $p, q \geq 1$. Let us describe how to form $T(n,p,q)$. First, $T(1,p,q) = M(1,p,q)$. Supposing that we already have $T(n,p,q)$ for some $n \geq 1$, we construct $T(n+1,p,q)$. In each case of the recursive step to form $M(n+1,p,q)$ described above, we connect v to certain vertices. Call the set of these vertices V. We assign numbers from 1 to q in that order to vertices in V. (If $|V| > q$, we repeat the numbering from 1 to q.) In forming $T(n+1,p,q)$ from $T(n,p,q)$, we form an edge between v and vertices that are numbered 1. See the Appendix for $(T(n,5,2))_{n=2}^{7}$.

1.2. Schreier Sets with Constant Gaps

Recall that an arithmetic progression of positive integers is a sequence that can be written as

$$a, a+d, a+2d, \ldots, a+kd, \ldots,$$

for some $a, d \in \mathbb{N}$. Here d is called the difference of the arithmetic progression. An interval is an arithmetic progression with difference 1. For $q \geq 1$, let

$$\mathcal{A}_q := \{F \subset \mathbb{N} : |F| = 1 \text{ or } F \text{ is an arithmetic progression with difference } q\}.$$

For $n, p, q \geq 1$, define

$$Sr(n,p,q) := \#\{F \subset [n] : p \min F \geq |F| \text{ and } F \in \mathcal{A}_q\}.$$

We are ready to state our main result.

Theorem 1. *For all $p, q, n \geq 1$, we have that*

$$Sr(n, p, q) \;=\; T(n+1, pq+1, q). \tag{2}$$

Remark 1. Plugging in $q = 1$ into (2), we have (1). Indeed, we have that $Sr(n, p, 1) = Sr(n, p)$ and by definitions,

$$T(n+1, p+1, 1) \;=\; M(n+1, p+1, 1) \;=\; M(n+1, p+1) \;=\; T(n+1, p+1).$$

Example 1. Choose $p = q = 2$. Let us consider the first few terms of two sequences $(Sr(n, 2, 2))_{n=1}^{\infty}$ and $(T(n+1, 5, 2))_{n=1}^{\infty}$ and confirm that these values are equal. A simple program gives $(Sr(n, 2, 2))$:

$$1, 2, 4, 6, 8, 11, 14, 18, 22, 26, 31, 36, 42, 48, 54, 61, 68, 76, 84, \ldots.$$

We include graphs $T(n, 5, 2)$ for small values of n in the Appendix.

The following theorem shows that we can write $Sr(n, p, q)$ as the partial sum of sequences.

Theorem 2. *Fix $p, q \geq 1$. We have that*

$$Sr(1, p, q) \;=\; 1 \text{ and } Sr(n+1, p, q) - Sr(n, p, q) \;=\; \left\lfloor \frac{p(n+q+1)}{pq+1} \right\rfloor, \text{ for all } n \geq 1.$$

Hence,

$$Sr(N, p, q) \;=\; 1 + \sum_{n=1}^{N-1} \left\lfloor \frac{p(n+q+1)}{pq+1} \right\rfloor.$$

2. A Combinatorial View of (1)

Let $p \geq 1$ be fixed. Our goal is to show that $Sr(n, p) = T(n+1, p+1)$ for all $n \geq 1$. The proof idea is simple: as we know $Sr(1, p) = T(2, p+1)$ from the definitions, we need only to verify that

$$Sr(n+1, p) - Sr(n, p) \;=\; T(n+2, p+1) - T(n+1, p+1), \text{ for all } n \geq 1. \tag{3}$$

Fix $n \geq 1$. Write $n = (p+1)\ell + k$ for some $\ell \geq 0$ and $0 \leq k \leq p$.

By definition, the left side of (3) counts the number of intervals $F \subset [n+1]$ such that $p \min F \geq |F|$ and $n+1 \in F$. Let $\mathcal{A}$ be the set of all these intervals and M be the largest interval in $\mathcal{A}$. (Note that all intervals in $\mathcal{A}$ contain $n+1$, so M is uniquely defined and is the interval with the smallest minimum.) Then

$$p \min M \;\geq\; |M| \;=\; n + 2 - \min M.$$

Hence,

$$\min M \;=\; \left\lceil \frac{n+2}{p+1} \right\rceil \text{ and } M \;=\; \left\{ \left\lceil \frac{n+2}{p+1} \right\rceil, \ldots, n+1 \right\}.$$

Since each $F \in \mathcal{A}$ is obtained from M by discarding the smallest numbers in M, it follows that

$$\begin{aligned} Sr(n+1,p) - Sr(n,p) \;&=\; n+2-\left\lceil \frac{n+2}{p+1} \right\rceil \;=\; n-\ell+2-\left\lceil \frac{k+2}{p+1} \right\rceil \\ &=\; \begin{cases} n-\ell & \text{if } k=p \\ n-\ell+1 & \text{if } k<p. \end{cases} \end{aligned} \tag{4}$$

We now consider the right side of (3). For the graph $T(n+1,p+1)$, there are $k+1$ classes that have $\ell+1$ vertices and $(p-k)$ classes that have ℓ vertices. If $k=p$, then adding one more vertex to any class increases the number of edges by

$$(\ell+1)k \;=\; (\ell+1)p \;=\; p\ell+k \;=\; n-\ell.$$

If $k<p$, then adding one more vertex to a class with ℓ vertices increases the number of edges by

$$\ell(p-k-1)+(\ell+1)(k+1) \;=\; n-\ell+1.$$

Therefore, we have that $Sr(n+1,p) - Sr(n,p) = T(n+2,p+1) - T(n+1,p+1)$, as desired.

3. Proof of Theorem 1

First, we need an easy lemma.

Lemma 1. *Given a set V of N vertices and a vertex $v \notin V$, if we number the vertices in V from 1 to q in that order and form an edge between v and each vertex that is numbered 1, then the number of edges is*

$$\left\lfloor \frac{N-1}{q} \right\rfloor + 1.$$

Proof. We can rephrase the problem as follows: given N integers, find the maximum possible number of integers that are congruent to 1 modulo q. The count is $\left\lfloor \frac{N-1}{q} \right\rfloor +$ 1. □

Proof of Theorem 1. Fix $p,q,n \geq 1$. Write $n=(pq+1)\ell+k$ for some $\ell \geq 0$ and $0 \leq k \leq pq$. Clearly, $Sr(1,p,q) \;=\; T(2,pq+1,q) = 1$. We need only to show that

$$Sr(n+1,p,q) - Sr(n,p,q) \;=\; T(n+2,pq+1,q) - T(n+1,pq+1,q).$$

We have that $Sr(n+1,p,q) - Sr(n,p,q)$ counts the number of sets $F \subset [n+1]$ with $\max F = n+1$, $p \min F \geq |F|$, and $F \in \mathcal{A}_q$. Let $\mathcal{A}$ be the set of all such F, and let M be the largest set in $\mathcal{A}$. (Note that M is uniquely defined because all sets in $\mathcal{A}$ have the same maximum and belong to $\mathcal{A}_q$. Hence, M is the set in $\mathcal{A}$ with the smallest minimum.) Then

$$p \min M \;\geq\; |M| \;=\; \frac{n+1-\min M}{q} + 1$$

and so,

$$\min M \;\geq\; \left\lceil \frac{n+1+q}{pq+1} \right\rceil.$$

Choosing $M = \{n+1-(\ell-1)q, \ldots, n+1-q, n+1\}$, we have that

$$n+1-(\ell-1)q \;\geq\; \left\lceil \frac{n+1+q}{pq+1} \right\rceil,$$

which gives

$$\ell \;\leq\; \left\lfloor \frac{1}{q}\left(n+1-\left\lceil \frac{n+1+q}{pq+1} \right\rceil\right)\right\rfloor + 1.$$

Because M is the largest,

$$|M| \;=\; \ell \;=\; \left\lfloor \frac{1}{q}\left(n+1-\left\lceil \frac{n+1+q}{pq+1} \right\rceil\right)\right\rfloor + 1.$$

Since each $F \in \mathcal{A}$ is obtained from M by discarding the smallest numbers in M, it follows that

$$\begin{aligned} Sr(n+1,p,q) - Sr(n,p,q) &= \left\lfloor \frac{1}{q}\left(n+1-\left\lceil \frac{n+1+q}{pq+1} \right\rceil\right)\right\rfloor + 1 \\ &= \left\lfloor \frac{1}{q}\left(n-\ell+1-\left\lceil \frac{k+1+q}{pq+1} \right\rceil\right)\right\rfloor + 1 \\ &= \begin{cases} \left\lfloor \frac{n-\ell-1}{q} \right\rfloor + 1 & \text{if } (p-1)q < k \leq pq \\ \left\lfloor \frac{n-\ell}{q} \right\rfloor + 1 & \text{if } k \leq (p-1)q. \end{cases} \end{aligned} \tag{5}$$

We now evaluate $T(n+2, pq+1, q) - T(n+1, pq+1, q)$. Again, $n+1 = (pq+1)\ell + (k+1)$, where $1 \leq k+1 \leq pq+1$. For the graph $T(n+1, pq+1, q)$, there are $k+1$ classes that have $\ell+1$ vertices and $(pq-k)$ classes that have ℓ vertices.

- `Case 1`: If $k = pq$, then the new vertex in forming $T(n+2, pq+1, q)$ can be added to any of the $pq+1$ classes. By the construction of $T(n+2, pq+1, q)$ from $T(n+1, pq+1, q)$ and Lemma 1, the number of new edges is

$$\left\lfloor \frac{pq(\ell+1)-1}{q} \right\rfloor + 1 \;=\; \left\lfloor \frac{n-\ell-1}{q} \right\rfloor + 1.$$

- `Case 2`: If $(p-1)q < k < pq$, then the new vertex in forming $T(n+2, pq+1, q)$ must be added to one of the $(pq-k)$ classes that have ℓ vertices. By the construction of $T(n+2, pq+1, q)$ from $T(n+1, pq+1, q)$ and Lemma 1, the number of new edges is

$$\left\lfloor \frac{\ell(pq-k)+(\ell+1)k-1}{q} \right\rfloor + 1 \;=\; \left\lfloor \frac{n-\ell-1}{q} \right\rfloor + 1.$$

- `Case 3`: If $k \le (p-1)q$, then the new vertex in forming $T(n+2, pq+1, q)$ must be added to one of the $(pq-k)$ classes that have ℓ vertices. By the construction of $T(n+2, pq+1, q)$ from $T(n+1, pq+1, q)$ and Lemma 1, the number of new edges is

$$\left\lfloor \frac{\ell(pq-k-1)+(\ell+1)(k+1)-1}{q} \right\rfloor + 1 \;=\; \left\lfloor \frac{n-\ell}{q} \right\rfloor + 1.$$

Therefore, we have that $Sr(n+1,p,q) - Sr(n,p,q) = T(n+2,pq+1,q) - T(n+1,pq+1,q)$, as desired. □

4. Proof of Theorem 2

Fix $p, q \ge 1$. We need the following lemma.

Lemma 2. *For $k \ge 1$ and $k \le (p-1)q$, we have that*

$$\left\lfloor \frac{k}{q} \right\rfloor \;=\; \left\lfloor \frac{pk+p-1}{pq+1} \right\rfloor.$$

Proof. Write $k = qs + t$, where $s \ge 0$ and $0 \le t < q$. Since $k \le (p-1)q$, we know that $s \le p-1$. Clearly, $\left\lfloor \frac{k}{q} \right\rfloor = s$. Also,

$$\left\lfloor \frac{pk+p-1}{pq+1} \right\rfloor \;=\; \left\lfloor \frac{(1+pq)s+(p-1)-s+pt}{pq+1} \right\rfloor \;=\; s + \left\lfloor \frac{(p-1)-s+pt}{pq+1} \right\rfloor \;=\; s,$$

where the last equality is due to the fact that $(p-1)-s+pt < pq+1$, which is equivalent to $-s < p(q-t-1)+2$. □

We want to show that

$$Sr(n+1,p,q) - Sr(n,p,q) \;=\; \left\lfloor \frac{p(n+q+1)}{pq+1} \right\rfloor, \text{ for all } n \ge 1.$$

By (5), we have that

$$Sr(n+1,p,q) - Sr(n,p,q) \;=\; \begin{cases} \left\lfloor \frac{n-\ell-1}{q} \right\rfloor + 1 & \text{if } (p-1)q < k \le pq \\ \left\lfloor \frac{n-\ell}{q} \right\rfloor + 1 & \text{if } k \le (p-1)q, \end{cases}$$

where $n = (pq+1)\ell+k$ for some $\ell \geq 0$ and $0 \leq k \leq pq$. Substituting $n = (pq+1)\ell+k$, we obtain

$$Sr(n+1,p,q) - Sr(n,p,q) \;=\; \begin{cases} p\ell + \left\lfloor \frac{k-1}{q} \right\rfloor + 1 & \text{if } (p-1)q < k \leq pq \\ p\ell + \left\lfloor \frac{k}{q} \right\rfloor + 1 & \text{if } k \leq (p-1)q. \end{cases} \tag{6}$$

On the other hand,

$$\left\lfloor \frac{p(n+q+1)}{pq+1} \right\rfloor \;=\; \left\lfloor \frac{p(((pq+1)\ell+k)+q+1)}{pq+1} \right\rfloor \;=\; p\ell+1+\left\lfloor \frac{pk+p-1}{pq+1} \right\rfloor. \tag{7}$$

By (6) and (7), we need only to confirm that

$$\left\lfloor \frac{pk+p-1}{pq+1} \right\rfloor \;=\; \begin{cases} \left\lfloor \frac{k-1}{q} \right\rfloor & \text{if } (p-1)q < k \leq pq \\ \left\lfloor \frac{k}{q} \right\rfloor & \text{if } k \leq (p-1)q. \end{cases}.$$

If $(p-1)q < k \leq pq$, it is easy to see that $\left\lfloor \frac{pk+p-1}{pq+1} \right\rfloor = \left\lfloor \frac{k-1}{q} \right\rfloor = p-1$. If $k \leq (p-1)q$, then we Lemma 2 and we are done.

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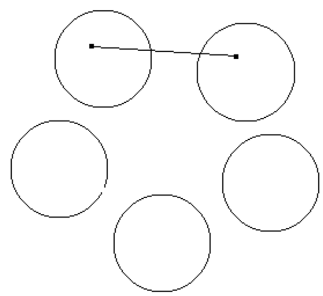
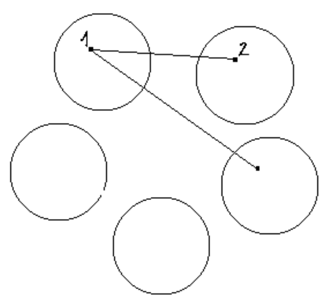
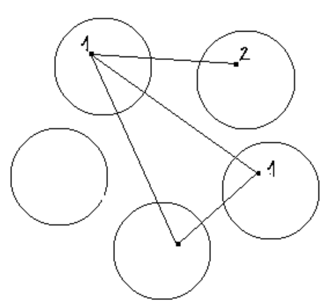
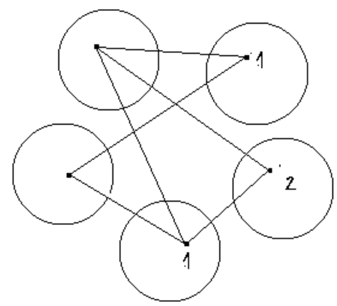
Appendix

The following are $T(n, 5, 2)$ for $2 \leq n \leq 7$.





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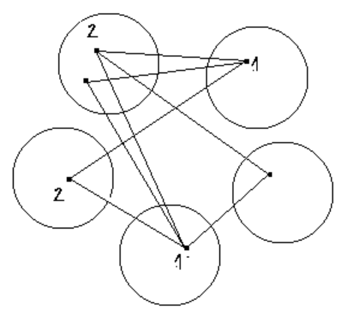
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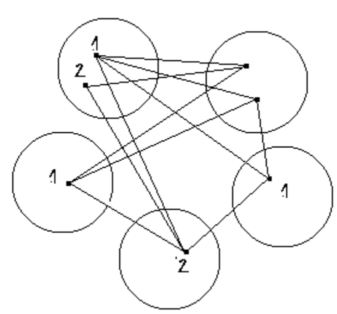
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