NEW FORMULAS FOR THE RIEMANN ZETA FUNCTION

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Abstract

A new method for continuing the usual Dirichlet series that defines the Riemann zeta function $\zeta(s)$ is presented. Numerical experiments demonstrating the computational efficacy of the resulting continuation are discussed.

1. Introduction

The usual Dirichlet series defining the Riemann Zeta Function $\zeta(s)$ is

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}, \tag{1}$$

which converges in the half-plane $\mathrm{Re}(s) > 1$ only. We construct a new family of linear combinations of subsums of Equation (1) along arithmetic progressions to achieve convergence in arbitrarily large half-planes.

To this end, let m be a positive integer, and let $d(m)$ denote the number of positive integer divisors of m. Our method enables continuing $\zeta(s)$ to the larger half-plane $\mathrm{Re}(s) > 2 - d(m)$, provided $d(m) \geq 4$.

An essential ingredient of our method is certain Dirichlet series weights b_k. These weights are m-periodic, so $b_{k+m} = b_k$ for all $k \geq 1$. The weights b_k may be found on demand by a routine calculation of nonzero vectors in the kernel of a certain $d(m) \times d(m)$ singular matrix A. Higher values of m with more divisors $d(m)$ allow for more complicated combinations and a larger half-plane of convergence.

The formulas we present may be reminiscent of the well-known formula [5, p.

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16],

$$\zeta(s)\left(1-\frac{2}{2^s}\right)=\sum_{n=0}^{\infty}\left(\frac{1}{(2n+1)^s}-\frac{1}{(2n+2)^s}\right), \tag{2}$$

which is convergent for $\text{Re}(s) > 0$. Our original motivation was, in fact, to generalize Equation (2) by considering analogies with numerical differentiation formulas.

Let D denote the set of positive divisors of m, and label the elements of D in increasing order, so $d_1 = 1$ and $d_{d(m)} = m$. These will be used to separate the integers from 1 to m into subsets according to $\gcd(k, m)$ for $1 \le k \le m$. Specifically, in Section 2, we will construct formulas of the form

$$\zeta(s)\cdot\left(\sum_{j=1}^{d(m)}\frac{a_j}{(d_j)^s}\right)=\sum_{n=0}^{\infty}\sum_{k=1}^{m}\frac{b_k}{(mn+k)^s}, \tag{3}$$

such that the right side is convergent in the half-plane $\text{Re}(s) > 2 - d(m)$. Note that the series on the right side is related to $\zeta(s)$ in a simple way; one may divide by the finite sum over j on the left (provided the sum is nonzero) to arrive at zeta.

In Section 4, we also study the error resulting from truncating the sum over n in Equation (3) at some $n = N$. An interesting finding concerns the convergence behavior on the critical line $\text{Re}(s) = 1/2$. Figure 1 displays a typical example, illustrating plateau-decay behavior, initially, and dramatic improvements in accuracy for some choices of the truncation point N. The last dramatic improvement occurs when N is around $\text{Im}(s)/(2\pi)$.

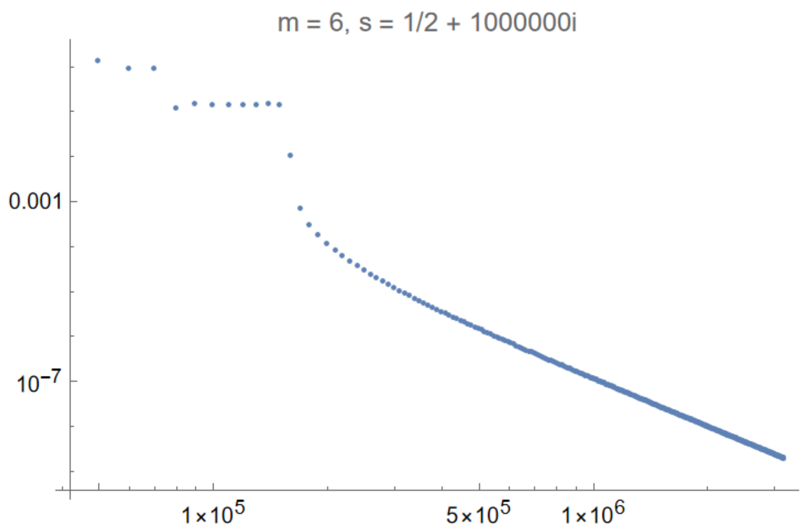
2. Derivation

2.1. Overview

Our goal is to determine coefficients $a_1, \ldots, a_{d(m)}$ and $b_1, \ldots, b_m$ such that Equation (3) holds and such that the right side in the formula converges in the half-plane $\text{Re}(s) > 2 - d(m)$.

Multiplying the Dirichlet series in Equation (1) term-by-term by $1/k^s$ "filters" the terms in the series to only those divisible by k. We will use this, multiplying $\zeta(s)$ by a_j/d_j^s for some set of coefficients a_j, where the d_j are in D. As will be explained later, the coefficients $a_1, \ldots, a_{d(m)}$ will in turn be used to determine the coefficients $b_1, \ldots, b_m$. This will result in a formula of the form of Equation (3) with the right side convergent far beyond the original half-plane of convergence.

Let us first motivate the conditions we intend to impose on the b_k's. Consider the outcome upon substituting non-positive integer values of s into the inner sum in Equation (3). In this case, the inner sum simplifies to a polynomial in n. Summing the values of this polynomial over n gives a divergent sum unless the polynomial in



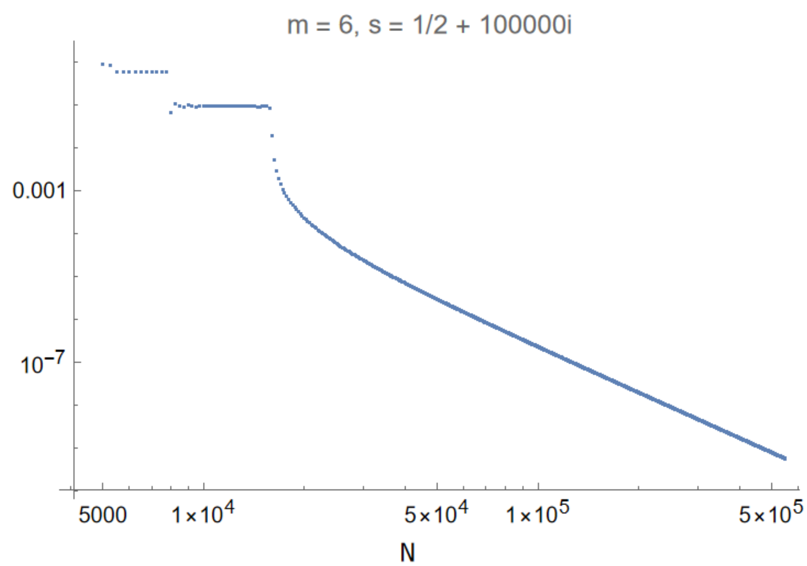


Figure 1: log-log plots of the truncation error against the number of terms N, where we used the formula from Equation (3) corresponding to the vector $\boldsymbol{a} = [1, -5, 5, -1]^T$. The range of N in the first graph is from $5 \cdot 10^4$ to $3.15 \cdot 10^6$, going in increments of 1000. The range of N in the second graph is from $5 \cdot 10^3$ to $5.5 \cdot 10^6$, going in increments of 250.

question is identically zero. For instance, if $m = 6$, then $d(m) = 4$ and the inner sum in Equation (3) becomes

$$b_1(6n+1)^{-s} + b_2(6n+2)^{-s} + \cdots + b_6(6n+6)^{-s}. \quad (4)$$

We would like this to be identically zero when $s = 0, -1, -2, -3$, if possible. It is enlightening to consider what conditions on the b_k's arise as we sequentially impose these requirements. When $s = 0$ and $n \geq 0$, Expression (4) turns into

$$b_1 + b_2 + \cdots + b_6.$$

To ensure that this expression is identically zero, we need $b_1 + b_2 + \cdots + b_6 = 0$. When $s = -1$, Expression (4) turns into the following polynomial in n.

$$6(b_1 + b_2 + \cdots + b_6)n + (b_1 + 2b_2 + \cdots + 6b_6).$$

So, in view of our earlier requirement that $b_1 + b_2 + \cdots + b_6 = 0$, to ensure that this last polynomial is identically zero we just need $b_1 + 2b_2 + \cdots + 6b_6 = 0$. Next, when $s = -2$, Expression (4) turns into

$$36(b_1 + b_2 + \cdots + b_6)n^2 + 12(b_1 + 2b_2 + \cdots + 6b_6)n + (b_1 + 2^2b_2 + \cdots + 6^2b_6).$$

So to ensure that this new polynomial in n is identically zero, we just need $b_1 + 2^2b_2 + \cdots + 6^2b_6 = 0$. Lastly, when $s = -3$, we obtain one additional condition that $b_1 + 2^3b_2 + \cdots + 6^3b_6 = 0$.

In summary, we obtain a system of equations in $b_1, \ldots, b_6$. This system can be represented by a matrix B of dimension $d(m) \times m = 4 \times 6$. (The matrix B is different from the matrix A mentioned in the introduction.) Hence, we just need to find nonzero vectors in the kernel of B. We find that the kernel of B is spanned by

$$b_1 = 4, \quad b_2 = -15, \quad b_3 = 20, \quad b_4 = -10, \quad b_5 = 0, \quad b_6 = 1.$$

and

$$b_1 = 1, \quad b_2 = -4, \quad b_3 = 6, \quad b_4 = -4, \quad b_5 = 1, \quad b_6 = 0.$$

In other words, the basis for the kernel is

$$\left\{ \begin{bmatrix} 4 \\ -15 \\ 20 \\ -10 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -4 \\ 6 \\ -4 \\ 1 \\ 0 \end{bmatrix} \right\}$$

Either of the vectors will accomplish the desired goal, as will any other nonzero vector in the kernel of B (thus any linear combination of them).

In general, as will transpire in the next subsection, for the double-sum in Equation (3) to be convergent in the larger half-plane $\mathrm{Re}(s) > 2-d(m)$, it is enough to ensure that

$$\sum_{k=1}^{m} b_k \cdot k^{-s} = 0 \qquad\qquad s = 0, -1, \ldots, 1-d(m). \tag{5}$$

Briefly, this is because for $s = 0, -1, \ldots, 1-d(m)$, if we expand $b_k(mn+k)^{-s}$ as a polynomial in n using the binomial theorem and sum across all k, then we can group the resulting terms by power of n. The coefficients of these powers of n are then of the form in Equation (5). Thus, in seeking convergence in the half-plane $\mathrm{Re}(s) > 2-d(m)$, we will want the coefficient of each power of n to be 0.

But even under these conditions, we cannot yet conclude that the right side of Equation (3) converges in the half-plane $\mathrm{Re}(s) > 2-d(m)$. This is because each inner sum on the right side of Equation (3) must be taken in full and cannot be truncated. So we cannot directly apply the well-known theorem that if a Dirichlet series converges at $s = x_0 + iy_0$, then it converges in the half-plane $\mathrm{Re}(s) > x_0$ and is analytic in that half-plane; see [1, Theorem 11.12] for an example statement of the said theorem.

2.2. The b_k Coefficients and Convergence

Let us denote the right side of Equation (3) by $Z_m(s)$. We prove the following.

Theorem 1. *If the coefficients $b_1, \ldots, b_m$ satisfy the conditions in Equation* (5)*, then the series for $Z_m(s)$ converges in the half-plane* $\mathrm{Re}(s) > 2-d(m)$ *and is analytic there.*

Proof. It is clear that the series for $Z_m(s)$ is absolutely convergent in the half-plane $\mathrm{Re}(s) > 1$. We now use the Taylor expansion to re-express the inner sums in $Z_m(s)$. We may restrict our analysis to inner sums with $n \geq n_0 \geq 3$, say. This restriction does not pose a problem for convergence of the series since $n = 0, 1, \ldots, n_0 - 1$ correspond to a finite subsum. So, let us write

$$\begin{aligned}\frac{1}{(mn+k)^s} &= \frac{1}{(mn+m/2+(k-m/2))^s}\\ &= \frac{1}{(mn+m/2)^s}\cdot\frac{1}{(1+z)^s}\bigg|_{z=\frac{k-m/2}{mn+m/2}}.\end{aligned}$$

We expand $(1+z)^{-s}$ in powers of z:

$$(1+z)^{-s} = \sum_{\ell=0}^{\infty} f_\ell(s) z^\ell. \tag{6}$$

So, $f_0(s) = 1$, $f_1(s) = -s$, $f_2(s) = s(s+1)/2$, and in general

$$f_\ell(s) = \frac{(-1)^\ell}{\ell!} \prod_{u=0}^{\ell-1} (s+u).$$

For s in compact sets, the coefficients $f_\ell(s)$ grow at most like a polynomial in ℓ as ℓ increases. More explicitly, we have $f_\ell(s) \ll (\ell+1)^{|s|+1}$, which can be seen with the aid of properties of the Γ-function; see [2, p. 73], for example. (If s is a nonpositive integer, then the $f_\ell(s)$ are eventually all zero.) Therefore, if $|z| < 1$, then the exponential decay due to $|z|^\ell$ will dominate the polynomial growth in $f_\ell(s)$ in Equation (6). Consequently, when $|z| < 1$, the expansion in Equation (6) converges absolutely for any value of s.

Since $n \geq n_0 \geq 3$ and $1 \leq k \leq m$, and considering that we plan to take $z = (k - m/2)/(mn + m/2)$, we see that the condition $|z| < 1$ is satisfied in our case. Therefore, we obtain

$$\sum_{n=n_0}^{\infty} \sum_{k=1}^{m} \frac{b_k}{(mn+k)^s} = \sum_{n=n_0}^{\infty} \sum_{k=1}^{m} b_k \sum_{\ell=0}^{\infty} \frac{f_\ell(s)}{(mn+m/2)^{s+\ell}} (k - m/2)^\ell, \tag{7}$$

where the sum over ℓ converges absolutely for any s.

Applying the binomial theorem to the $(k - m/2)^\ell$ term in Equation (7), then interchanging the order of summation in the absolutely convergent double-sum over ℓ and k, and finally grouping the resulting terms by degree, gives that the right side in Equation (7) is equal to

$$\sum_{n=n_0}^{\infty} \sum_{\ell=0}^{\infty} \frac{f_\ell(s)}{(mn+m/2)^{s+\ell}} \sum_{r=0}^{\ell} \binom{\ell}{r} (-m/2)^{\ell-r} \sum_{k=1}^{m} b_k k^r. \tag{8}$$

We now appeal to the conditions from Equation (5); namely, that for each integer r satisfying $0 \leq r \leq d(m) - 1$,

$$\sum_{k=1}^{m} b_k k^r = 0.$$

Using this, we see that the sum over $r \in [0, \ell]$ in Equation (8) vanishes if $\ell < d(m)$. Therefore, the expression in Equation (8) is equal to

$$\sum_{n=n_0}^{\infty} \sum_{\ell=d(m)}^{\infty} \frac{f_\ell(s)}{(mn+m/2)^{s+\ell}} \sum_{r=d(m)}^{\ell} \binom{\ell}{r} (-m/2)^{\ell-r} \sum_{k=1}^{m} b_k k^r. \tag{9}$$

We claim that the sum in Equation (9) is absolutely convergent for any s in the half-plane $\mathrm{Re}(s) + d(m) > 2$, and hence the series for $Z_m(s)$ converges in that half-plane and is analytic there. To see this, by the geometric-arithmetic mean inequality, and since $n \geq n_0 \geq 3$,

$$mn + \frac{m}{2} = m(n - n_0 + 1) + \left(n_0 - \frac{1}{2}\right) m \geq 2m\sqrt{n - n_0 + 1}.$$

Thus, for $\ell \geq d(m)$ and $\mathrm{Re}(s) + d(m) > 0$,

$$\left(mn + \frac{m}{2}\right)^{\mathrm{Re}(s)+\ell} \geq (n - n_0 + 1)^{\frac{\mathrm{Re}(s)+d(m)}{2}} (2m)^{\mathrm{Re}(s)+\ell}.$$

Hence, by the triangle inequality, the sum in Equation (9) is bounded in size by

$$\begin{aligned}
&\zeta\left(\frac{\mathrm{Re}(s) + d(m)}{2}\right) \sum_{\ell=d(m)}^{\infty} \frac{|f_\ell(s)|}{(2m)^{\mathrm{Re}(s)+\ell}} \sum_{r=d(m)}^{\ell} \binom{\ell}{r} (m/2)^{\ell-r} \sum_{k=1}^{m} |b_k| k^r \\
&\leq \zeta\left(\frac{\mathrm{Re}(s) + d(m)}{2}\right) \sum_{k=1}^{m} |b_k| \sum_{\ell=d(m)}^{\infty} |f_\ell(s)| \frac{(k + m/2)^\ell}{(2m)^{\mathrm{Re}(s)+\ell}} \\
&\leq \zeta\left(\frac{\mathrm{Re}(s) + d(m)}{2}\right) \left(\sum_{k=1}^{m} |b_k|\right) (2m)^{|\mathrm{Re}(s)|} \sum_{\ell=d(m)}^{\infty} |f_\ell(s)| \left(\frac{3}{4}\right)^\ell,
\end{aligned}$$

and, as pointed out earlier, the sum over ℓ converges absolutely for any s. Hence, as claimed, the sum in Equation (9) converges absolutely in the half-plane $\mathrm{Re}(s) + d(m) > 2$. □

Remark 1. In view of Equation (5), the first nonzero term in the sum over ℓ in Equation (9) is

$$\frac{f_{d(m)}(s)}{(mn + m/2)^{s+d(m)}} \sum_{k=1}^{m} b_k k^{d(m)}.$$

All subsequent terms have exponents with larger real part than $\mathrm{Re}(s) + d(m)$ in the denominator. So it is probable that convergence occurs in the larger half-plane $\mathrm{Re}(s) > 1 - d(m)$. Numerical experiments that we carried out seem consistent with this.

2.3. Solving for the a_j Coefficients

We now consider the left side of Equation (3). For $\mathrm{Re}(s) > 1$, we have

$$\begin{aligned}
\zeta(s) \cdot \left(\sum_{j=1}^{d(m)} \frac{a_j}{(d_j)^s}\right) &= \sum_{j=1}^{d(m)} \left(\frac{a_j}{(d_j)^s} + \frac{a_j}{(2d_j)^s} + \frac{a_j}{(3d_j)^s} \cdots\right) \\
&= \frac{a_1}{1^s} + \frac{\sum_{d_j \mid 2} a_j}{2^s} + \frac{\sum_{d_j \mid 3} a_j}{3^s} \cdots,
\end{aligned} \tag{10}$$

where we used absolute convergence to rearrange the sum. On the other hand, the quantity

$$\sum_{d_j\,|h} a_j, \qquad h \geq 1,$$

satisfies

$$\sum_{d_j\,|h} a_j = \sum_{d_j\,|h+m} a_j.$$

Thus, this quantity is periodic with period m. So, by absolute convergence, we may rearrange the sum and write

$$\zeta(s) \sum_{j=1}^{d(m)} \frac{a_j}{(d_j)^s} = \sum_{n=0}^{\infty} \sum_{k=1}^{m} \frac{b_k}{(mn+k)^s},$$

where

$$b_k = \sum_{d_j\,|k} a_j, \qquad k = 1, \ldots, m. \tag{11}$$

Therefore, each b_k is the sum of the a_j with the property that the j-th divisor of m divides k. Hence, in terms of the a_j, the conditions from Equation (5) read

$$\sum_{k=1}^{m} \sum_{d_j\,|k} a_j k^r = 0, \qquad r = 0, \ldots, d(m) - 1. \tag{12}$$

For example, when $m = 6$, the coefficient b_1 of the $(6n+1)^{-s}$ term is equal to a_1, because only $d_1 = 1$ divides $k = 1$ and so only a_1 contributes to b_1. In another case, the coefficient b_3 of the $(6n+3)^{-s}$ term is $a_1 + a_3$, as only d_1 and d_3 (which equal 1 and 3, respectively) divide $k = 3$. Put together, we can easily compute that

$$b_1 = a_1, \quad b_2 = a_1 + a_2, \quad b_3 = a_1 + a_3, \quad b_4 = a_1 + a_2.$$

Substituting these back into the condition given by Equation (5), we get

$$\frac{a_1}{1^s} + \frac{a_1 + a_2}{2^s} + \frac{a_1 + a_3}{3^s} + \frac{a_1 + a_2}{4^s} + \frac{a_1}{5^s} + \frac{a_1 + a_2 + a_3 + a_4}{6^s} = 0.$$

Thus, rearranging, we get

$$a_1 \left(\frac{1}{1^s} + \cdots + \frac{1}{6^s} \right) + a_2 \left(\frac{1}{2^s} + \frac{1}{4^s} + \frac{1}{6^s} \right) + a_3 \left(\frac{1}{3^s} + \frac{1}{6^s} \right) + a_4 \left(\frac{1}{6^s} \right) = 0.$$

We want the last equation to hold for each $s = 0, -1, -2, -3$. This results in a linear system represented by a 4-dimensional square matrix A and we want to solve the matrix equation $A\boldsymbol{a} = \mathbf{0}$, where $\boldsymbol{a} = [a_1, a_2, a_3, a_4]^T$. We will later show that if $d(m) \geq 4$ then A is singular, and so we will obtain a nonzero solution for $\boldsymbol{a}$, and consequently for the b_k's.

In summary, we construct a $d(m)$-dimensional square matrix $A = (A_{ij})$, where $1 \le i, j \le d(m)$. The entries of A are given by

$$A_{ij} = \sum_{n=1}^{\frac{m}{d_j}} (d_j \cdot n)^{i-1}. \tag{13}$$

The formula in Equation (13) arises from the linear constraints imposed in Equation (12). Moreover, in view of the findings in the next subsection, provided $d(m) \ge 4$, we can find a nonzero vector $\boldsymbol{a} = [a_1, \ldots, a_{d(m)}]^T$ such that $A\boldsymbol{a} = \mathbf{0}$. Given such a vector $\boldsymbol{a}$, we can solve for the b_k using Equation (11) and hence obtain a formula of the form

$$\zeta(s) \cdot \left(\sum_{j=1}^{d(m)} \frac{a_j}{(d_j)^s} \right) = Z_m(s).$$

Although this formula was derived for $\mathrm{Re}(s) > 1$, it follows from Theorem 1 that the equality holds by analytic continuation throughout the half-plane $\mathrm{Re}(s) > 2 - d(m)$.

Remark 2. It is natural to consider an extension of our method to Dirichlet L-functions $L(s, \chi)$. An immediate obstacle arises in Equation (10). The Dirichlet convolution of $L(s, \chi)$ with $\sum_{1 \le j \le d(m)} a_j (d_j)^{-s}$ results in coefficients of the form $\sum_{d_j | h} \chi(h/d_j) a_j$, which are not periodic in h with period m in general. Nevertheless, it is plausible that our method can be extended to Dirichlet L-function after appropriate adjustments.

2.4. Singularity of A

We show that A is singular, so the kernel of A is nonzero. Suppose m is such that $d(m) \ge 4$. Then A has at least 4 columns and rows. We claim that the nonzero vector

$$\boldsymbol{c} = \left[0, m^2, -3m, 2, 0, 0, \cdots, 0\right]$$

is in the left-kernel of A. This will follow on showing that for each $j = 1, \ldots, m$, the dot product of $\boldsymbol{c}$ with the j-th column of A, denoted by A_j, is zero.

Using the formula in Equation (13), the second, third, and fourth entries of the j-th column A_j are given by

$$A_{2j} = \sum_{n=1}^{\frac{m}{d_j}} (d_j \cdot n) = d_j \cdot \frac{1}{2} \cdot \frac{m}{d_j} \left(\frac{m}{d_j} + 1 \right) = \frac{m(m + d_j)}{2d_j},$$

$$A_{3j} = \sum_{n=1}^{\frac{m}{d_j}} (d_j \cdot n)^2 = (d_j)^2 \cdot \frac{1}{6} \cdot \frac{m}{d_j} \left(\frac{m}{d_j} + 1 \right) \left(\frac{2m}{d_j} + 1 \right) = \frac{m(m + d_j)(2m + d_j)}{6d_j},$$

$$A_{4j} = \sum_{n=1}^{\frac{m}{d_j}} (d_j \cdot n)^3 = (d_j)^3 \cdot \frac{1}{4} \cdot \left(\frac{m}{d_j} \right)^2 \left(\frac{m}{d_j} + 1 \right)^2 = \frac{m^2(m + d_j)^2}{4d_j}.$$

Thus, taking a dot product $\boldsymbol{c} \cdot A_j$ we get

$$\begin{aligned}\boldsymbol{c} \cdot A_j &= \frac{m^3(m+d_j)}{2d_j} - \frac{m^2(m+d_j)(2m+d_j)}{2d_j} + \frac{m^2(m+d_j)^2}{2d_j} \\ &= \frac{m(m+d_j)}{2d_j}(m^2 - m(2m+d_j) + m(m+d_j)) = 0.\end{aligned}$$

So the left-kernel of the square matrix A is nonzero, and so A must be singular.

For the cases where $d(m) < 4$, we do find that the generated matrix is nonsingular. The two cases to consider are $m = p$ for p prime, which gives $d(m) = 2$, and $m = p^2$ for p a prime, which gives $d(m) = 3$. In the first case, the matrix A reduces to

$$A = \begin{bmatrix} p & 1 \\ \frac{p(p+1)}{2} & p \end{bmatrix},$$

which is nonsingular. In the second case, the matrix reduces to

$$A = \begin{bmatrix} p^2 & p & 1 \\ \frac{p^2(p^2+1)}{2} & p\frac{p(p+1)}{2} & p^2 \\ \frac{p^2(p^2+1)(2p^2+1)}{6} & p^2\frac{p(p+1)(2p+1)}{6} & p^4 \end{bmatrix}.$$

Taking the determinant, we get $\det(A) = \frac{p^8}{12} - \frac{p^7}{6} + \frac{p^5}{6} - \frac{p^4}{12}$. This is

$$\frac{p^4}{12}(p^4 - 2p^3 + 2p - 1) = \frac{p^4}{12}(p^2-1)(p^2-2p+1).$$

So again, the determinant is nonzero. Thus, if $d(m) < 4$, the generated matrix is nonsingular, and our method is not applicable. For example, the formula in Equation (2) falls outside the scope of our method. But if $d(m) \geq 4$, the generated matrix A will be singular and our method works.

3. Example with $m = 24$

To clarify each step, we provide an example when $m = 24$. In this case, $d(m) = 8$ and $D = \{1, 2, 3, 4, 6, 8, 12, 24\}$. So $d_1 = 1, d_2 = 2, \ldots, d_8 = 24$. Using the formula in Equation (13) for A_{ij} we find that our $d(m)$-dimensional, or 8-dimensional, matrix $A = (A_{ij})$ is given by

$$A = \begin{bmatrix} 24 & 12 & 8 & \ldots & 1 \\ 300 & 156 & 108 & \ldots & 24 \\ 4900 & 2600 & 1836 & \ldots & 576 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \sum_{n=1}^{24}(n)^7 & \sum_{n=1}^{12}(2n)^7 & \sum_{n=1}^{8}(3n)^7 & \ldots & (24n)^7 \end{bmatrix}.$$

Using a computer algebra system to compute the so-called row-reduced echelon form of this matrix, see [4] for example, the result is

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 3 & 56 & 2761 \\ 0 & 1 & 0 & 0 & 0 & \frac{91}{4} & -407 & \frac{-78085}{4} \\ 0 & 0 & 1 & 0 & 0 & 47 & 792 & 36685 \\ 0 & 0 & 0 & 1 & 0 & \frac{-67}{2} & -517 & \frac{-45793}{2} \\ 0 & 0 & 0 & 0 & 1 & \frac{29}{4} & 77 & \frac{11891}{4} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Thus, one of the nonzero vectors in the kernel is

$$\boldsymbol{c} = \begin{bmatrix} 56 & -407 & 792 & -517 & 77 & 0 & -1 & 0 \end{bmatrix}^T.$$

Matching this vector with the divisors d_j, in accordance with formula in Equation (3), we get

$$\zeta(s)\left(56 - \frac{407}{2^s} + \frac{792}{3^s} - \frac{517}{4^s} + \frac{77}{6^s} - \frac{1}{12^s}\right) = Z_{24}(s).$$

To write down the series representation of $Z_{24}(s)$, we use formula in Equation (11) to calculate the b_k. For example, to calculate b_k when $k = 16$, we add the coefficients a_j corresponding to the divisors $d_j \in D$ such that $d_j|16$. Those divisors are $1, 2, 4$, and 8, and the corresponding a_j are $56, -407, -517$, and 0. The resulting coefficient is therefore $b_{16} = 56 - 407 - 517 + 0 = -868$. Repeating this process for every b_k, the end result is

$$\begin{aligned} Z_{24}(s) = \sum_{n=0}^{\infty} \Bigg(& \frac{56}{(24n+1)^s} - \frac{351}{(24n+2)^s} + \frac{848}{(24n+3)^s} - \frac{868}{(24n+4)^s} + \frac{56}{(24n+5)^s} \\ & + \frac{518}{(24n+6)^s} + \frac{56}{(24n+7)^s} - \frac{868}{(24n+8)^s} + \frac{848}{(24n+9)^s} - \frac{351}{(24n+10)^s} + \frac{56}{(24n+11)^s} \\ & + \frac{0}{(24n+12)^s} + \frac{56}{(24n+13)^s} - \frac{351}{(24n+14)^s} + \frac{848}{(24n+15)^s} - \frac{868}{(24n+16)^s} + \frac{56}{(24n+17)^s} \\ & + \frac{518}{(24n+18)^s} + \frac{56}{(24n+19)^s} - \frac{868}{(24n+20)^s} + \frac{848}{(24n+21)^s} - \frac{351}{(24n+22)^s} + \frac{56}{(24n+23)^s} \\ & + \frac{0}{(24n+24)^s} \Bigg). \end{aligned}$$

Combining, we get

$$\zeta(s) = \frac{Z_{24}(s)}{56 - \frac{407}{2^s} + \frac{792}{3^s} - \frac{517}{4^s} + \frac{77}{6^s} - \frac{1}{12^s}},$$

whenever the denominator on the right side is nonzero, and this converges for $\mathrm{Re}(s) > 2 - d(m) = -6$.

4. Numerical Experiments with $m = 6, 24, 60$

Wolfram Mathematica was used to verify the accuracy of the generated Dirichlet series and to check the rate of convergence. To do so, we computed highly accurate values of $\zeta(\frac{1}{2}+it)$ for $t = 10^4, 10^5, 10^6, 10^7$ and compared the outcome to the result of our method with $m = 60$, $m = 24$, and $m = 6$, as well as to the well-known continuation through the Dirichlet Eta function given in Equation (2).

We find the series representation corresponding to $m = 60$ is given by

$$\begin{aligned}
Z_{60}(s) = \sum_{n=0}^{\infty}\Bigg(&\frac{61768}{(60n+1)^s} - \frac{506228}{(60n+2)^s} + \frac{1657604}{(60n+3)^s} - \frac{2557849}{(60n+4)^s} + \frac{1354748}{(60n+5)^s}\\
&+\frac{754819}{(60n+6)^s} + \frac{61768}{(60n+7)^s} - \frac{2557849}{(60n+8)^s} + \frac{1657604}{(60n+9)^s} + \frac{791167}{(60n+10)^s} + \frac{61768}{(60n+11)^s}\\
&-\frac{1297395}{(60n+12)^s} + \frac{61768}{(60n+13)^s} - \frac{506228}{(60n+14)^s} + \frac{2950584}{(60n+15)^s} - \frac{2557849}{(60n+16)^s} + \frac{61768}{(60n+17)^s}\\
&+\frac{754819}{(60n+18)^s} + \frac{61768}{(60n+19)^s} - \frac{1260454}{(60n+20)^s} + \frac{1657604}{(60n+21)^s} - \frac{506228}{(60n+22)^s} + \frac{61768}{(60n+23)^s}\\
&-\frac{1297395}{(60n+24)^s} + \frac{1354748}{(60n+25)^s} - \frac{506228}{(60n+26)^s} + \frac{1657604}{(60n+27)^s} - \frac{2557849}{(60n+28)^s} + \frac{61768}{(60n+29)^s}\\
&+\frac{2052214}{(60n+30)^s} + \frac{61768}{(60n+31)^s} - \frac{2557849}{(60n+32)^s} + \frac{1657604}{(60n+33)^s} - \frac{506228}{(60n+34)^s} + \frac{1354748}{(60n+35)^s}\\
&-\frac{1297395}{(60n+36)^s} + \frac{61768}{(60n+37)^s} - \frac{506228}{(60n+38)^s} + \frac{1657604}{(60n+39)^s} - \frac{1260454}{(60n+40)^s} + \frac{61768}{(60n+41)^s}\\
&+\frac{754819}{(60n+42)^s} + \frac{61768}{(60n+43)^s} - \frac{2557849}{(60n+44)^s} + \frac{2950584}{(60n+45)^s} - \frac{506228}{(60n+46)^s} + \frac{61768}{(60n+47)^s}\\
&-\frac{1297395}{(60n+48)^s} + \frac{61768}{(60n+49)^s} + \frac{791167}{(60n+50)^s} + \frac{1657604}{(60n+51)^s} - \frac{2557849}{(60n+52)^s} + \frac{61768}{(60n+53)^s}\\
&+\frac{754819}{(60n+54)^s} + \frac{1354748}{(60n+55)^s} - \frac{2557849}{(60n+56)^s} + \frac{1657604}{(60n+57)^s} - \frac{506228}{(60n+58)^s} + \frac{61768}{(60n+59)^s}\\
&+\frac{0}{(60n+60)^s}\Bigg).
\end{aligned}$$

Therefore,

$$\zeta(s) = \frac{Z_{60}(s)}{61768 - \frac{567996}{2^s} + \frac{1595836}{3^s} - \frac{2051621}{4^s} + \frac{1292980}{5^s} - \frac{334789}{6^s} + \frac{4415}{10^s} - \frac{593}{12^s}}.$$

The series representation obtained when $m = 24$ was derived in the section prior, and is given by

$$\frac{Z_{24}(s)}{56 - \frac{407}{2^s} + \frac{792}{3^s} - \frac{517}{4^s} + \frac{77}{6^s} - \frac{1}{12^s}}.$$

The series representation when $m = 6$ is

$$\frac{\sum_{n=0}^{\infty}\left(\frac{1}{(6n+1)^s} - \frac{4}{(6n+2)^s} + \frac{6}{(6n+3)^s} - \frac{4}{(6n+4)^s} + \frac{1}{(6n+5)^s}\right)}{1 - \frac{5}{2^s} + \frac{5}{3^s} - \frac{1}{6^s}}.$$

And the series representation from Equation (2) is

$$\frac{\sum_{n=0}^{\infty}\left(\frac{1}{(2n+1)^s}-\frac{1}{(2n+2)^s}\right)}{1-\frac{2}{2^s}}.$$

In order to test the convergence rate of these series, we computed the summation over n in Equation (3) using various truncation thresholds N. The choices of N that we made were the minimum necessary to be within a prescribed desired accuracy of less than 0.001,0.0001 and 0.00001. The minimum N that achieved this is given in the displayed tables. Note, however, that our criterion for choosing N could be occasionally inconsistent. For example, for some t there could be N that by chance brings the sum to within the prescribed accuracy. This appears to be the case for $m = 2$ when t is small. Nevertheless, one can still glean distinct patterns despite the occasional inconsistency.

$s = 1/2 + it$	Minimum N necessary for error of magnitude < 0.001			
t	$m = 60$	$m = 24$	$m = 6$	$m = 2$
10^4	2.4×10^2	4.5×10^2	2×10^3	7.5×10^4
10^5	1.7×10^3	4.25×10^3	1.8×10^4	6.5×10^4
10^6	2.7×10^4	4.3×10^4	1.7×10^5	2.4×10^5
10^7	2.7×10^5	4.1×10^5	1.7×10^6	2.4×10^6

$s = 1/2 + it$	Minimum N necessary for error of magnitude < 0.0001			
t	$m = 60$	$m = 24$	$m = 6$	$m = 2$
10^4	3.2×10^2	8.1×10^2	2.9×10^3	7×10^6
10^5	2.7×10^3	8×10^3	2.3×10^4	5.1×10^6
10^6	3.2×10^4	8×10^4	2.1×10^5	6×10^6
10^7	3.2×10^5	8×10^5	1.8×10^6	7.3×10^7

$s = 1/2 + it$	Minimum N necessary for error of magnitude < 0.00001			
t	$m = 60$	$m = 24$	$m = 6$	$m = 2$
10^4	3.3×10^2	8.8×10^3	5×10^3	7×10^8
10^5	3.2×10^3	8.3×10^3	3.7×10^4	5.1×10^8
10^6	3.2×10^4	8.2×10^4	3.1×10^5	6×10^8
10^7	3.2×10^5	8×10^5	2.4×10^6	7.3×10^9

Table 1: Minimum number of terms needed for various m and s to be within an error of magnitude under $0.001, 0.0001$, and 0.00001.

The $m = 2$ case, the Dirichlet Eta Function, is known to converge on the critical line with error term of order $\frac{1}{\sqrt{N}}$; see [3] for example. This is reflected in the tables, as decreasing the error by a factor of 10 (from .001 to .0001, or from .0001 to .00001) for the same value of t requires 100 times as many terms. In comparison, for each of the $m = 6$, $m = 24$, and $m = 60$ cases, the minimum N needed does not increase nearly as fast as the prescribed accuracy is decreased. Moreover, in almost all cases, for a given prescribed accuracy, the number of terms needed scales approximately linearly with t (or with the magnitude of s).

Remark 3. The number N refers to the number of inner sums being added in the right side of Equation (3), and so is the upper limit of the summation over n. To get the total number of individual terms added one should multiply by m (since each inner sum has m terms).

4.1. Error Analysis

In this section, we will approximate the error resulting from truncating our formula for $Z_m(s)$ at $n = N$. Consider

$$\frac{f_{d(m)}(s)}{(mn + m/2)^{s+d(m)}} \sum_{k=1}^{m} b_k k^{d(m)}. \tag{14}$$

As pointed out in the remark following Theorem 1, this is the first nonzero term in the Taylor expansion used in the proof of the theorem. For $\mathrm{Re}(s) > 2 - d(m)$, we use monotonicity to estimate

$$\begin{aligned}\int_{n=N}^{\infty} \frac{1}{(mn + m/2)^{\mathrm{Re}(s)+d(m)}}\, dn &< \sum_{n=N}^{\infty} \frac{1}{(mn + m/2)^{\mathrm{Re}(s)+d(m)}} \\ &< \int_{n=N-1}^{\infty} \frac{1}{(mn + m/2)^{\mathrm{Re}(s)+d(m)}}\, dn\,.\end{aligned}$$

Therefore,

$$\begin{aligned}\sum_{n=N}^{\infty} \frac{1}{(mn + m/2)^{\mathrm{Re}(s)+d(m)}} &< \frac{1}{m(\mathrm{Re}(s) + d(m) - 1)(mN - \frac{m}{2})^{\mathrm{Re}(s)+d(m)-1}} \\ &=: \mathcal{T}(s, N).\end{aligned}$$

If the behavior of the error is mainly determined by Equation (14), as expected, then the truncation error resulting from using $n = N$ terms in the formula for $Z_m(s)$ – that is, the difference between the actual value of $\zeta(s)$ and our truncated formula – is approximately

$$\left| \mathcal{T}(s, N) \cdot \frac{\prod_{u=0}^{d(m)-1}(s+u)}{d(m)!} \cdot \frac{\sum_{k=1}^{m} b_k k^{d(m)}}{\sum_{j=1}^{d(m)} a_j (d_j)^{-s}} \right|. \tag{15}$$

We simplify the estimate from Equation (15) some more by specializing to the parameters used in our set of numerical experiments, which we conducted on the critical line, including for heights t not too small. Since $|s|$ is significantly larger than m in our experiments, it is reasonable to approximate $|\prod_{u=0}^{d(m)-1}(s+u)| \approx |s|^{d(m)}$. Also, since we are working on the critical line, we approximate $|\mathcal{T}(s, N)| \approx (m(mN)^{d(m)-1/2})^{-1}$. So, on the critical line, the estimate from Equation (15) behaves like

$$\left| \frac{1}{m(mN)^{d(m)-1/2}} \cdot \frac{|s|^{d(m)}}{d(m)!} \cdot \frac{\sum_{k=1}^{m} b_k k^{d(m)}}{\sum_{j=1}^{d(m)} a_j (d_j)^{-s}} \right|. \tag{16}$$

Using Equation (16), we see why in our set of experiments with $m = 6, 24, 60$, increasing t tenfold requires about a tenfold increase in N to maintain the same level of accuracy. Increasing t in such a way multiplies $|s|$ by approximately 10, which multiplies the numerator in Equation (16) by approximately $10^{d(m)}$. On the other hand, increasing N tenfold multiplies the denominator by a factor of $10^{d(m)-1/2}$. For the same accuracy, then, it suffices to multiply N by about 10. This behavior is clearly reflected in the tables from the prior subsection.

We can also interpret the rate of convergence observed in our experiments using the estimate from Equation (16). Generally, to improve the accuracy by a multiplicative factor $1/\eta$, we need to multiply N by a factor κ such that $\kappa^{d(m)-1/2} = \eta$, so $\kappa = \eta^{\frac{1}{d(m)-1/2}}$. Table 2 demonstrates that this expected rate of convergence is at work.

For example, for $\eta = 10$, we consider the corresponding κ's for various m. When $m = 6$, we get $\kappa = 10^{2/7} \approx 1.93$. When $m = 24$, we get $\kappa = 10^{2/15} \approx 1.36$. And when $m = 60$, we get $\kappa = 10^{2/23} \approx 1.22$. We see that these κ's empirically match these expected values fairly well. When $m = 6$, we get $4.70/2.43 \approx 1.93$ in one case and $2.43/1.27 \approx 1.91$ in another case. Similarly, when $m = 24$, we get $1.96/1.46 \approx 1.34$ in one case, and $1.46/1.14 \approx 1.28$ in another case. And when $m = 60$ we get $4.74/4.08 \approx 1.16$ and $4.08/3.61 \approx 1.13$.

	Minimum N necessary to achieve a specified accuracy		
Accuracy	$m = 60$	$m = 24$	$m = 6$
10^{-6}	3.33×10^3	9.4×10^3	6.7×10^4
10^{-7}	3.61×10^3	1.14×10^4	1.27×10^5
10^{-8}	4.08×10^3	1.46×10^4	2.43×10^5
10^{-9}	4.74×10^3	1.96×10^4	4.70×10^5

Table 2: Minimum N to achieve a prescribed accuracy for $s = \frac{1}{2} + 10^5 i$. The case $m = 2$ was excluded as its rate of convergence is known.

When compared to the Dirichlet Eta Function ($m = 2$), representations with higher m converge much faster. For $m = 2$, demanding that the accuracy of the output is better by a factor of 10 requires increasing the number of terms N by a factor of 100. Whereas for $m = 60$, once N is large enough, our formula appears to satisfy the same demand on increasing the number of terms by only a factor of about 1.22. Therefore, the series in our formulas offer more efficient and fairly simple ways to compute the zeta function.

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