



GAPS IN INTERVALS OF N -EXPANSIONS

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Abstract

For $N \in \mathbb{N}_{\geq 2}$ and $\alpha \in \mathbb{R}$ such that $0 < \alpha \leq \sqrt{N} - 1$, the continued fraction map $T_\alpha : [\alpha, \alpha + 1] \rightarrow [\alpha, \alpha + 1]$ is defined as $T_\alpha(x) := N/x - d(x)$, where $d : [\alpha, \alpha + 1] \rightarrow \mathbb{N}$ is defined by $d(x) := \lfloor N/x - \alpha \rfloor$. A maximal open interval $(a, b) \subset I_\alpha$ is called a *gap* of I_α if for almost every $x \in I_\alpha$ there is an $n_0(x) \in \mathbb{N}$ such that $x_n \notin (a, b)$ for all $n \geq n_0$. In this paper, all conditions are given in which I_α is gapless. For $\alpha = \sqrt{N} - 1$ it is shown that the number of gaps is a finite, monotonically non-decreasing and unbounded function of N .

1. Introduction

Recently, H. Nakada and the authors of the present paper studied N -expansions with finitely many digits in [1]. These N -expansions are continued fraction expansions, introduced in [2] (see [3]), and defined as follows.

Definition 1. For $N \in \mathbb{N}_{\geq 2}$ and $\alpha \in \mathbb{R}$ such that $0 < \alpha \leq \sqrt{N} - 1$, let $I_\alpha := [\alpha, \alpha + 1]$ and $I_\alpha^- := [\alpha, \alpha + 1)$. We define the N -expansion map $T_\alpha : I_\alpha \rightarrow I_\alpha^-$ as

$$T_\alpha(x) := \frac{N}{x} - d(x), \tag{1}$$

where $d : I_\alpha \rightarrow \mathbb{N}$ is defined by

$$d(x) := \left\lfloor \frac{N}{x} - \alpha \right\rfloor, \quad \text{if either } x \in (\alpha, \alpha + 1] \text{ or both } x = \alpha \text{ and } \frac{N}{\alpha} - \alpha \notin \mathbb{Z}.$$

So as to avoid having cylinders consisting of one point, we also have

$$d(\alpha) = \left\lfloor \frac{N}{\alpha} - \alpha \right\rfloor - 1, \quad \text{if } \frac{N}{\alpha} - \alpha \in \mathbb{Z}.$$

For a fixed $\alpha \in (0, \sqrt{N} - 1]$ and $x \in I_\alpha$, we define

$$d_n = d_n(x) := d(T_\alpha^{n-1}(x)) \text{ for } n \in \mathbb{N}.$$

Throughout this paper, we assume that $N \in \mathbb{N}_{\geq 2}$. We will write α_i for $T_\alpha^i(\alpha)$ and β_i for $T_\alpha^i(\alpha + 1)$, with $i \in \mathbb{N}$; we also often write β for $\alpha + 1$. Applying (1), for every $x \in I_\alpha$ we obtain a continued fraction expansion of the form

$$x = T_\alpha^0(x) = \frac{N}{d_1 + T_\alpha(x)} = \frac{N}{d_1 + \frac{N}{d_2 + \frac{N}{d_3 + \ddots}}}.$$

Throughout this paper, we will write $x = [d_1, d_2, d_3, \dots]_{N, \alpha}$ to represent the above equation. For reasons of legibility, we will usually omit suffixes such as ‘(N)’, ‘(N, α)’ or ‘(N, d)’. We will call the numbers d_i , with $i \in \mathbb{N}$, the *partial quotients* or *digits* of this *N-continued fraction expansion* of x . Since $0 \notin I_\alpha$, this expansion is infinite for every $x \in I_\alpha$. We write $x_n := T_\alpha^n(x)$, with $n \in \mathbb{N} \cup \{0\}$. The sequence x_n , $n = 0, 1, 2, \dots$, is called the *orbit of x under T_α* . When $x = [d_1, d_2, d_3, \dots]$ and there are smallest $h, k \in \mathbb{N}$ such that $d_{h+i} = d_{h+nk+i}$ for all $n \in \mathbb{N}$ and $i \in \{0, \dots, k - 1\}$, we call the expansion *eventually periodic* with *period length $|x| = k$* and denote the periodic part as $\overline{d_h, \dots, d_{h+k-1}}$. If $h = 1$, we write $x = [d_1, \dots, d_k]$. In this latter case, x is called a *periodic point* with a *purely periodic expansion* and *periodic orbit*.

Let $N \in \mathbb{N}_{\geq 2}$ and $\alpha \in (0, \sqrt{N} - 1]$. Then in each *cylinder set* $\Delta_i := \{x \in I_\alpha; d(x) = i\}$ of rank 1, with $d_{\min} \leq i \leq d_{\max}$, where $d_{\max} := d(\alpha)$ is the largest partial quotient¹ and $d_{\min} := d(\alpha + 1)$ the smallest one, the map T_α obviously has one *fixed point* f_i . From now on, we will write simply ‘cylinder set’ for ‘cylinder set of rank 1’. It is easy to see that

$$f_i = f_i(N) := \frac{\sqrt{4N + i^2} - i}{2}, \text{ for } d_{\min} \leq i \leq d_{\max}.$$

Note that $N/\alpha - \alpha \in \mathbb{Z}$ if and only if for some $d \geq 2$ we have that $d + 1 = \max\{d_i\}$ for any $\alpha_0 < \alpha$, i.e., $\Delta_{d+1} \neq \emptyset$ for $\alpha_0 < \alpha$, and $\alpha = f_{d+1}$. Moreover, we can write $f_i = [\tilde{i}]$. Note that fixed points are periodic points with period length 1.

Given N , we let $\alpha_{\max} = \sqrt{N} - 1$ be the largest value of α we consider, so as to avoid 0 being a partial quotient as well. Since $T'_\alpha(x) = -N/x^2$ and because

¹Note that the number of occurring digits is finite for $\alpha > 0$.

$0 < \alpha \leq \sqrt{N} - 1$, we have $|T'_\alpha(x)| > 1$ on I_α^- . From this, it follows that the fixed points act as *repellers* and that the maps T_α are *expanding* when $0 < \alpha \leq \sqrt{N} - 1$.

As in the previous paper (see [1]), we are interested in subsets of I_α that we will call *gaps*, defined as follows.

Definition 2. A maximal open interval $(a, b) \subset I_\alpha$ is called a *gap* of² I_α if for almost every³ $x \in I_\alpha$ there is an $n_0(x) \in \mathbb{N}$ such that $x_n \notin (a, b)$ for all $n \geq n_0$.

Figure 1 is an example of an interval I_α having two gaps. In this case, we have $N = 100$ and $\alpha \approx 8.983$. It appears that in this example the boundaries of the gaps are the first two images of α and β , respectively, under T_α . In Theorem 5, we show that the boundaries between gaps and non-gaps are *always* images of α and β .

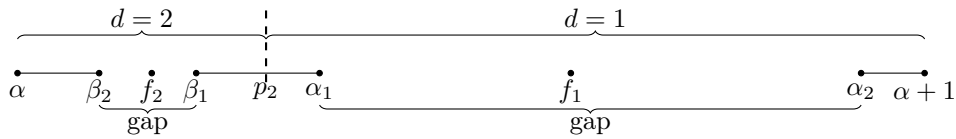


Figure 1: An interval I_α with two gaps. Here $N = 100$ and $\alpha \approx 8.983$

Each pair of consecutive cylinder sets (Δ_i, Δ_{i-1}) is separated by a *discontinuity point* $p_i(N, \alpha)$ of T_α , satisfying $N/p_i - i = \alpha$, so $p_i = N/(\alpha + i)$. In Figure 1, α is such that $\alpha_3 = p_2$, implying $\alpha \approx 8.983$. A cylinder set Δ_i is called *full* if $T_\alpha(\Delta_i) = I_\alpha^-$ (or $T_\alpha(\Delta_d) = I_\alpha$ in case $d = d(\alpha)$ and $T_\alpha(\alpha) = \beta$). When a cylinder set is not full, it contains either α (in which case $T_\alpha(\alpha) < \alpha + 1$) or $\alpha + 1$ (in which case $T_\alpha(\alpha + 1) > \alpha$), and is called *incomplete*. On account of our definition of T_α , cylinder sets will always be an interval, and *never* consist of one single point. An interval I_α together with its cylinder sets, associated fixed points and discontinuity points, is called an *arrangement* of I_α , depending on N . An arrangement is called *full* when its cylinder sets are all full. Finally, a cylinder set is called *gapless* if it has empty intersection with any gap.

In [1], some results are obtained that can quite easily be extended to a complete overview of gaplessness; we write $d := d(\alpha)$.

Theorem 1. *Let $I_\alpha = \Delta_d \cup \Delta_{d-1}$. Then I_α is gapless if and only if $\alpha_1 \geq f_{d-1}$ and $\beta_1 \leq f_d$.*

Proof. The condition for gaplessness is proved in [1, Theorem 5]. So let $I_\alpha = \Delta_d \cup \Delta_{d-1}$ and suppose $\alpha_1 < f_{d-1}$. Since T_α is expanding and $\alpha_1 < f_{d-1}$, we find

²We will usually omit the addition ‘of I_α ’.

³Here we use ‘for almost all x ’ (and not ‘for all x ’) because we want to exclude fixed points and pre-images of fixed points, i.e., points that T_α maps to a fixed point, which may never leave an interval (a, b) . All ‘for all’ statements in this paper are with respect to Lebesgue measure.

that $f_{d-1} < \alpha_2$. For the same reason, we have that $(\beta_1, \alpha_3) \subset [\alpha, \alpha_1]$. Consequently, $T_\alpha^n(\Delta_d) \subset I_\alpha \setminus (\alpha_1, \alpha_2)$ for $n \geq 1$, with $f_{d-1} \in (\alpha_1, \alpha_2)$. Now let $x \in I_\alpha \setminus \{f_{d-1}\}$. Then there is a smallest non-negative integer n_0 (possibly 0) such that $T_\alpha^{n_0}(x) \in \Delta_d$. Since $T_\alpha^n(\Delta_d) \subset I_\alpha \setminus (\alpha_1, \alpha_2)$ for $n \geq 1$, it follows that $T_\alpha^n(x) \notin (\alpha_1, \alpha_2)$ for $n \geq n_0$. We conclude that (α_1, α_2) is a gap containing f_{d-1} . The reasoning is completely similar in case $\beta_1 > f_d$. \square

Theorem 2. *Let $I_\alpha = \Delta_d \cup \dots \cup \Delta_{d-m}$, with $m \in \{2, 3\}$. Then I_α is gapless if and only if $\alpha_1 \geq f_{d-1}$ or $\beta_1 \leq f_{d-m+1}$.*

Proof. The condition for gaplessness is proved in [1, Theorem 6]. We consider here the case $m = 2$; the case $m = 3$ is proved in a similar way. So let $I_\alpha = \Delta_d \cup \Delta_{d-1} \cup \Delta_{d-2}$, and suppose that $\alpha_1 < f_{d-1} < \beta_1$. Then $T_\alpha((p_d, \alpha_1)) = [\alpha_2, \alpha + 1)$ and $T_\alpha([\beta_1, p_{d-1})) = (\alpha, \beta_2]$ (*). Let $a = \max\{\alpha_1, \beta_2\}$ and $b = \min\{\alpha_2, \beta_1\}$ and let $x \in I_\alpha \setminus \{f_{d-1}\}$. Then there exists a smallest n_0 (possibly 0) depending on x , for which $T_\alpha^{n_0}(x) \in I_\alpha \setminus (a, b)$. But then it follows from (*) that we have $T_\alpha^n(x) \in I_\alpha \setminus (a, b)$ for all $n \geq n_0$, i.e., $T_\alpha^n(x) \notin (a, b)$. \square

In [1], the following two results were obtained (see [1, Theorems 7 and 4]).

Theorem 3. *Let $N \in \mathbb{N}_{\geq 2}$ and $I_\alpha = \Delta_d \cup \Delta_{d-1} \cup \Delta_{d-2} \cup \Delta_{d-3}$. Then there is a gap in I_α if and only if $N = 2k^2 + 2k - i$, with $k > 1$ and $i \in \{1, 2, 3\}$. Moreover, if there is a gap in I_α , the gap contains f_{d-1} and f_{d-2} , while Δ_d and Δ_{d-3} are gapless.*

Theorem 4. *Let I_α consist of five cylinders or more. Then I_α has no gaps.*

Knowing all about *gaplessness*, we want to start a thorough investigation of the occurrence of gaps. We will make a lot of use of the graphs of T_α , which are drawn in the square⁴ $\Upsilon_{N,\alpha} := I_\alpha \times I_\alpha^-$. This square is divided into rectangular sets of points $\square_i := \{(x, y) \in \Upsilon_\alpha : d(x) = i\}$. We identify these two-dimensional *fundamental regions* \square_i with the one-dimensional cylinder sets Δ_i we already use. It is obvious that the graph of T_α has one fixed point $F_i := (f_i, f_i)$ in each \square_i . We will denote the dividing line between \square_i and \square_{i-1} by ℓ_i , which is the line segment $\{p_i\} \times [\alpha, \alpha + 1)$, where p_i is the discontinuity point between Δ_i and Δ_{i-1} . Depending on the fullness of a cylinder set Δ_i , we will call the concerning branch of the graph of T_α in \square_i *complete* or *incomplete*. Finally, we will use the word ‘arrangement’ in a similar way for Υ_α together with its cylinder sets, fixed points and dividing lines as for I_α . Figure 2 shows the two-dimensional version of Figure 1. In grey, we have drawn the domain for *cobweb plots* of points outside the gaps of I_α , which we will call the *roadmap* for I_α or $\Upsilon_{N,\alpha}$.

Our approach sometimes requires a shift of focus from gaps to gapless intervals of I_α , the union of which is called the *attractor* of I_α , denoted by A_α . The maximal

⁴We have $\Upsilon_{N,\alpha} := I_\alpha \times I_\alpha$ in case $N/\alpha - \alpha \in \mathbb{Z}$.

and $T_\alpha^2(\Delta_{d-2}) \setminus (\Delta_{d-2} \cup [\beta_1, p_{d-1}] \cup \Delta_d) = (p_d, \beta_2]$, the image of which is $[\beta_3, \beta)$. Second, $f_d \leq \beta_2 \leq p_d$, in which case $T_\alpha^2(\Delta_{d-2}) \setminus (\Delta_{d-2} \cup [\beta_1, p_{d-1}] \cup \Delta_d) = \emptyset$. If this is the case, all orbits except the fixed points pass through $[\beta_1, p_{d-1}]$ and Δ_d . Third, $\beta_2 < f_d$. Then $T_\alpha^2(\Delta_{d-2}) \setminus (\Delta_{d-2} \cup [\beta_1, p_{d-1}]) = [\alpha, \beta_2]$.

We conclude that all orbits except the fixed points will eventually pass through $\Delta_d, \Delta_{d-2}, [\alpha, \alpha_3], [\alpha_2, \beta], [\beta_3, \beta)$ and $[\alpha, \beta_2]$. The establishment of the actual attractor A_α will then depend on the position of the images of these last four intervals, all having points in the orbits of images of α and β as endpoints. These intervals will expand under T_α up to the point that an image contains a discontinuity point $p \in \{p_d, p_{d-1}\}$. Since we just saw that the image of an interval with p as left endpoint has β as right endpoint and an interval with p as right endpoint has α as left endpoint, all orbits except the fixed points will eventually only be in intervals with images of α and β as endpoints. This finishes the proof of Theorem 5. \square

When studying gaps and attractors, we are mainly interested in the *characteristic part* of continued fraction expansions and their orbits, defined as follows.

Definition 3. Let $x = [d_1, d_2, d_3, \dots]_{N,\alpha}$ be the N -expansion of x . The *characteristic part* of this expansion, CPE in short, is $x_n = T_\alpha^n(x) = [d_{n+1}, d_{n+2}, \dots]_{N,\alpha}$, where $n \in \mathbb{N} \cup \{0\}$ is the smallest number such that $x_n \in A_\alpha$.

Investigating gaps and attractors involves determining not only the orbits of α and β , but also comparing them with certain purely periodic orbits. The reason for this is that if one of the points of a periodic orbit is in a gap, all of its points are. In view of the previous theorems on gaplessness, we can confine our investigation of gaps to arrangements with two and three cylinders. For these arrangements we will find a formula for the number of gaps $G(N, \alpha)$. In this paper we will confine ourselves to the cases where $\alpha = \alpha_{\max}$. In a forthcoming paper, we will consider arrangements with two cylinders where $\alpha < \alpha_{\max}$.

2. Gaps in I_α when $\alpha = \alpha_{\max}$

Throughout the rest of this paper, we take $\alpha = \sqrt{N} - 1$. Applying Theorems 1 and 2, it is easily found that all arrangements for $N \in \{2, 3, 4, 5, 6, 7, 8\}$ are gapless. Nonetheless, our approach involves that we consider the cases $N \in \{4, 5, 6, 7, 8\}$ as well. Note that in case $N = 4$, we have a full arrangement. For $N \geq 5$, we have

$$d = d_{\max} = \left\lfloor \frac{N}{\sqrt{N} - 1} - (\sqrt{N} - 1) \right\rfloor = \left\lfloor \frac{2\sqrt{N} - 1}{\sqrt{N} - 1} \right\rfloor = 2$$

and

$$d_{\min} = \left\lfloor \frac{N}{\sqrt{N}} - (\sqrt{N} - 1) \right\rfloor = 1.$$

This means that we are solely dealing with arrangements $I_\alpha = \Delta_2 \cup \Delta_1$, the right one of which is full, yielding $\beta_1 = \alpha$. Therefore, referring to the proof of Theorem 5, we know that, β excluded, the endpoints of all intervals contained in A_α are images of α only.

The main goal of this paper is proving the following theorem, stating that in arrangements where $\alpha = \alpha_{\max}$, the number of gaps is an unbounded, stepwise, monotonically non-decreasing function of N .

Theorem 6. *Let $N \in \mathbb{N}_{\geq 4}$. Set $\alpha := \sqrt{N} - 1$ and $\alpha_i := T_\alpha^i(\alpha)$, for $i \geq 1$. Define $a := 2, 1, 1$ and $b := 2, 2$ as strings of partial quotients so as to recursively define the N -expansions c_n as follows. First, $c_0 := [\overline{1, 2}]$, $c_1 := [\overline{1}]$, $c_2 := [\overline{2}]$, $c_3 := [\overline{a}]$ and $c_4 := [\overline{a, b}]$. For $n \geq 5$, if $c_{n-1} = [\overline{j, k}]$, then $c_n = [\overline{j, k, j, \ell}]$, with j a string of partial quotients and $\{k, \ell\} = \{a, b\}$. Let $m \in \mathbb{N}$ be such that*

$$\alpha_{|c_i|} < \alpha_{2^{i-1}} \text{ for } i \in \{2, \dots, m\} \text{ and } \alpha_{|c_{m+1}|} > \alpha_{2^m}.$$

Let \mathcal{G}_N be the collection of gaps of I_α . Then every gap $G \in \mathcal{G}_N$ contains exactly one point from one of the orbits in $\bigcup_{i=1}^m \{c_i\}$. Conversely, each of the points of the orbits⁵ in $\bigcup_{i=1}^m \{c_i\}$ is contained in some (unique) $G \in \mathcal{G}_N$. The number of gaps $G(N) = |\mathcal{G}_N|$ is therefore $\sum_{i=1}^m |c_i|$, which can be expressed more explicitly as

$$G(N) = \begin{cases} \frac{2^{m+1} - 1}{3}, & \text{when } m \text{ is odd;} \\ \frac{2^{m+1} - 2}{3}, & \text{when } m \text{ is even.} \end{cases}$$

In particular, G is a finite, monotonically non-decreasing and unbounded function of N .

Our approach of the proof of Theorem 6 is based on extending a property of fixed points to periodic points, or rather considering the fixed points as special cases of periodic points. This property is that if an interval $J \subset I_\alpha$ contains a periodic point, the interval expands along its orbit. We will use this for a classification of attractors according to the number of disjoint intervals they consist of. We define

$$C_0 := \{4\} \text{ and } C_n := \{N \in \mathbb{N}_{\geq 5} \mid c_i \notin A_\alpha \text{ for } 0 \leq i < n, \text{ and } c_n \in A_\alpha\}. \quad (2)$$

We claim that for each class C_n there is a unique periodic point c_n such that all gaps of I_α contain exactly one orbit point of $\bigcup_{i=1}^{n-1} c_i$. We call this point c_n the *characteristic point* of C_n . For the orbit points of c_n , we will use the notation $c'_n = T_\alpha(c_n), \dots, c'''_n = T_\alpha^3(c_n)$ and $c_n^{(i)} = T_\alpha^{(i)}(c_n)$ for $i \geq 4$. We will shortly introduce some equations and inequalities as tools for our proof. Then we will

⁵Note that for $N = 2, \dots, 8$ the sets \mathcal{G}_N and $\bigcup_{i=1}^m \{c_i\}$ are empty.

explain our approach for the determination of the attractors for $4 \leq N \leq 532$, divided into five classes, corresponding with the following values: $N = 4$; $5 \leq N \leq 8$; $9 \leq N \leq 32$; $33 \leq N \leq 132$ and $133 \leq N \leq 532$, with characteristic points c_0 through c_4 , respectively. Finally, we will generalize our approach and prove Theorem 6 by induction.

Since the composition of the attractors depends completely on the expansiveness of T_α , and all expansions take place in Δ_1 or Δ_2 , we provide some useful tools for estimating expansion factors of intervals. As a start, let interval $M \subset I_\alpha$ be any interval not containing p_2 , with endpoints j and k , where $j < k$, or having p_2 as an endpoint. Then the expansion factor $g(M)$ under one iteration of T_α is the geometric mean of $|T'_\alpha(j)|$ and $|T'_\alpha(k)|$, which we can write as follows⁶:

$$|T_\alpha(M)| = \sqrt{|T'_\alpha(j)| \cdot |T'_\alpha(k)|} \cdot |M| = \frac{k-j}{kj} N = \frac{N}{jk} \cdot |M|,$$

so that

$$g(j, k) = \frac{N}{jk}. \tag{3}$$

For $M \subset [f_2, p_2]$, we define $g_\ell(M)$ as the multiplication factor under *two* iterations of T_α . We find:

$$g_\ell(M) = \frac{N^2}{(N-2j)(N-2k)}. \tag{4}$$

Taking j and k as small as possible and as large as possible, respectively, i.e., taking limits in Equation (4) of both j and k to f_2 and p_2 , respectively, we find a lower and an upper bound for g_ℓ .

$$\frac{(\sqrt{N+1}+1)^4}{N^2} < g_\ell(M) < \frac{(\sqrt{N}+1)^4}{(N-1)^2}. \tag{5}$$

For $M \subset (p_2, f_1]$, we define $g_2(M)$ as the multiplication factor under *two* iterations of T_α . We find:

$$g_2(M) = \frac{N^2}{(N-j)(N-k)}. \tag{6}$$

Taking j and k as small as possible and as large as possible, respectively, in Equation (6), i.e., taking limits to p_2 and f_1 , respectively, we find a lower and an upper bound for g_2 .

$$\left(1 + \frac{1}{\sqrt{N}}\right)^2 < g_2(M) < \left(1 + \frac{1}{\sqrt{N}} + \frac{1}{N}\right)^2. \tag{7}$$

For $M \subset (p_2, f_1]$, we define $g_r(M)$ as the multiplication factor under *three* iterations of T_α , provided that $T_\alpha^2(M) \subset \Delta_2$, i.e., $k \leq N/(N/(p_2+1)+1) = (N^2 + N\sqrt{N} +$

⁶This formula is not limited to $\alpha = \alpha_{\max}$.

$N)/(N\sqrt{N} + 2N + \sqrt{N} + 1)$. We find:

$$g_r(M) = \frac{N^3}{((N + 1)j - N)((N + 1)k - N)}. \tag{8}$$

Taking j and k as large as possible and as small as possible, respectively, in Equation (8), i.e., taking limits to $N/(N/(p_2 + 1) + 1)$ and p_2 , respectively, we find a lower and an upper bound for g_r .

$$\frac{(\sqrt{N} + 1)^4}{N^2} < g_r(M) < \frac{(\sqrt{N} + 1)^4}{(N - 1)^2}. \tag{9}$$

Remark 1. Note that all bounds are decreasing functions of N , reflecting the decrease of expansiveness of T_α on I_α as N increases.

Remark 2. For N increasing, with $N \geq 33$, the gaps containing the fixed points increase and all disjoint maximal subsets of A_α are images of the maximal subset of A_α containing p_2 . As a consequence, the upper bounds approximate the actual factors generally (much) better than the lower bounds do.

Remark 3. Note that in the double inequalities (5), (7) and (9) only one side is optimal/sharp. The other bounds are chosen so as to ease computations, and are sufficient for this purpose; see also Remark 4. Moreover, the upper bound we use for g_ℓ is equal to the upper bound we use for g_r . Finally, the lower bound we use for g_r is actually $(T'_\alpha(p_2))^2$. The lower bound for g_ℓ is larger than that, but for our purposes it is useful to always use the lower bound of g_r only.

Remark 4. If for $x \in (p_2, f_1)$ such that $x_2 < p_2$ (implying $p_2 < x < N/(N/(p_2 + 1) + 1)$) we take $J_r = (p_2, x]$ and $J_\ell = [2p_2 - x, p_2]$ (implying $|J_\ell| = |J_r|$), we have

$$g_2(J_r) = \frac{N + \sqrt{N}}{N - x},$$

and since $g((j, k)) = N/(jk)$ (see Equation (3)), we find

$$g(J_\ell) = \frac{(\sqrt{N} + 1)^2}{2N - x(\sqrt{N} + 1)}. \tag{10}$$

It is not hard to show that $g(J_\ell) > g_2(J_r)$, where the difference between the two factors is comparatively small. With some intricate but otherwise basic computations⁷ it can be found that for $x = \alpha_1$ the factor $g(J_\ell)$ is then only

$$\frac{2(\sqrt{N} - 1)}{N^2 - N\sqrt{N} - 2\sqrt{N}} \cdot 100$$

percent larger than $g_2(J_r)$; for $N \geq 35$ this percentage is even smaller than 1.

⁷For almost all computations in this paper we made use of Wolfram's *Mathematica*.

We will now continue with the preparation of the proof of Theorem 6. First, we note that by definition of A_α , for $x \in I_\alpha$ and interval $J \subset A_\alpha$, we have $x \in A_\alpha$ if and only if there is an integer n – equal to 0 if $x \in J$ – such that $x \in T_\alpha^n(J)$. If J contains p_2 , say $J = J_\ell \cup J_r$, where p_2 is the right endpoint of J_ℓ and the left endpoint of J_r , the two subsets expand with different expansion factors. Note that both $T_\alpha(J_\ell)$ and $T_\alpha^2(J_r)$ have α as left endpoint, so expansion of J after multiple iterations of T_α depends on the expansion of the larger one of $T_\alpha(J_\ell)$ and $T_\alpha^2(J_r)$ or, equivalently, on the expansion of $J' = T_\alpha(J_\ell) \cup T_\alpha^2(J_r)$. Now let m be an integer with either $m = 0$ or $m > 0$ such that $p_2 \in T_\alpha^m(J')$ and $p_2 \notin T_\alpha^i(J')$ for $0 \leq i < m$. We define

$$E(J) = \frac{\max\{|T_\alpha^{m+1}(J_\ell)|, |T_\alpha^{m+2}(J_r)|\}}{\max\{|J_\ell|, |J_r|\}}.$$

Then $E(J) > 2$ is clearly a sufficient condition for J to expand after $m+1$ or $m+2$ iterations of T_α . In the rest of this paper we will always use J, J_ℓ, J_r, J' and m in the previous sense.

Remark 5. Although $E(J) > 2$ is not a very sharp condition for the growth of J under T_α , it is useful for the determination of the classes C_2, C_3 and C_4 (see (2)). Once we have done that, we are able to focus on the induction step of the proof of Theorem 6.

We will show that for all N in C_2, C_3 and C_4 , the characteristic points of the related arrangements are c_2, c_3 and c_4 , respectively. Although this approach (with minor adaptations) is also valid for C_0 and C_1 , we grant these classes only two observations. First, although all arrangements for $4 \leq N \leq 8$ are gapless, the fullness of the arrangement for $N = 4$ makes an important difference: the periodic points $c_0 = [1, 2]$ and $c'_0 = [2, 1]$ exist in I_α only if $N = 4$. Secondly, only for $4 \leq N \leq 8$ the periodic point c_1 is not contained in a gap. This is mainly why C_0 consists of $N = 4$ only and C_1 consists of $5 \leq N \leq 8$.

For all $N \geq 9$ the interval (α_1, α_2) is a gap; see Figure 3. As a consequence, the attractor A_α consists of two components. Moreover, in CPEs the sequence 2, 1 is always followed by the digit 1, implying that, apart from $c_2 = f_2$ (with period length 1), periodic point $c_3 = [2, 1, 1]$ has the smallest period length.

Next assuming that $N \geq 9$, we investigate the expansion of open intervals $J \subset (I_\alpha \setminus (\alpha_1, \alpha_2))$. If $c_2 = [2] \in J_\ell$, there is an n such that $\Delta_2 \subset T_\alpha^n(J)$. Since $T_\alpha(\Delta_2) = \Delta_2 \cup (p_2, \alpha_1]$, we find that J expands to $I_\alpha \setminus (\alpha_1, \alpha_2)$. If $c_2 \notin J'$ and $p_2 \in T_\alpha(J')$, then $(p_2, \alpha_1] \subset T_\alpha(J')$. Since $T_\alpha((p_2, \alpha_1]) = [\alpha_2, \beta)$ and $T_\alpha([\alpha_2, \beta)) = (\alpha, \alpha_3]$, we find that c_2 is contained in an image of J only if $\alpha_3 \geq c_2$, which is equivalent to $\alpha_3 \geq \alpha_4$. Disregarding that $N \in \mathbb{N}$, we solve $\alpha_3 = \alpha_4$ and find $N = 32.9777\dots$. Note that in this case, we would have $\alpha_3 = \alpha_4 = c_2 = f_2$, so we could have also solved $\alpha_3 = f_2$. If $p_2 \notin T_\alpha(J')$, then $m \geq 3$, yielding $E(J) > (1 + 1/\sqrt{N})^6$, which is larger than 2 for $N \leq 66$ and amply larger than 2 for $N = 32$. We conclude that

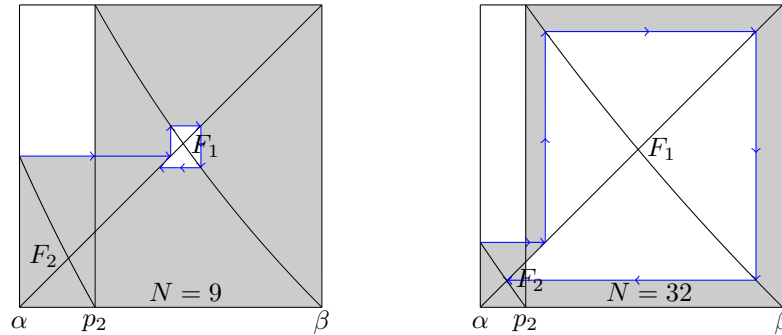


Figure 3: Arrangements for $N = 9$ and $N = 32$

any interval $J \subset (I_\alpha \setminus (\alpha_1, \alpha_2))$ expands to containing c_2 for $9 \leq N \leq 32$ and that for $N \geq 33$ there is a gap (α_3, α_4) , so $C_2 = \{9, \dots, 32\}$.

Remark 6. It is not hard to find that the length of the gap (α_1, α_2) , existing for all $N \geq 9$, is

$$1 - \frac{N}{N\sqrt{N} - 3N + 4\sqrt{N} - 2}.$$

This is an increasing function of N , with limit 1 if $N \rightarrow \infty$, implying $|\Delta_2| \rightarrow 0$ if $N \rightarrow \infty$. As a consequence, all other gaps will eventually become infinitely small. In particular, the length of the gap (α_3, α_4) , existing for all $N \geq 33$, is given by

$$\frac{N^3 - 8N^2\sqrt{N} + 16N^2 - 20N\sqrt{N} + 16N - 8\sqrt{N} + 4}{N^3\sqrt{N} - 5N^3 + 12N^2\sqrt{N} - 18N^2 + 20N\sqrt{N} - 16N + 8\sqrt{N} - 4},$$

which has a maximum of approximately 0.0566 for $N = 104$, after which it decreases to 0 for $N \rightarrow \infty$.

In the following, the next straightforward results will become increasingly useful.

Lemma 1. Let $x = [d_1, \dots, d_k, 1, \dots]$ and $x' = [d_1, \dots, d_k, 2, \dots]$. Then $x < x'$ if and only if k is odd.

The proof of Lemma 1 follows immediately from the inequality $|x - x'| < 1$ and the equivalence of $a < b$ and $c/a > c/b$ for positive numbers a, b and c . From this lemma the next two corollaries follow almost immediately.

Corollary 1. Let $c_0, c_1, c_2 \dots$ be the sequence of periodic points as defined in Theorem 6. Then $k > \ell \geq 1$ if and only if $c_k < c_\ell$.

Corollary 2. Let c_n be a periodic point as defined in Theorem 6. Then for $n \geq 1$ the number c_n is the smallest of all points in the orbit of c_n .

As of $N = 33$, we have (α_1, α_2) and (α_3, α_4) as gaps. As a consequence, for $N \geq 33$ each attractor has a component containing p_2 that is bounded on both sides by a gap; we will denote this interval by P_2 . For C_3 – yet to be determined – the attractor A_α consists of three components, each one containing one of the points in the orbit of the characteristic point $c_3 = [2, 1, 1]$. We will see that this is typical for odd n , whereas for even n the rightmost component does not contain an orbit point of c_n . The class $C_2 = \{9, \dots, 32\}$, for example, has two components, but the single orbit point $c_2 = f_2$ is in the left-hand component only.

As N increases, the maximum number of consecutive digits 2 in CPEs becomes more limited. It decreases from 21, for $N = 33$, to 3 for $N \geq 69$. We note that for $N \geq 69$, we have $\alpha_6 > p_2$, implying that all CPEs consist of sequences 1, 1, 2 and 2, 2, while every sequence 1, 2, 2 is followed by 2, 1, 1. Some more observations are useful. In order to find the largest N in C_2 , we could have confined ourselves to checking for which N we have $f_2 = c_2 \in T_\alpha^2((p_2, \alpha_1]) = (\alpha_0, \alpha_3]$ and finding that this is the case for $N \leq 32$. Where we assumed J' (see page 9) to be such that $p_2 \notin T_\alpha(J')$, we found expansion factors larger than 2 involved for $N \leq 32$, excluding any gaps but the one containing c_2 . We will apply these considerations for $N \geq 33$ as follows: we take $J = [\alpha_4, c'_3]$ (similar to taking $J = (c_2, \alpha_1]$ in C_2) and find that $c_3 \in T_\alpha(J_\ell)$ if and only if $\alpha_5 \geq c_3$, which is equivalent to $\alpha_5 \geq \alpha_8$. Once more disregarding that N -expansions involve positive integers for N , we solve $\alpha_5 = c_3 = \alpha_8$ (with ‘solution’ $N = 132.876\dots$) and find that for $33 \leq N \leq 132$ we have $\alpha_5 > \alpha_8$, while for $N \geq 133$ we have $\alpha_5 < \alpha_8$; see Figure 4, where we have only drawn the bottom left parts of the arrangements for visual reasons.

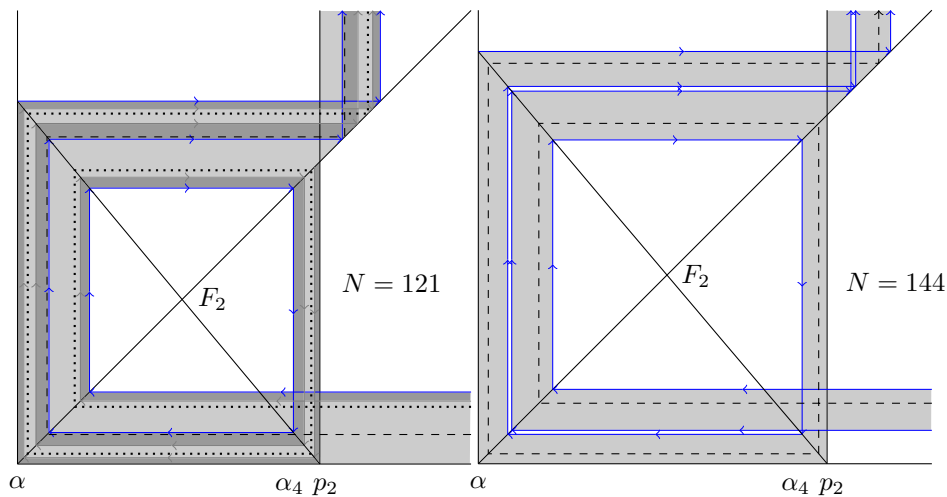


Figure 4: Two arrangements, one with $c_3 \in A_\alpha$, one with $c_3 \notin A_\alpha$

In the left arrangement of Figure 4 we have drawn the orbit of c_3 with dashed line segments and the orbit of c_4 with dotted line segments. The dark grey stripes illustrate the expansion of an interval $[x, p_2]$, with $c_4^{(4)} < x < p_2$. This is to visualize our approach as follows: if the expansion under T_α of an interval $K = [x, p_2] \subset J_\ell$ that contains $c_4^{(4)}$ is such that $J_\ell \subset T_\alpha^n(K)$ for some positive integer n , only for $33 \leq N \leq 132$ we have $c_3 \in T_\alpha(J_\ell)$. If $c_4 \notin K$, such as illustrated with the small dark grey stroke left of p_2 in the left arrangement of Figure 4, we rule out gaps other than $[\alpha_3, \alpha_4]$ and $[\alpha_1, \alpha_2]$ by computing $n \in \mathbb{N}$ such that $p_2 \in T_\alpha^n(K)$. We find $n \geq 2|c_4| = 10$, involving an expansion factor larger than $(1 + 1/\sqrt{N})^{16}$, which is amply larger than 2 for $33 \leq 132$. We conclude that $C_3 = \{33, \dots, 132\}$. Note that, for visual reasons, in Figure 4 only the images of K up to its seventh expansion are drawn.

Before we investigate the cases $N \geq 133$, we make some generalizing remarks. Only if $N = 4$, we have $\{c_0 = [2, 1], c'_0 = [1, 2]\} \cap I_\alpha \neq \emptyset$. This means that for $N \geq 5$ all CPEs have a limited number of consecutive sequences 1, 2. Then, as of $N = 9$, all CPEs also have a limited number of consecutive digits 1 and, as of $N = 33$, also a limited number of consecutive digits 2. It follows immediately that for $N \geq 33$ the periodic point in A_α with shortest period length is $c_3 = [2, 1, 1]$. We already noted that in CPEs with $N \geq 69$ digit 1 only occurs in pairs and each pair is followed by either one or three digits 2. If now, for $N \geq 133$, the number of consecutive sequences 2, 1, 1 is also limited, it follows that every CPE has a sequence 2, 1, 1, 2, 2. This implies that there is a set of $N \geq 133$ for which the periodic point with shortest period length is $c_4 = [2, 1, 1, 2, 2]$; in the right arrangement of Figure 4, where $N = 144$, we have added the orbit of c_4 .

Assuming that $N \geq 133$, we know that $[\alpha_5, \alpha_8]$, $[\alpha_3, \alpha_4]$, $[\alpha_9, \alpha_6]$, $[\alpha_1, \alpha_2]$ and $[\alpha_7, \alpha_{10}]$ are gaps, which implies that all attractors have at least six components. To determine C_4 , we take $J = (c_4^{(4)}, \alpha_9]$ (similar to taking $J = (c_2, \alpha_1]$ in C_2 and $J = [\alpha_4, c'_3)$ in C_3) and find that $c_4 \in T_\alpha^2(J_r)$ if and only if $\alpha_{11} \geq c_4$, which is equivalent to $\alpha_{11} \geq \alpha_{16}$. Once more disregarding that N -expansions involve positive integers for N , we solve $\alpha_{11} = c_4 = \alpha_{16}$ (with ‘solution’ $N = 532.497\dots$) and find that for $133 \leq N \leq 532$ we have $\alpha_{11} > \alpha_{16}$, while for $N \geq 533$ we have $\alpha_{11} < \alpha_{16}$; see Figure 5. The existence of more than five gaps for $133 \leq N \leq 532$ is ruled out as follows: if an interval $K = (p_2, x] \subset J_r$ containing $c_5^{(9)}$ expands such that $J_r \subset T_\alpha^n(K)$ for some positive integer n , then $c_4 \in T_\alpha^2(J_r)$ only for $133 \leq N \leq 532$. If $c_5 \notin K$, then $p_2 \in T_\alpha^n(K)$ only if $n \geq 2|c_5| = 22$, involving an expansion factor larger than $(1 + 1/\sqrt{N})^{32}$, which is amply larger than 2 for $133 \leq 532$. We conclude that $C_4 = \{133, \dots, 532\}$. Note that in C_4 , the orbit of c_4 has one point in all components except the rightmost one, which is because $c_4^{(4)}$ is the pre-image of c_4 .

We will now commence the induction part of our proof, in which we will also show that for $n \in \mathbb{N}$ the orbit of c_n has one point in each component of C_n in case n is odd, and one point in each component except the rightmost one in case

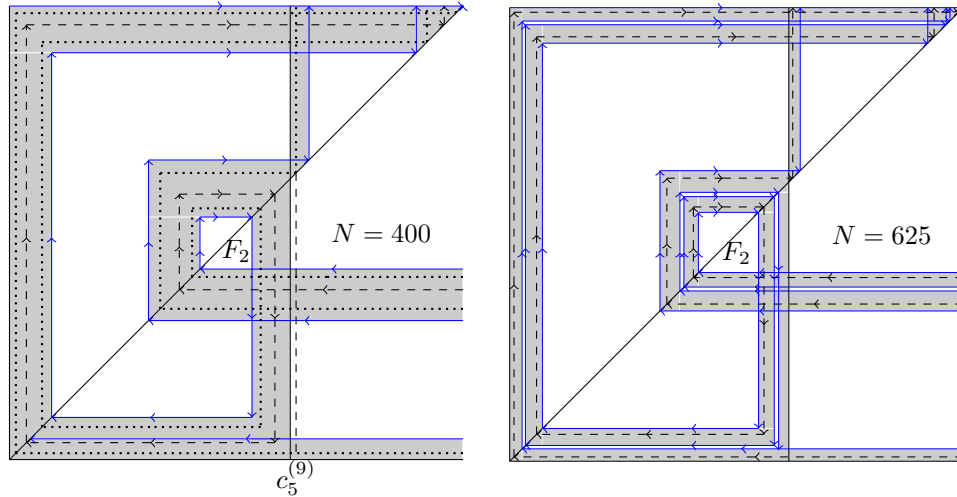


Figure 5: The arrangements, not to scale, for $N = 400$, where $\alpha_{11} > \alpha_{16}$, and $N = 625$, where $\alpha_{11} < \alpha_{16}$

n is even. First we observe that from Theorem 5 it follows that for $N \geq 33$ we have $P_2 = [\alpha_i, \alpha_j]$, for certain $i, j \in \mathbb{N}$. Moreover, P_2 is an image of the leftmost component K_ℓ of A_α , i.e., the component with α as left endpoint. Note that in terms of J, J_ℓ, J_r, J' (see page 9) we have $J = P_2, J_\ell = [\alpha_i, p_2], J_r = (p_2, \alpha_j]$ and $J' = K_\ell$. Let $x_0 \in K_\ell$ be in the orbit of a periodic point with the *smallest length possible*. Then there is an $n \in \mathbb{N}$ such that $x_0^* := T_\alpha^n(x_0) \in P_2$ and $T_\alpha^i(x_0) \notin P_2$ for $0 \leq i < n$. Then either $x_0^* \in [\alpha_i, p_2]$ or $x_0^* \in (p_2, \alpha_j]$. In the first case, we have $T_\alpha(x_0^*) \in K_\ell$ and $T_\alpha(x_0^*) = x_0$, since the orbit is the shortest possible; in the second case, we have $T_\alpha^2(x_0^*) = x_0$. Note that x_0 is the root of the equation $T_\alpha^{n+1}(x) = x$ or $T_\alpha^{n+2}(x) = x$, with $x \in K_\ell$. Solving these is complicated⁸, and we will avoid this by solving equations of the form $\alpha_n = \alpha_m$ instead, as we did for C_3 and C_4 . For $33 \leq N \leq 132$ we have $x_0 = c_3$ and $x_0^* = c'_3$, while for $133 \leq N \leq 532$ we have $x_0 = c_4$ and $x_0^* = c'_4$. We will show that x_0 , similarly defined for larger N , is a characteristic point for $N \geq 533$ as well.

So, let $N \geq 533$ and $C = \{N_1, \dots, N_k\}$ be the class of all N such that $P_2 =$

⁸Heuristically, the existence of these shortest periodic orbits for large N can be understood as follows: we could start with any x somewhere in the middle of the leftmost component K_ℓ . After n iterations of T_α the orbit of x has moved along with the images of K_ℓ . Depending on whether $T_\alpha^n(x) \in [\alpha_i, p_2]$ or $T_\alpha^n(x) \in (p_2, \alpha_j]$, we find that either $T_\alpha^{n+1}(x) \in K_\ell$ or $T_\alpha^{n+2}(x) \in K_\ell$. If neither is equal to x , we choose some x' slightly left or right from x to perform the same procedure so as to find that the image after $n + 1$ or $n + 2$ iterations of T_α on x' yields an image in K_ℓ closer to x' than the image of x in K_ℓ is to x . It will then be clear what digits should be used to draw up the equation $T_\alpha^{n+1}(x) = x$ or $T_\alpha^{n+2}(x) = x$, with $x \in K_\ell$.

$[\alpha_i, \alpha_j]$; referring to Remark 1, we can take $N_{i+1} = N_i + 1$ for $i \in \{1, \dots, k - 1\}$. Let $a := 2, 1, 1$ and $b := 2, 2$. Let s be a string of partial quotients beginning with a . First, suppose that $x_0 = \overline{[s, b]}$ is the periodic point with smallest possible period length in A_α . Then $x_0^* = x_0^{(|x_0|-1)} \in (\alpha_i, p_2)$, and x_0 is in the component of A_α with α as left endpoint. Let $C' = \{N_{k+1}, \dots, N_t\}$ be the class of all N such that each orbit point of x_0 is contained in a gap. Let $M = (x_0^*, p_2)$. Then

$$T_\alpha(M) = (\alpha, x_0), \dots, T_\alpha^{|x_0|}(M) = (\alpha_{|x_0|-1}, x_0^*), T_\alpha^{|x_0|+1}(M) = (x_0, \alpha_{|x_0|}) \dots,$$

where M expands along the orbit of x_0 . Note that $p_2 \in T_\alpha^{2|x_0|}(M)$ and $p_2 \notin T_\alpha^i(M)$ for $0 \leq i < 2|x_0|$. This means that two possible expansions exist for points in M : $[s, b, s, b, \dots]$ and $[s, b, s, a, \dots]$. For the periodic point with smallest possible period length in A'_α (the attractor in C') only $\overline{[s, b, s, a]}$ qualifies, since $\overline{[s, b, s, b]} = \overline{[s, b]} = x_0$ does not.

Next, suppose that $x_0 = \overline{[s, a]}$ is the periodic point with smallest possible period length in A_α . Then $x_0^* = x_0^{(|x_0|-2)} \in (p_2, \alpha_j)$, and x_0 is in the component of A_α with α as left endpoint. Let $C' = \{N_{k+1}, \dots, N_p\}$ be the class of all N such that each orbit point of x_0 is contained in a gap. Let $M = (p_2, x_0^*)$. Then

$$T_\alpha^2(M) = (\alpha, x_0), \dots, T_\alpha^{|x_0|}(M) = (x_0^*, \alpha_{|x_0|-2}), T_\alpha^{|x_0|+2}(M) = (\alpha_{|x_0|}, x_0) \dots,$$

where M expands along the orbit of x_0 . Note that $p_2 \in T_\alpha^{2|x_0|}(M)$ and $p_2 \notin T_\alpha^i(M)$ for $0 \leq i < 2|x_0|$. This means that two possible expansions exist for points in M : $[s, a, s, a, \dots]$ and $[s, a, s, b, \dots]$. For the periodic point with smallest possible period length in A'_α (the attractor in C') only $\overline{[s, a, s, b]}$ qualifies, since $\overline{[s, a, s, a]} = \overline{[j, a]} = x_0$ does not.

We have thus obtained the following lemma.

Lemma 2. *Let $N \geq 33$ and let C_n be the class with $P_2 = [\alpha_i, \alpha_j]$, where P_2 is the component containing p_2 . Let c_n^* be the orbit point of c_n in P_2 .*

If n is odd, then $c_n^ \in (p_2, \alpha_j]$, $c_{n+1}^* \in [\alpha_i, p_2)$ and $P_2 = [\alpha_i, \alpha_{i+|c_{n+1}|}]$ for $N \in C_{n+1}$.*

If n is even, then $c_n^ \in [\alpha_i, p_2]$, $c_{n+1}^* \in [p_2, \alpha_j)$ and $P_2 = [\alpha_{j+|c_{n+1}|}, \alpha_j]$ for $N \in C_{n+1}$.*

The only thing left to prove Theorem 6 is finding an explicit formula for $|c_n|$. Note that by definition, we have $|c_{n+1}| = 2|c_n| + (-1)^n$ for $n \in \mathbb{N}$, so

$$|c_n| = 2(2|c_{n-2}| + (-1)^{n-2}) + (-1)^{n-1} = 4|c_{n-2}| + (-1)^{n-2}.$$

Applying induction, we obtain the following.

Lemma 3. *Let c_n be a periodic point as defined in Theorem 6, with $n \geq 1$. If n is even, then $|c_n| = \frac{2^n - 1}{3}$; if n is odd, then $|c_n| = \frac{2^n + 1}{3}$.*

Proof. What follows is the final step of the proof of Theorem 6. First, there are no gaps in I_α for $4 \leq N \leq 8$. For $9 \leq N \leq 32$, there is only one gap in I_α , containing c_1 . This satisfies the formula for the number of gaps for odd m , $G(N) = \frac{2^{m+1}-1}{3}$, according to which $G(N) = \frac{2^2-1}{3} = 1$ if $9 \leq N \leq 32$. For $33 \leq N \leq 132$, there are two gaps in I_α , one containing c_1 , the other c_2 . This satisfies the formula for the number of gaps in I_α for even m , $G(N) = \frac{2^{m+1}-2}{3}$, according to which $G(N) = \frac{2^3-2}{3} = 2$ if $33 \leq N \leq 132$. Finally, for $133 \leq N \leq 532$, there are five gaps in I_α , one containing c_1 , one containing c_2 , and the other three gaps each containing one point of the orbit of $c_3 = [2, 1, 1]$. This satisfies the formula for the number of gaps in I_α for odd m , according to which $G(N) = \frac{2^4-1}{3} = 5$ for $133 \leq N \leq 532$.

As for the induction step for $N \geq 533$, we note that the specific composition of CPEs for $N \geq 69$ with nothing but blocks of sequences 2, 2 and 2, 1, 1 allows us to apply the inequalities (5) and (9). Regarding Remark 1, we know that for $N \in C_{m+1}$ the expansiveness of T_α is such that P_2 has no images under T_α containing c_i^* for $3 \leq i \leq m$. Since the difference between the number of components and the number of gaps equals 1, we see that for the number of gaps G we have $G(N) = |c_{m+1}|$ if m is odd, since in that case every component of A_α except the rightmost one contains a point in the orbit of c_{m+1} ; if m is even, we have $G(N) = |c_{m+1}| - 1$, since in that case every component of A_α contains a point in the orbit of c_{m+1} . Applying Lemma 3, we indeed find the formula for $G(N)$:

$$G(N) = \begin{cases} \frac{2^{m+1} - 1}{3}, & \text{when } m \text{ is odd;} \\ \frac{2^{m+1} - 2}{3}, & \text{when } m \text{ is even.} \end{cases}$$

□

As we noted before (see Remark 2), the function $(\sqrt{N} + 1)^4 / (N - 1)^2$ provides a good indication of the expansion factors of iterations of T_α related to the sequences 2, 2 and 2, 1, 1. As to this approximating factor we note that each class C_{n+1} involves a factor twice as large as C_n . While for all N in $C_2 = \{9, \dots, 32\}$ and for $N = 33$ as well we have $(\sqrt{N} + 1)^4 / (N - 1)^2 > 2$, for all N in $C_3 \setminus \{33\} = \{34, \dots, 132\}$ and for $N = 133$ as well we have $((\sqrt{N} + 1)^4 / (N - 1)^2)^2 > 2$, and for all N in $C_4 \setminus \{133\} = \{134, \dots, 532\}$ and for $N = 533$ as well we have $((\sqrt{N} + 1)^4 / (N - 1)^2)^4 > 2$.

It seems that the equation

$$\left(\frac{(\sqrt{N} + 1)^4}{(N - 1)^2} \right)^{2^{n-2}} = 2, \text{ implying } N = N(n) := \left(\frac{2^{\frac{1}{2^{2n-2+1}} + 1}}{2^{\frac{1}{2^{2n-2+1}} - 1}} \right)^2, \text{ with } n \in \mathbb{N}_{\geq 2},$$

provides very good indications for the boundary values of N between two classes C_n and C_{n+1} - once more disregarding that $N \in \mathbb{N}$. In Table 1 the number of gaps in

I_α for $4 \leq N \leq 34\,919\,576$ is shown. Indeed, it is striking how well $N(n)$ approaches the maximum value for each class. We also note that the ratio $|C_{n+1}|/|C_n|$ very rapidly approaches 4 as n increases. It is $4.0012515\dots$ for $n = 5$, for example, already 4 for $n \in \{3, 6, 7, 10\}$, 3.9999804 for $n = 8$ and $4.0000004\dots$ for $n = 11$.

n	$ C_n $	up to $N = \dots$	$\lfloor N(n) \rfloor$	# gaps	condition
1	4	8	not defined	0	$\alpha_1 > \alpha_2$
2	24	32	33	1	$\alpha_3 > \alpha_4$
3	100	132	133	2	$\alpha_5 > \alpha_8$
4	400	532	533	5	$\alpha_{11} > \alpha_{16}$
5	1598	2 130	2 131	10	$\alpha_{21} > \alpha_{32}$
6	6394	8 524	8 525	21	$\alpha_{43} > \alpha_{64}$
7	25 576	34 100	34 101	42	$\alpha_{85} > \alpha_{128}$
8	102 304	136 404	136 405	85	$\alpha_{171} > \alpha_{256}$
9	409 214	545 618	545 619	170	$\alpha_{341} > \alpha_{512}$
10	1 636 855	2 182 473	2 182 474	341	$\alpha_{683} > \alpha_{1\,024}$
11	6 547 420	8 729 893	8.729894	682	$\alpha_{1\,365} > \alpha_{2\,048}$
12	26 189 683	34 919 576	34 919 577	1 365	$\alpha_{2\,731} > \alpha_{4\,096}$
≥ 13	not computed	not computed	139 678 308	$\geq 2\,730$	$\alpha_{2\,731} < \alpha_{4\,096}$

Table 1: The relation between the number of gaps and N

Above, in Remark 6, we observed that as N tends to infinity, the ‘first’ gap (i.e., (α_1, α_2)) tends to ‘swallow up’ I_α . At the same time Theorem 6 states that the number of gaps grows also to infinity as N tends to infinity. So we have the remarkable fact that although the length of $I_\alpha \setminus (\alpha_1, \alpha_2)$ tends to zero as $N \rightarrow \infty$, it will be punctuated by an increasing number of ever smaller gaps.

Acknowledgement. We are very grateful to Pjotr Buys for making very useful simulations that give insight in the distribution of gaps in I_α for $\alpha \in (0, \sqrt{N} - 1]$. These will be especially useful in a forthcoming paper about gaps in I_α for $\alpha < \alpha_{\max}$.

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