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ON THE NONINTEGRALITY OF CERTAIN GENERALIZED BINOMIAL SUMS

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Abstract

We consider certain generalized binomial sums $\mathcal{S}_{(r,n)}(\ell)$ and discuss the nonintegrality of their values for integral parameters $n, r \geq 1$ and $\ell \in \mathbb{Z}$ in several cases using p-adic methods. In particular, we show some properties of the denominator of $\mathcal{S}_{(r,n)}(\ell)$. Viewed as polynomials, the sequence $(\mathcal{S}_{(r,n)}(x))_{n \geq 0}$ forms an Appell sequence. The special case $\mathcal{S}_{(r,n)}(2)$ reduces to the sum $\sum_{k=0}^{n} \binom{n}{k} \frac{r}{r+k}$, which has recently received some attention from several authors regarding the conjectured nonintegrality of its values. So far, only a few cases have been proved. The generalized results imply, among other things, for even $|\ell| \geq 2$ that $\mathcal{S}_{(r,n)}(\ell) \notin \mathbb{Z}$ when $\binom{r+n}{r}$ is even, e.g., r and n are odd. Although there exist exceptions where $\mathcal{S}_{(r,n)}(\ell) \in \mathbb{Z}$, "almost all" values of $\mathcal{S}_{(r,n)}(\ell)$ for $n, r \geq 1$ are nonintegral for any fixed $|\ell| \geq 2$. Subsequently, we also derive explicit inequalities between the parameters for which $\mathcal{S}_{(r,n)}(\ell) \notin \mathbb{Z}$. Especially, this is shown for certain small values of ℓ for $r \geq n$ and $n > r \geq \frac{1}{5}n$. As a supplement, we finally discuss exceptional cases where $\mathcal{S}_{(r,n)}(\ell) \in \mathbb{Z}$.

> – *If you cannot solve a problem, then try*
> *to solve a more general problem.*
> Pólya [16]

1. Introduction

Define the monic polynomial

$$\mathcal{S}_{(r,n)}(x) = \sum_{k=0}^{n} \binom{n}{k} (-1)^k \, x^{n-k} \binom{r+k}{r}^{-1} \in \mathbb{Q}[x] \tag{1.1}$$

of degree n for integers $n, r \geq 0$. Trivial cases are given by

$$\mathcal{S}_{(0,n)}(x) = (x-1)^n \quad \text{and} \quad \mathcal{S}_{(r,0)}(x) = 1. \tag{1.2}$$

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Therefore, we assume that $n, r \geq 1$ for the rest of the paper.

As we shall see later, the polynomial (1.1) can be expressed in several different ways that lead to various properties. As a surprising relation, we have

$$\mathcal{S}_{(r,n)}(2) = \sum_{k=0}^{n} \binom{n}{k} \frac{r}{r+k}. \tag{1.3}$$

The above sum has received some attention in recent times, where it is conjectured that (1.3) only takes nonintegral values. This has been shown for $1 \leq r \leq 22$ and for $1 \leq n < r$ in that case. See [2, 9, 11–13, 24] for the history and results.

Interestingly, the following sum, related to (1.3) with alternating signs,

$$\mathcal{S}_{(r,n)}(0) = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \frac{r}{r+k} \notin \mathbb{Z},$$

can be evaluated instantly, since it can be interpreted as a finite difference as well as a partial fraction decomposition (see Corollary 2.2 and Section 5).

However, a generalized conjecture of (1.3) cannot be established without further study, since there are several exceptions where in fact $\mathcal{S}_{(r,n)}(\ell) \in \mathbb{Z}$ for certain $\ell \in \mathbb{Z}$ as listed in the two tables below. See Section 7 for more results.

Parameters (r, n, ℓ)					
$(2,4,-49)$	$(2,4,-34)$	$(2,4,-19)$	$(2,4,-4)$	$(2,4,11)$	$(2,4,26)$
$(2,4,41)$	$(2,8,-17)$	$(2,8,28)$	$(2,12,-38)$	$(2,16,-16)$	$(2,16,35)$
$(2,20,-5)$	$(2,40,-40)$	$(3,7,-17)$	$(3,7,43)$	$(4,6,43)$	$(4,24,46)$

Table 1.1: Exceptions where $\mathcal{S}_{(r,n)}(\ell) \in \mathbb{Z}$ in the range $1 \leq |\ell|, n, r \leq 50$.

One observes that the exceptions in Table 1.1 have the property that $r < n$ and ℓ is relatively small. In contrast, Table 1.2 shows exceptions of the opposite case $n \leq r$, which reveals that the least positive ℓ can be arbitrarily large.

Parameters (r, n, ℓ)				
$(9,6,1002)$	$(10,5,2003)$	$(12,8,50\,389)$	$(12,9,41\,991)$	$(16,12,4\,345\,966)$

Table 1.2: Exceptions where $\mathcal{S}_{(r,n)}(\ell) \in \mathbb{Z}$ with $1 \leq n \leq r \leq 16$ and least positive ℓ.

The purpose of the paper is to discuss the phenomenon of the nonintegrality of the sum $\mathcal{S}_{(r,n)}(\ell)$ in spite of exceptions and to derive explicit conditions for its parameters. Indeed, the motivation for the generalized results was induced by the above quotation of Pólya, since the sum (1.3) sheds no light on its behavior when viewed individually.

The paper is organized as follows. The next section presents some basic properties of the polynomial $\mathcal{S}_{(r,n)}(x)$ and its values, while Section 3 contains the main results. Subsequently, Section 4 is devoted to preliminaries and some known results in number theory. Sections 5 and 6 contain the proofs of the theorems. The last section discusses the case of exceptions.

2. Basic Properties

Let $(n)_k$ denote the falling factorial such that $\binom{n}{k} = (n)_k/k!$. Let $\mathrm{denom}(\cdot)$ be the denominator of a rational polynomial or number. For properties of Appell polynomials, see [1,18,19]. The following theorem shows some basic properties of $\mathcal{S}_{(r,n)}(x)$.

Theorem 2.1. *Let $n, r \geq 1$. There are the following identities:*

(i) $$\mathcal{S}_{(r,n)}(x) = \sum_{k=0}^{n} \frac{(n)_k}{(r+k)_k} (-1)^k x^{n-k};$$

(ii) $$\mathcal{S}_{(r,n)}(x) = r \int_0^1 (x-t)^n (1-t)^{r-1} dt.$$

The polynomial $\mathcal{S}_{(r,n)}(x)$ is an Appell polynomial satisfying the equivalent relations

(iii) $$\mathcal{S}_{(r,n)}(x)' = n\, \mathcal{S}_{(r,n-1)}(x);$$

(iv) $$\mathcal{S}_{(r,n)}(x+y) = \sum_{k=0}^{n} \binom{n}{k} \mathcal{S}_{(r,k)}(x)\, y^{n-k}.$$

The denominator of $\mathcal{S}_{(r,n)}(x)$ and its values for $\ell \in \mathbb{Z}$ have the properties

(v) $$\mathrm{denom}\left(\mathcal{S}_{(r,n)}(x)\right) = \binom{r+n}{r};$$

(vi) $$\mathrm{denom}\left(\mathcal{S}_{(r,n)}(\ell)\right) \;\mid\; \binom{r+n}{r}.$$

Evaluating the integral formula and using the Appell properties of $\mathcal{S}_{(r,n)}(x)$ easily imply the following results.

Corollary 2.2. *Let $n, r \geq 1$. We have*

(i) $$\mathcal{S}_{(r,n)}(x+1) = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} \frac{r}{r+k}.$$

Special values are given as follows:

(ii) $$\mathcal{S}_{(r,n)}(0) = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \frac{r}{r+k} = (-1)^n \binom{r+n}{r}^{-1} \notin \mathbb{Z};$$

(iii) $$\mathcal{S}_{(r,n)}(1) = \frac{r}{r+n} \notin \mathbb{Z};$$

(iv) $$\mathcal{S}_{(r,n)}(2) = \sum_{k=0}^{n} \binom{n}{k} \frac{r}{r+k}.$$

Moreover, the values of $\mathcal{S}_{(r,n)}(x)$ *have the properties*

(v) $\mathcal{S}_{(r,n)}(x) > 0 \quad (x \geq 1)$ *and* $(-1)^n \mathcal{S}_{(r,n)}(x) > 0 \quad (x \leq 0)$.

It turns out that $\mathcal{S}_{(r,n)}(-1)$ is related to partial sums of binomial coefficients in a row of Pascal's triangle. So far, no closed forms are known for such sums according to [7, Sec. 5.1, pp. 165–167].

Theorem 2.3. *Let* $n, r \geq 1$*. There are the following identities:*

(i) $$\mathcal{S}_{(r,n)}(x) = \binom{r+n}{r}^{-1} \sum_{k=0}^{n} \binom{r+n}{k} (-1)^{n-k} x^k;$$

(ii) $$\mathcal{S}_{(r,n)}(-1) = (-1)^n \binom{r+n}{r}^{-1} \sum_{k=0}^{n} \binom{r+n}{k}.$$

(iii) We have the reciprocity relation

$$(-1)^n \mathcal{S}_{(r,n)}(-1) + (-1)^r \mathcal{S}_{(n,r)}(-1) = 2^{r+n} \binom{r+n}{r}^{-1} + 1.$$

(iv) At least one of the values of $\left\{\mathcal{S}_{(r,n)}(-1), \mathcal{S}_{(n,r)}(-1)\right\}$ *is not in* $\mathbb{Z}$*. In particular,*

$$\mathcal{S}_{(n,n)}(-1),\ \mathcal{S}_{(n+1,n)}(-1),\ \mathcal{S}_{(n,n+1)}(-1) \notin \mathbb{Z}.$$

In contrast, the related sum to $\mathcal{S}_{(r,n)}(-1)$ with alternating signs,

$$\mathcal{S}_{(r,n)}(1) = \binom{r+n}{r}^{-1} \sum_{k=0}^{n} \binom{r+n}{k} (-1)^{n-k} = \frac{r}{r+n},$$

is solvable at once by Corollary 2.2(iii).

Corollary 2.4. *Let* $n, r \geq 1$*. We have*

$$\begin{aligned} \sum_{k=0}^{n} \binom{r+n}{k} &= r \binom{r+n}{r} \int_0^1 (1+t)^n (1-t)^{r-1} dt \\ &= r \binom{r+n}{r} \sum_{k=0}^{n} \binom{n}{k} (-1)^k \frac{2^{n-k}}{r+k}. \end{aligned}$$

The reciprocity relation of Theorem 2.3 can be given in a generalized form, which then has a different shape. Define the reciprocal polynomial

$$\mathcal{S}^{\star}_{(r,n)}(x) = x^n \mathcal{S}_{(r,n)}(x^{-1}).$$

Theorem 2.5. *Let $n, r \geq 1$. We have the reciprocity relation*

$$\mathcal{S}_{(r,n)}(x) + x^n \mathcal{S}^{\star}_{(n,r)}(x) = (-1)^r (x-1)^{r+n} \binom{r+n}{r}^{-1} + x^n.$$

For the next applications, we need some recurrence formulas.

Proposition 2.6. *Let $n, r \geq 1$. There are the following recurrence formulas:*

(i) $$\mathcal{S}_{(r,n)}(x) = (x-1)\,\mathcal{S}_{(r,n-1)}(x) + \frac{r}{r+1}\,\mathcal{S}_{(r+1,n-1)}(x);$$

(ii) $$\mathcal{S}_{(r,n)}(x) = x^n - \frac{n}{r+1}\,\mathcal{S}_{(r+1,n-1)}(x);$$

(iii) $$\mathcal{S}_{(r+1,n)}(x) = \frac{r+1}{r+n+1}\left(x^{n+1} - (x-1)\,\mathcal{S}_{(r,n)}(x)\right);$$

(iv) $$\mathcal{S}_{(r,n+1)}(x) = \frac{r}{r+n+1} x^{n+1} + \frac{n+1}{r+n+1}(x-1)\,\mathcal{S}_{(r,n)}(x).$$

To tackle the problem of the nonintegrality and to obtain divisibility properties, it is convenient to find a further representation of $\mathcal{S}_{(r,n)}(x)$ as follows.

Theorem 2.7. *Let $n, r \geq 1$. We have*

$$\mathcal{S}_{(r,n)}(x) = (-1)^r r!\,\frac{(x-1)^{r+n} - x^{n+1}\,\psi_{(r,n)}(x)}{(n+1)\cdots(n+r)}, \tag{2.1}$$

where

$$\psi_{(r,n)}(x) = \sum_{k=0}^{r-1} \binom{n+k}{k} (-1)^k (x-1)^{r-1-k} \tag{2.2}$$

$$= \sum_{k=0}^{r-1} \binom{n+r}{k} (-1)^k x^{r-1-k}. \tag{2.3}$$

In particular, there are the special cases:

(i) $$\mathcal{S}_{(1,n)}(x) = \frac{x^{n+1} - (x-1)^{n+1}}{n+1};$$

(ii) $$\mathcal{S}_{(r,1)}(x) = x - \frac{1}{r+1};$$

(iii) $$\mathcal{S}_{(r,n)}(2) = (-1)^r r!\,\frac{1 - 2^{n+1}\sum_{k=0}^{r-1}(-1)^k\binom{n+k}{k}}{(n+1)\cdots(n+r)}.$$

3. Main Results

In this section, we derive several conditions on the nonintegrality of $\mathcal{S}_{(r,n)}(\ell)$. Let p always denote a prime. Let $\mathrm{ord}_p(n)$ and $s_p(n)$ be the p-adic valuation and the sum of base-p digits of n, respectively. The notation $p^e \parallel n$ means that $p^e \mid n$ but $p^{e+1} \nmid n$, i.e., $\mathrm{ord}_p(n) = e$. The following two results give conditions to test the (non-) integrality via congruences.

Proposition 3.1. *Let $n, r \geq 1$ and $\ell \in \mathbb{Z}$. Then $\mathcal{S}_{(r,n)}(\ell) \in \mathbb{Z}$ if and only if*

$$\sum_{k=0}^{n-1} \binom{r+n}{k} (-\ell)^k \equiv 0 \pmod{\binom{r+n}{r}}.$$

Proposition 3.2. *Let $n, r \geq 1$ and $\ell \in \mathbb{Z} \setminus \{0, 1\}$. If there exists an index $d \in \{1, \ldots, r\}$ where*

$$r!\,(\ell-1)^{r-d}\left((\ell-1)^{n+d} - \ell^{n+d}\right) \not\equiv 0 \pmod{n+d},$$

then $\mathcal{S}_{(r,n)}(\ell) \notin \mathbb{Z}$.

Regarding the properties of $\mathcal{S}_{(r,n)}(\ell)$, we have a kind of reciprocity relation between the parameters n and r, as well as a symmetry relation of ℓ.

Theorem 3.3. *Let $n, r \geq 1$ and $\ell \in \mathbb{Z}$. Set $g = \gcd\left(\binom{r+n}{r}, \ell\right)$ and $e_p = \mathrm{ord}_p\left(\binom{r+n}{r}\right)$. Assume that one of the following conditions holds:*

(i) $r = 1$ *or* $n = 1$;

(ii) $r = n$;

(iii) $r + n$ *is a prime power;*

(iv) r *and* n *are odd, and* ℓ *is even;*

(v) $\ell \in \{-1, 0, 1\}$;

(vi) $g \neq 1$.

Then we have that $\mathcal{S}_{(r,n)}(\pm\ell) \notin \mathbb{Z}$ and $\mathcal{S}_{(n,r)}(\pm\ell) \notin \mathbb{Z}$, except for the case when only condition (v) holds with $\ell = \pm 1$, where at least $\mathcal{S}_{(r,n)}(-1) \notin \mathbb{Z}$ or $\mathcal{S}_{(n,r)}(-1) \notin \mathbb{Z}$. Moreover, if $g \neq 1$, then we have for each prime divisor $p \mid g$ that

$$p^{e_p} \parallel \mathrm{denom}(\mathcal{S}_{(r,n)}(\pm\ell)) \quad \textit{and} \quad p^{e_p} \parallel \mathrm{denom}(\mathcal{S}_{(n,r)}(\pm\ell)).$$

The diagonal case $r = n$ can be handled in more detail as follows.

Theorem 3.4. *Let $n \geq 1$. We have*

$$\mathcal{S}_{(n,n)}(x) = \binom{2n}{n}^{-1} \sum_{k=0}^{n} \binom{2n}{k} (-1)^{n-k}\, x^k,$$

which obeys the recurrence

$$2\mathcal{S}_{(n+1,n+1)}(x) = \frac{n+1}{2n+1}(x-1)\left(x^{n+1} - (x-1)\mathcal{S}_{(n,n)}(x)\right) + x^{n+1}$$

with $\mathcal{S}_{(1,1)}(x) = x - \frac{1}{2}$*. For* $\ell \in \mathbb{Z}$*, we have* $\mathcal{S}_{(n,n)}(\ell) \notin \mathbb{Z}$*. More precisely,*

$$\operatorname{ord}_2(\operatorname{denom}(\mathcal{S}_{(n,n)}(\ell))) = \begin{cases} 1, & \text{if } \ell \text{ is odd;} \\ s_2(n), & \text{if } \ell \text{ is even.} \end{cases}$$

Since for any given positive integer ℓ, almost all binomial coefficients (in the sense of a density) are divisible by ℓ (this is due to Singmaster; see Theorem 4.3), this implies the following corollary of Theorem 3.3.

Corollary 3.5. *Define the following sets for* $m \geq 2$ *and* $\ell \in \mathbb{Z}$*:*

$$\mathcal{N}_m(\ell) = \left\{(r,n) \in \mathbb{Z}^2 : n, r \geq 1,\ 1 \leq r+n \leq m, \text{ and } \mathcal{S}_{(r,n)}(\ell) \notin \mathbb{Z}\right\}.$$

If $|\ell| \geq 2$*, then we have the density*

$$\lim_{m\to\infty} \#\mathcal{N}_m(\ell) \Big/ \binom{m}{2} = 1,$$

which implies that almost all values of $\mathcal{S}_{(r,n)}(\ell)$ *are nonintegral for* $n, r \geq 1$*.*

Using Theorem 3.3 for even $\ell \in \mathbb{Z} \setminus \{0\}$, we arrive at a particular case. If $\binom{r+n}{r}$ is even (e.g., r and n are odd), then $\mathcal{S}_{(r,n)}(\ell) \notin \mathbb{Z}$. This coincides with Pascal's triangle modulo 2, which is known as the Sierpiński gasket [21]. See Figure 3.1, where small black triangles represent the odd binomial coefficients, and the blanks represent the even ones.

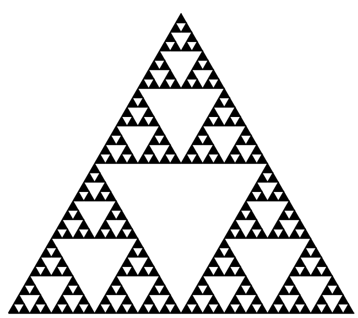

Figure 3.1: Sierpiński gasket.

At the end of this section, we consider the situation of inequalities between the parameters of $\mathcal{S}_{(r,n)}(\ell)$, supplementing the results of Theorem 3.3. We use several known results on primes in short intervals, which will be introduced in Section 4.

Theorem 3.6. *Let $n, r \geq 2$ and $\ell \in \mathbb{Z} \setminus \{0,1\}$. Set $g = \gcd\left(\binom{r+n}{r}, \ell - 1\right)$ and $\mathcal{P} = \left\{p : p > \frac{3}{2}n,\ p \mid \binom{r+n}{r},\ p \nmid \ell - 1\right\}$. We have $\mathcal{S}_{(r,n)}(\ell) \notin \mathbb{Z}$ if one of the following mutually exclusive conditions holds:*

(i) $n > r \geq \frac{1}{5}n$ where $n \geq |\ell - 1|$ (if $n \geq 89\,693$, then $r \geq \frac{1}{5}n$ can be improved by $r > n/\log^3 n$);

(ii) $r > n$ where $n \geq \frac{2}{3}|\ell - 1|$ or $g = 1$ or $\mathcal{P} \neq \emptyset$.

The exceptions $(r, n, \ell) \in \{(2, 4, -4), (2, 20, -5)\}$ of Table 1.1 show that the conditions of Theorem 3.6(i), namely, $n > r \geq \frac{1}{5}n$ and $n \geq |\ell - 1|$, are essentially needed. Regarding part (ii), the exceptions of Table 1.2 imply that the condition $r > n$ generally requires an additional condition on ℓ. The special case $r > n \geq 1$ for $\ell = 2$ was proved by López-Aguayo and Luca [12] for the sum (1.3). For small values $\ell \in \mathcal{L}$, where

$$\mathcal{L} = \{-3, -2, -1, 2, 3, 4, 5\},$$

we finally achieve the following result with simpler conditions.

Corollary 3.7. *Let $n, r \geq 1$ and $\ell \in \mathcal{L}$. If $r \geq n$ or $n > r \geq \frac{1}{5}n$, then $\mathcal{S}_{(r,n)}(\ell) \notin \mathbb{Z}$.*

If one could remove the above restriction $r \geq \frac{1}{5}n$, then this would prove the existing conjecture of the nonintegrality of $\mathcal{S}_{(r,n)}(\ell)$ for $\ell = 2$ as well as for other small values of ℓ. We may raise the extended conjecture as follows.

Conjecture 3.8. If $n, r \geq 1$ and $\ell \in \mathcal{L}$, then $\mathcal{S}_{(r,n)}(\ell) \notin \mathbb{Z}$.

We conclude with the following question.

Question. For which numbers $\ell \in \mathbb{Z} \setminus \{0,1\}$ does $\mathcal{S}_{(r,n)}(\ell)$ take only nonintegral values for all $n, r \geq 1$?

4. Preliminaries

Let $\mathbb{Z}_p$ be the ring of p-adic integers and $\mathbb{Q}_p$ be the field of p-adic numbers. Extend $\operatorname{ord}_p(s)$ as the p-adic valuation of $s \in \mathbb{Q}_p$. Let $\mathbb{F}_p$ be the finite field with p elements.

Applying Legendre's formula [10, pp. 8–10]

$$\operatorname{ord}_p(n!) = \frac{n - s_p(n)}{p - 1}$$

to binomial coefficients provides that

$$\operatorname{ord}_p\left(\binom{n}{k}\right) = \frac{s_p(k) + s_p(n-k) - s_p(n)}{p - 1}. \tag{4.1}$$

Lemma 4.1 ([17, Sec. 5.1, p. 37]). *If $n \geq 1$, then*

$$\operatorname{ord}_p\left(\sum_{\nu=0}^{n} x_\nu\right) \geq \min_{0 \leq \nu \leq n} \operatorname{ord}_p(x_\nu) \quad (x_\nu \in \mathbb{Q}_p),$$

where equality holds, if there exists an index m such that $\operatorname{ord}_p(x_m) < \operatorname{ord}_p(x_\nu)$ for all $\nu \neq m$.

Lemma 4.2. *If $n = p^e$ with p a prime and $e \geq 1$, then*

$$\operatorname{ord}_p\left(\binom{n}{k}\right) \geq 1 \quad (0 < k < n).$$

Proof. This follows from applying the Frobenius endomorphism in $\mathbb{F}_p$ iteratively such that

$$(x+y)^{p^\nu} = x^{p^\nu} + y^{p^\nu} \quad (\nu \geq 1).$$ □

Theorem 4.3 (Singmaster [22]). *Let $d, m \geq 1$. Define the sets*

$$B_m(d) = \left\{(j,k) \in \mathbb{Z}^2 : j,k \geq 0,\ 0 \leq j+k < m, \text{ and } d \mid \binom{j+k}{k}\right\}.$$

Then we have the density

$$\lim_{m \to \infty} \#B_m(d) \Big/ \binom{m+1}{2} = 1,$$

which implies that almost all binomial coefficients are divisible by d.

Theorem 4.4 (Faulkner [5]). *If $n \geq 2k \geq 2$, then $\binom{n}{k}$ has a prime divisor $p \geq \frac{7}{5}k$, where the factor $\frac{7}{5}$ is best possible.*

Theorem 4.5 (Hanson [8]). *If $n \geq 2k \geq 2$, then $\binom{n}{k}$ has a prime divisor $p > \frac{3}{2}k$, except for the cases $(n,k) \in \{(4,2),(9,2),(10,5)\}$.*

Theorems 4.4 and 4.5 are stronger versions of a theorem of Sylvester [23], independently discovered by Schur [20], which states that if $n \geq 2k \geq 2$ then $\binom{n}{k}$ has a prime divisor $p > k$. A simple proof was given by Erdős [4]. Considering the special case $\binom{2n}{n}$ for $n \geq 2$ implies Bertrand's postulate that there always exists a prime p with $n < p < 2n$. We need the following improvements.

Theorem 4.6 (Nagura [14]). *If $n \geq 25$, then there exists a prime p such that $n < p < \frac{6}{5}n$.*

Theorem 4.7 (Dusart [3]). *If $n \geq 89\,693$, then there exists a prime p such that $n < p < (1 + \log^{-3} n)\, n$.*

Lemma 4.8. *Let $n, r \geq 2$. Then there exists an odd prime p with $p \mid \binom{r+n}{n}$. In particular,*

$$2^{r+n} \Big/ \binom{r+n}{n} \notin \mathbb{Z}.$$

Proof. Since $\binom{r+n}{n} = \binom{r+n}{r}$ by symmetry, we can assume that $r \geq n$. Thus, we have $r + n \geq 2n$ and $n \geq 2$. By Theorem 4.4 there exists a prime $p \geq \frac{7}{5}n$, so $p \geq 3$, that divides $\binom{r+n}{n}$. $\square$

For a polynomial $f(x) \in \mathbb{Q}[x]$, its denominator $\mathrm{denom}(f(x))$ is the smallest positive integer d such that $d \cdot f(x) \in \mathbb{Z}[x]$, the latter polynomial having coprime coefficients. This definition includes the usual definition of $\mathrm{denom}(q)$ for $q \in \mathbb{Q}$. In particular, $\mathrm{denom}(q) = 1$ if and only if $q \in \mathbb{Z}$.

Lemma 4.9. *Let $n \geq 0$ and define the polynomial*

$$f(x) = \sum_{\nu=0}^{n} a_\nu \, x^\nu$$

with rational coefficients a_ν. Then

$$\mathrm{denom}(f(x)) = \mathrm{lcm}(\mathrm{denom}(a_0), \ldots, \mathrm{denom}(a_n)).$$

For $\ell \in \mathbb{Z}$, we have

$$\mathrm{denom}(f(\ell)) \mid \mathrm{denom}(f(x)).$$

Proof. This follows from $\mathrm{ord}_p(f(x)) = \min\limits_{0 \leq \nu \leq n} \mathrm{ord}_p(a_\nu)$ and $\mathrm{ord}_p(a_\nu \, \ell^\nu) \geq \mathrm{ord}_p(a_\nu)$ for any prime p. $\square$

Lemma 4.10. *Let $n, r \geq 1$. For $0 \leq k \leq n$, we have*

$$\mathrm{denom}\left(\frac{(n)_k}{(r+k)_k} \right) \mid \mathrm{denom}\left(\frac{(n)_n}{(r+n)_n} \right).$$

If $n + r = p^e$ with p a prime and $e \geq 1$, then we have for $0 \leq k < n$ the strict inequalities

$$\mathrm{ord}_p\left(\frac{(r+n)_n}{n!} \right) > \mathrm{ord}_p\left(\frac{(r+k)_k}{(n)_k} \right).$$

Proof. Let $k \in \{0, \ldots, n\}$. We then have

$$\frac{(r+n)_n}{n!} = \frac{(r+n)_{n-k}}{(n-k)!} \, \frac{(r+k)_k}{(n)_k} = \binom{r+n}{n-k} \frac{(r+k)_k}{(n)_k}.$$

For any prime p, we have $\binom{r+n}{n-k} \in \mathbb{Z}_p$. This shows that

$$\mathrm{ord}_p\left(\frac{(r+n)_n}{n!} \right) \geq \mathrm{ord}_p\left(\frac{(r+k)_k}{(n)_k} \right),$$

implying the first claim. Now, if $n+r=p^e$, then by Lemma 4.2 we have that $\binom{r+n}{n-k} \in p\mathbb{Z}_p$ for $0 \le k < n$, proving the second claim. $\square$

Lemma 4.11. *For $\ell \in \mathbb{Z}$ and $n \ge 2$, we have*

$$\frac{\ell^n - (\ell-1)^n}{n} \notin \mathbb{Z}.$$

Proof. The cases $\ell \in \{0,1\}$ are trivial. Define $f_n(\ell) = \ell^n - (\ell-1)^n$. It is easy to see that we have a reflection relation by

$$f_n(\ell) = (-1)^{n+1} f_n(1-\ell).$$

Thus, there remains to consider the integers $\ell \ge 2$. Now fix $\ell, n \ge 2$ and assume to the contrary that

$$\ell^n \equiv (\ell-1)^n \pmod{n}.$$

We have $g = \gcd(n, \ell(\ell-1)) = 1$. Otherwise, $p \mid g$ would imply a congruence of the type $0 \equiv (\pm 1)^n \pmod{p}$. Since $g=1$ and $2 \mid \ell(\ell-1)$, we have $2 \nmid n$. Next we choose the smallest prime divisor $p \ge 3$ of n. We then obtain $b \equiv \ell/(\ell-1) \not\equiv 1 \pmod{p}$ and arrive at $b^n \equiv 1 \pmod{p}$. By Fermat's little theorem, we have $b^e \equiv 1 \pmod{p}$ with a minimal exponent $e = (p-1)/d > 1$ and $d \mid p-1$. As a consequence, we infer that $e \mid n$, but this contradicts the assumption that p is the smallest prime divisor of n. $\square$

Remark. The special case $\ell = 2$ of Lemma 4.11 was handled in [11], but without giving a proof. Actually, a proof was given in the same issue as a solution to the initial problem of Chiriţă [2]. (The editors noted there that the fact that $n \nmid 2^n - 1$ for $n > 1$ goes back to a proposed problem in 1972.) A further proof of that case was also given later in [24].

Euler's beta function is defined by

$$\mathrm{B}(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt$$

for $\mathrm{Re}(x) > 0$ and $\mathrm{Re}(y) > 0$, which satisfies the identity

$$\mathrm{B}(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)},$$

where Γ is the gamma function.

5. Proofs of Basic Theorems

Lemma 5.1. *Let $n, r \geq 1$ and $x \in \mathbb{C}$. Then*

$$\mathcal{S}_{(r,n)}(x) = r \int_0^1 (x-t)^n (1-t)^{r-1} dt. \tag{5.1}$$

In particular, we have

$$\mathcal{S}_{(1,n)}(x) = \frac{x^{n+1} - (x-1)^{n+1}}{n+1}, \tag{5.2}$$

$$\mathcal{S}_{(r,1)}(x) = x - \frac{1}{r+1}, \tag{5.3}$$

and

$$\mathcal{S}_{(r,n)}(1) = \frac{r}{r+n}. \tag{5.4}$$

Proof. Using the beta function, we infer that

$$\int_0^1 (x-t)^n (1-t)^{r-1} dt = \sum_{k=0}^{n} \binom{n}{k} (-1)^k \, x^{n-k} \, \mathrm{B}(k+1, r).$$

Since $r \, \mathrm{B}(k+1,r) = 1/\binom{r+k}{r}$, this establishes (5.1) using (1.1). Direct evaluations provide that

$$\mathcal{S}_{(1,n)}(x) = \int_0^1 (x-t)^n dt = - \left. \frac{(x-t)^{n+1}}{n+1} \right|_0^1 = \frac{x^{n+1} - (x-1)^{n+1}}{n+1}$$

and

$$\mathcal{S}_{(r,n)}(1) = r \int_0^1 (1-t)^{r+n-1} dt = -r \left. \frac{(1-t)^{r+n}}{r+n} \right|_0^1 = \frac{r}{r+n}.$$

Formula (5.3) is given by (1.1) with $n = 1$. □

Proof of Theorem 2.1. We have to show six parts.

(i). This follows from (1.1) and using $\binom{n}{k} = \frac{(n)_k}{k!}$ and $\binom{r+k}{r} = \binom{r+k}{k} = \frac{(r+k)_k}{k!}$.

(ii). This is given by Lemma 5.1 and (5.1).

(iii), (iv). Differentiating (5.1) with respect to x yields $\mathcal{S}_{(r,n)}(x)' = n \, \mathcal{S}_{(r,n-1)}(x)$. Together with $\mathcal{S}_{(r,0)}(x) = 1$ by (1.2), the polynomials $\mathcal{S}_{(r,n)}(x)$ for $n \geq 0$ form an Appell sequence. As a consequence, part (iv) is equivalent to part (iii), see [1].

(v), (vi). We use part (i) and apply Lemmas 4.9 and 4.10. This shows part (v). Part (vi) follows from using Lemma 4.9 again. This proves the theorem. □

Proof of Corollary 2.2. The results are derived from Theorem 2.1(ii) and (iv). We show the claims in order of their dependencies.

(iii). Evaluating the integral (5.1), Lemma 5.1 gives the result by (5.4).

(i). It then follows that

$$\mathcal{S}_{(r,n)}(x+1) = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} \mathcal{S}_{(r,k)}(1).$$

(ii), (iv). Note that $\mathcal{S}_{(r,n)}(0) = (-1)^n \binom{r+n}{r}^{-1}$ by (1.1). The identities follow from taking $x = \pm 1$.

(v). We consider the integrand of (5.1). For $t \in (0,1)$, we have $(1-t)^{r-1} > 0$, as well as $(x-t)^n > 0$ for $x \geq 1$ and $(-1)^n (x-t)^n > 0$ for $x \leq 0$. $\square$

Remark. The partial fraction decomposition

$$\sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k}{x+k} = \frac{1}{x} \binom{x+n}{n}^{-1}$$

and its inversion

$$\frac{x}{x+n} = \sum_{k=0}^{n} \binom{n}{k} (-1)^k \binom{x+k}{k}^{-1}$$

are well known, cf. [7, Sec. 5.3, p. 196] and [15, § 4, p. 54]. Instead of using finite differences, the identities are derived here from the integral (5.1) and the Appell properties of $\mathcal{S}_{(r,n)}(x)$.

Proof of Theorem 2.3. Let $n, r \geq 1$. We have to show four parts.

(i), (ii). It is easy to verify that

$$\binom{n}{k} \binom{r+n-k}{r}^{-1} = \binom{r+n}{k} \binom{r+n}{n}^{-1}.$$

We reverse the summation of (1.1) and use the above identity. Thus,

$$\mathcal{S}_{(r,n)}(x) = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} x^k \binom{r+n-k}{r}^{-1} = \binom{r+n}{n}^{-1} \sum_{k=0}^{n} \binom{r+n}{k} (-1)^{n-k} x^k.$$

By taking $x = -1$, the formula for $\mathcal{S}_{(r,n)}(-1)$ follows.

(iii). Due to the symmetry of the binomial coefficients, we sum from the left-hand and right-hand side in a row of Pascal's triangle. Therefore, this yields

$$\sum_{k=0}^{n} \binom{r+n}{k} + \sum_{k=0}^{r} \binom{r+n}{k} = 2^{r+n} + \binom{r+n}{n}, \tag{5.5}$$

where $\binom{r+n}{n} = \binom{r+n}{r}$ is counted twice. Considering the sign and the extra factor of $\mathcal{S}_{(r,n)}(-1)$, we finally obtain the reciprocity relation

$$(-1)^n \mathcal{S}_{(r,n)}(-1) + (-1)^r \mathcal{S}_{(n,r)}(-1) = 2^{r+n} \binom{r+n}{n}^{-1} + 1. \tag{5.6}$$

(iv). First we have $\mathcal{S}_{(r,1)}(-1) \notin \mathbb{Z}$ by (5.3). Further we infer from applying Lemmas 4.11 and 5.1 that $\mathcal{S}_{(1,n)}(-1) \notin \mathbb{Z}$. Thus, we can now assume that $n, r \geq 2$. Lemma 4.8 shows that the right-hand side of (5.6) is not integral, implying that $\mathcal{S}_{(r,n)}(-1)$ and $\mathcal{S}_{(n,r)}(-1)$ cannot be both integers. As a consequence, $\mathcal{S}_{(n,n)}(-1) \notin \mathbb{Z}$ for $n \geq 1$. Further, direct computations via (5.5) and (5.6) imply that

$$(-1)^n \mathcal{S}_{(n+1,n)}(-1) = 2^{2n} \binom{2n+1}{n}^{-1} \notin \mathbb{Z}$$

and

$$(-1)^{n+1} \mathcal{S}_{(n,n+1)}(-1) = 2^{2n} \binom{2n+1}{n}^{-1} + 1 \notin \mathbb{Z},$$

using the same arguments from above. This proves the theorem. □

Proof of Corollary 2.4. The first equation follows from combining Theorem 2.1(ii) and Theorem 2.3(ii), and the second one from Corollary 2.2(i) with $x = -2$. □

Proof of Theorem 2.5. Let $n, r \geq 1$. We introduce the notation

$$(x+y)^{n,m} = \sum_{k=0}^{m} \binom{n}{k} x^{n-k} y^k \quad (0 \leq m \leq n)$$

for partial sums of the binomial identity, which is not commutative in general. It is easy to see that

$$(x+y)^{n,m} + (y+x)^{n,n-m} = (x+y)^n + \binom{n}{m} x^{n-m} y^m. \tag{5.7}$$

From Theorem 2.3(i), it then follows that

$$\begin{aligned} \mathcal{S}_{(r,n)}(x) &= (-1)^r \binom{r+n}{r}^{-1} (-1+x)^{r+n,n}, \\ x^r \mathcal{S}^{\star}_{(r,n)}(x) &= (-1)^n \binom{r+n}{r}^{-1} (x-1)^{r+n,n}. \end{aligned}$$

Using (5.7), we finally derive that

$$\mathcal{S}_{(r,n)}(x) + x^n \mathcal{S}^{\star}_{(n,r)}(x) = (-1)^r (x-1)^{r+n} \binom{r+n}{r}^{-1} + x^n. \qquad \square$$

Proof of Proposition 2.6. Let $n, r \geq 1$. We have to show four parts, where we make use of the integral formula (5.1).

(i). Rewriting the integrand by

$$\begin{aligned} (x-t)^n (1-t)^{r-1} &= (x-1+1-t)(x-t)^{n-1}(1-t)^{r-1} \\ &= (x-1)(x-t)^{n-1}(1-t)^{r-1} + (x-t)^{n-1}(1-t)^r \end{aligned}$$

implies the formula

$$\mathcal{S}_{(r,n)}(x) = (x-1)\,\mathcal{S}_{(r,n-1)}(x) + \frac{r}{r+1}\,\mathcal{S}_{(r+1,n-1)}(x). \tag{5.8}$$

(ii). We use the integration by parts formula

$$f(t)g(t)\Big|_0^1 = \int_0^1 f(t)g'(t)dt + \int_0^1 f'(t)g(t)dt.$$

Set $f(t) = -(x-t)^n$ and $g(t) = (1-t)^r$. Then we obtain the equation

$$x^n = \mathcal{S}_{(r,n)}(x) + \frac{n}{r+1}\,\mathcal{S}_{(r+1,n-1)}(x). \tag{5.9}$$

(iii). Subtracting (5.8) from (5.9) and shifting the index by $n \mapsto n+1$ yield

$$\mathcal{S}_{(r+1,n)}(x) = \frac{r+1}{r+n+1}\left(x^{n+1} - (x-1)\mathcal{S}_{(r,n)}(x)\right).$$

(iv). Multiply (5.8) by n and (5.9) by r, respectively, and subtract the equations. Divide the resulting equation by $n+r$ and shift the index by $n \mapsto n+1$. Finally, this gives the equation

$$\mathcal{S}_{(r,n+1)}(x) = \frac{r}{r+n+1}x^{n+1} + \frac{n+1}{r+n+1}(x-1)\mathcal{S}_{(r,n)}(x),$$

completing the proof. □

For $n, r \geq 1$, recall by (2.2) the function

$$\psi_{(r,n)}(x) = \sum_{k=0}^{r-1} \binom{n+k}{k}(-1)^k(x-1)^{r-1-k}, \tag{5.10}$$

which satisfies the recurrence

$$\psi_{(r+1,n)}(x) = (x-1)\psi_{(r,n)}(x) + (-1)^r\binom{n+r}{r}. \tag{5.11}$$

Lemma 5.2. *Let $n, r \geq 1$ and $\ell \in \mathbb{Z}$. For $d \in \{1, \ldots, r\}$, we have*

$$r!\,\psi_{(r,n)}(\ell) \equiv r!\,\ell^{d-1}(\ell-1)^{r-d} \pmod{n+d}.$$

Proof. Let $n, r \geq 1$ and $1 \leq d \leq r$. We infer for $k \geq 0$ that

$$(n+k)_k(-1)^k \equiv (d-1)_k \pmod{n+d}.$$

Using (5.10), we obtain for $\ell \in \mathbb{Z}$ that

$$\begin{aligned} r!\,\psi_{(r,n)}(\ell) &\equiv \sum_{k=0}^{r-1} \frac{r!}{k!}(n+k)_k(-1)^k(\ell-1)^{r-1-k} \\ &\equiv \sum_{k=0}^{r-1} \frac{r!}{k!}(d-1)_k(\ell-1)^{r-1-k} \\ &\equiv r!\,(\ell-1)^{r-d}\sum_{k=0}^{d-1}\binom{d-1}{k}(\ell-1)^{d-1-k} \\ &\equiv r!\,\ell^{d-1}(\ell-1)^{r-d} \pmod{n+d}, \end{aligned}$$

as desired. □

Proof of Theorem 2.7. Let $n, r \geq 1$. We first show that

$$\mathcal{S}_{(r,n)}(x) = (-1)^r\binom{n+r}{r}^{-1}\left((x-1)^{n+r} - x^{n+1}\psi_{(r,n)}(x)\right). \tag{5.12}$$

Now, fix n. We use induction on r. By Lemma 5.1 and (5.2), we have

$$\mathcal{S}_{(1,n)}(x) = \frac{x^{n+1}-(x-1)^{n+1}}{n+1},$$

which coincides with (5.12) in the case $r = 1$. Inductive step: we assume that (5.12) holds for r and prove for $r+1$. We use the recurrence formula of Proposition 2.6(iii). Thus, we obtain

$$\begin{aligned} \frac{r+n+1}{r+1}\mathcal{S}_{(r+1,n)}(x) &= x^{n+1} - (x-1)\mathcal{S}_{(r,n)}(x) \\ &= (-1)^r\binom{n+r}{r}^{-1}\Bigg((-1)^r\binom{n+r}{r}x^{n+1} - (x-1)^{n+r+1} \\ &\quad + x^{n+1}(x-1)\psi_{(r,n)}(x)\Bigg) \\ &= (-1)^{r+1}\binom{n+r}{r}^{-1}\left((x-1)^{n+r+1} - x^{n+1}\psi_{(r+1,n)}(x)\right), \end{aligned}$$

where the last equation follows from (5.11).

This implies that $\mathcal{S}_{(r+1,n)}(x)$ is equal to (5.12) in the case $r+1$. Finally, identities (2.1) and (2.2) are established. To show the alternative identity (2.3), we have by Theorem 2.5 that

$$\mathcal{S}_{(r,n)}(x) + x^n\mathcal{S}^{\star}_{(n,r)}(x) = (-1)^r(x-1)^{r+n}\binom{r+n}{r}^{-1} + x^n.$$

By Theorem 2.3(i), we can write

$$\begin{aligned} x^n \mathcal{S}^{\star}_{(n,r)}(x) &= (-1)^r x^n \binom{r+n}{r}^{-1} \sum_{k=0}^{r} \binom{r+n}{k} (-1)^k \, x^{r-k} \\ &= x^n + (-1)^r x^{n+1} \binom{r+n}{r}^{-1} \widetilde{\psi}_{(r,n)}(x), \end{aligned}$$

where

$$\widetilde{\psi}_{(r,n)}(x) = \sum_{k=0}^{r-1} \binom{r+n}{k} (-1)^k x^{r-1-k}. \tag{5.13}$$

Putting all together shows (2.1), but holding with (2.3). Consequently, we obtain

$$\psi_{(r,n)}(x) = \widetilde{\psi}_{(r,n)}(x).$$

Lastly, parts (i) and (ii) are given by (5.2) and (5.3), respectively. Part (iii) follows from (5.12) by taking $x = 2$. This proves the theorem. □

Remark. Searching for identities similar to (5.12) in the literature, one finds the following identity in Gould's tables of combinatorial identities of 1972 (see [6, Eq. (4.13), p. 47]) that

$$\sum_{k=0}^{n} \binom{n}{k} \frac{x^k}{\binom{k+r}{k}} = 1 + \frac{(x+1)^{n+r} - \sum_{k=0}^{r} \binom{n+r}{k} x^k}{x^r \binom{n+r}{n}},$$

from which one can also deduce formula (5.12) with (5.13).

6. Proofs of Main Theorems

Proof of Proposition 3.1. This easily follows from Theorem 2.3(i). □

Proof of Proposition 3.2. Let $n, r \geq 1$ and $\ell \in \mathbb{Z} \setminus \{0, 1\}$. We use Theorem 2.7 and assume that $\mathcal{S}_{(r,n)}(\ell) \in \mathbb{Z}$. Then the numerator of (2.1) must be divisible by each factor $n + d$ of the denominator for $1 \leq d \leq r$. Using Lemma 5.2, we infer the necessary but not sufficient conditions that

$$\begin{aligned} 0 &\equiv r! \left((\ell - 1)^{r+n} - \ell^{n+1} \, \psi_{(r,n)}(\ell) \right) \\ &\equiv r! \, (\ell - 1)^{r-d} \left((\ell - 1)^{n+d} - \ell^{n+d} \right) \equiv a_d \pmod{n+d} \end{aligned}$$

for $1 \leq d \leq r$. Conversely, if one $a_d \not\equiv 0 \pmod{n+d}$, then $\mathcal{S}_{(r,n)}(\ell) \notin \mathbb{Z}$. □

Proof of Theorem 3.3. Let $n, r \geq 1$ and $\ell \in \mathbb{Z}$. We show six conditions in order of their dependencies that imply $\mathcal{S}_{(r,n)}(\ell) \notin \mathbb{Z}$ and also $\mathcal{S}_{(n,r)}(\ell) \notin \mathbb{Z}$ by symmetry of $\binom{r+n}{r} = \binom{n+r}{n}$. It is easy to see that the results also hold for $-\ell$ except for part (v).

(i). This follows from Lemmas 4.11 and 5.1.

(iii). By Theorem 2.1 we have

$$\mathcal{S}_{(r,n)}(\ell) = \sum_{k=0}^{n} (-1)^k \, \ell^{n-k} \, \frac{(n)_k}{(r+k)_k}.$$

Since $r + n = p^e$ is a prime power with $e \geq 1$, Lemma 4.10 shows that the last summand satisfies that

$$v = \operatorname{ord}_p\left(\frac{n!}{(r+n)_n}\right) < \operatorname{ord}_p\left(\frac{(n)_k}{(r+k)_k}\right) \leq \operatorname{ord}_p\left(\ell^{n-k} \frac{(n)_k}{(r+k)_k}\right)$$

for $0 \leq k < n$. By Lemma 4.2, we have $(r+n)_n/n! = \binom{r+n}{n} \in p\mathbb{Z}_p$, so $v < 0$. Using Lemma 4.1, the result follows.

(v). This is given by Corollary 2.2(ii) and (iii) for $\ell \in \{0, 1\}$, and by Theorem 2.3(iv) for $\ell = -1$.

(vi). Let $b = \binom{r+n}{r}$ and $g = \gcd(b, \ell) > 1$. Note that case $\ell = 0$ is compatible with Corollary 2.2(ii), since $g = b$. So we assume that $|\ell| \geq 2$. Now fix a prime divisor p of g. We use Theorem 2.7 to derive that

$$(-1)^r \mathcal{S}_{(r,n)}(\ell) = \frac{(\ell-1)^{r+n}}{b} - \frac{\ell^{n+1} \, \psi_{(r,n)}(\ell)}{b} = f_1 - f_2.$$

Assume that $\psi_{(r,n)}(\ell) \neq 0$; otherwise, we are done. Since $p \mid g$ and $\gcd(\ell - 1, \ell) = 1$, we obtain for the fractions that

$$-\operatorname{ord}_p(b) = \operatorname{ord}_p(f_1) < \operatorname{ord}_p(f_2).$$

From Lemma 4.1, we finally infer that

$$\operatorname{ord}_p(\operatorname{denom}(\mathcal{S}_{(r,n)}(\ell))) = \operatorname{ord}_p(b) > 0.$$

(ii). The case $r = n$ is postponed and borrowed from the proof of Theorem 3.4 below, which uses the independent part (vi).

(iv). Assume that r and n are odd, and ℓ is even. The case $\ell = 0$ is handled by part (v), so $|\ell| \geq 2$. Let $b = \binom{r+n}{r}$ and $g = \gcd(b, \ell)$. Since

$$b \equiv \binom{r+n-1}{r-1} \frac{r+n}{r} \equiv 0 \pmod{2},$$

we have $2 \mid g$, and we can apply part (vi). This completes the proof. $\square$

Proof of Theorem 3.4. The first formula is given by Theorem 2.3(i), where

$$\mathcal{S}_{(1,1)}(x) = x - \tfrac{1}{2}.$$

The recurrence formula follows from Proposition 2.6(iii) and (iv). Hence,

$$\mathcal{S}_{(n+1,n+1)}(x) = A_n(x) + \tfrac{1}{2}x^{n+1} \tag{6.1}$$

with

$$A_n(x) = \frac{n+1}{2n+1}\frac{x-1}{2}\left(x^{n+1} - (x-1)\mathcal{S}_{(n,n)}(x)\right).$$

Now, let $\ell \in \mathbb{Z}$ be odd. We use Lemma 4.1 implicitly. We have $\mathcal{S}_{(1,1)}(\ell) \in \frac{1}{2}\mathbb{Z}_2 \backslash \mathbb{Z}_2$. We use induction on n. Assume that $\mathcal{S}_{(n,n)}(\ell) \in \frac{1}{2}\mathbb{Z}_2 \backslash \mathbb{Z}_2$. Since $\ell - 1$ is even, it follows that $A_n(\ell) \in \mathbb{Z}_2$. From (6.1) we deduce that $\mathcal{S}_{(n+1,n+1)}(\ell) \in \frac{1}{2}\mathbb{Z}_2 \setminus \mathbb{Z}_2$. Finally, we obtain that $\operatorname{ord}_2(\mathcal{S}_{(n,n)}(\ell)) = -1$ for all $n \geq 1$.

In the other case, where $\ell \in \mathbb{Z}$ is even, we use Theorem 3.3(vi). For $n \geq 1$, we have $\binom{2n}{n} = 2\binom{2n-1}{n-1}$, and thus $2 \mid g = \gcd\big(\binom{2n}{n}, \ell\big)$. Using (4.1), we then infer that

$$\operatorname{ord}_2(\mathcal{S}_{(n,n)}(\ell)) = -\operatorname{ord}_2\left(\binom{2n}{n}\right) = -(2s_2(n) - s_2(2n)) = -s_2(n).$$

As a result, $\mathcal{S}_{(n,n)}(\ell) \notin \mathbb{Z}$ for $n \geq 1$ and $\ell \in \mathbb{Z}$. This proves the theorem. □

Remark. Theorem 3.4 implies for $n \geq 1$ and odd $\ell \in \mathbb{Z}$ that

$$\mathcal{S}_{(n,n)}(\ell) - \mathcal{S}_{(n,n)}(1) = \binom{2n}{n}^{-1} \sum_{k=0}^{n} \binom{2n}{k} (-1)^{n-k} \left(\ell^k - 1\right) \in \mathbb{Z}_2.$$

However, a direct proof via 2-adic valuation of the above summands seems to be complicated. More generally, it follows for odd $\ell, \ell_2 \in \mathbb{Z}$ that

$$\mathcal{S}_{(n,n)}(\ell) - \mathcal{S}_{(n,n)}(\ell_2) \in \mathbb{Z}_2.$$

Proof of Corollary 3.5. We first consider Theorem 4.3. By a simple counting argument and excluding those binomial coefficients that equal 1, we arrive at an equivalent formulation as follows. For $d \geq 1$ and $m \geq 2$ define the sets

$$\widetilde{B}_m(d) = \left\{(r,n) \in \mathbb{Z}^2 : n, r \geq 1,\ 1 \leq r+n \leq m, \text{ and } \ d \mid \binom{r+n}{r}\right\}.$$

Then we also have the density

$$\lim_{m\to\infty} \#\widetilde{B}_m(d) \Big/ \binom{m}{2} = 1.$$

Let $d = |\ell| \geq 2$. By Theorem 3.3(vi), we have that

$$\gcd\left(\binom{r+n}{r}, d\right) > 1 \quad \text{implies} \quad \mathcal{S}_{(r,n)}(\ell) \notin \mathbb{Z}.$$

The stronger condition also shows that

$$d \mid \binom{r+n}{r} \quad \text{implies} \quad \mathcal{S}_{(r,n)}(\ell) \notin \mathbb{Z}.$$

Therefore, we infer that $\#\mathcal{N}_m(\ell) \geq \#\widetilde{B}_m(d)$, implying the result. $\square$

Proof of Theorem 3.6. Let $n, r \geq 2$ and $\ell \in \mathbb{Z} \setminus \{0, 1\}$. We have to show two parts.

(i). Assume that $n > r \geq \frac{1}{5}n$ and $n \geq |\ell - 1|$. First we consider the case $n \geq 25$. Using Theorem 4.6, we infer that there exists a prime p satisfying $n < p < n + r$, since $n + r \geq (1 + \frac{1}{5})n$. By assumption we have

$$r!\,(\ell - 1) \not\equiv 0 \pmod{p} \quad \text{and} \quad (\ell - 1)^p - \ell^p \equiv -1 \pmod{p}.$$

Hence, applying Proposition 3.2 yields that $\mathcal{S}_{(r,n)}(\ell) \notin \mathbb{Z}$ for $n \geq 25$. Checking Table 1.1 reveals that the result also holds for the remaining case where $1 < n < 25$. As a refinement, Theorem 4.7 allows us to take the condition $r > n/\log^3 n$ for $n \geq 89\,693$.

(ii). Assume that $r > n$. We apply Theorem 4.5 to $\binom{r+n}{n}$, where the exceptions

$$\binom{r+n}{n} \in \left\{ \binom{4}{2}, \binom{9}{2}, \binom{10}{5} \right\}$$

are ruled out by the excluded condition $r = n$, and by Theorem 3.3(iii) that $r + n$ is a prime power. Therefore, we can continue without restrictions. Then there exists a prime $p > \frac{3}{2}n$ that divides exactly one of the numbers $r + 1, \ldots, r + n$, say $r + d$ with $d \in \{1, \ldots, n\}$. This also implies that $p \nmid r$. We split the proof into two cases as follows.

`Case` $p \nmid \ell - 1$. By Corollary 2.2(i), we have that

$$\mathcal{S}_{(r,n)}(\ell) = \sum_{k=0}^{n} \binom{n}{k} (\ell - 1)^{n-k} \frac{r}{r + k}. \tag{6.2}$$

All summands of (6.2) are p-integral, except for $k = d$ where

$$\operatorname{ord}_p \left(\binom{n}{d} (\ell - 1)^{n-d}\, r \right) = 0 \quad \text{and} \quad \operatorname{ord}_p \left(\frac{1}{r + d} \right) < 0. \tag{6.3}$$

Thus, Lemma 4.1 implies that $\operatorname{ord}_p(\mathcal{S}_{(r,n)}(\ell)) < 0$ and finally $\mathcal{S}_{(r,n)}(\ell) \notin \mathbb{Z}$.

`Case` $p \mid \ell - 1$. We infer from (6.2) and (6.3) that if

$$(n - d) \operatorname{ord}_p(\ell - 1) < \operatorname{ord}_p(r + d)$$

(being always true for $d = n$), then $\mathcal{S}_{(r,n)}(\ell) \notin \mathbb{Z}$. Conversely, if and only if

$$d \neq n \quad \text{and} \quad (n - d) \operatorname{ord}_p(\ell - 1) \geq \operatorname{ord}_p(r + d),$$

then $\mathcal{S}_{(r,n)}(\ell) \in \mathbb{Z}_p$.

To prevent the latter case $\mathcal{S}_{(r,n)}(\ell) \in \mathbb{Z}_p$, which can happen for sufficiently large ℓ, we have to require that $p \nmid \ell - 1$. A priori, this is satisfied if $p > |\ell - 1|$, which is handled by the condition $n \geq \frac{2}{3}|\ell - 1|$. Furthermore, the condition $\gcd\bigl(\binom{r+n}{r}, \ell - 1\bigr) = 1$ also ensures that $p \nmid \ell - 1$, but it may exclude the allowed cases in which a prime $q < p$ satisfies $q \mid \ell - 1$ and $q \mid \binom{r+n}{r}$. Thus, an improved condition, involving such primes, defines the set

$$\mathcal{P} = \left\{ p : p > \tfrac{3}{2}n,\, p \mid \binom{r+n}{r},\, p \nmid \ell - 1 \right\},$$

which has to be nonempty. This completes the proof of the theorem. $\square$

Proof of Corollary 3.7. Theorem 3.3(i) and (ii) cover the cases $r = 1$, $n = 1$, and $r = n$. Let $n, r \geq 2$, $r \neq n$, and $\ell \in \mathcal{L}$. We consider the two parts of the proof of Theorem 3.6. In both cases there exists a prime $p \geq 5 > |\ell - 1|$, from which the result then follows. (i). We have $p > n > r \geq 2$. (ii). We have $p > \frac{3}{2}n \geq 3$. $\square$

7. Exceptions

Let $n, r \geq 1$ and $\ell \in \mathbb{Z}$. The necessary and sufficient condition for exceptional cases, where $\mathcal{S}_{(r,n)}(\ell) \in \mathbb{Z}$, can be reformulated by Proposition 3.1 as a congruence of an incomplete binomial sum such that

$$\sum_{k=0}^{n-1} \binom{r+n}{k} (-\ell)^k \equiv 0 \pmod{\binom{r+n}{r}}. \tag{7.1}$$

Theorem 7.1. *Let $n, r \geq 1$ and $\ell \in \mathbb{Z}$. If $\mathcal{S}_{(r,n)}(\ell) \in \mathbb{Z}$, then there exist positive integers a and b such that*

$$\mathcal{S}_{(r,n)}(a + b\lambda) \in \mathbb{Z} \quad (\lambda \in \mathbb{Z}),$$

where $b = \binom{r+n}{r}$ and $1 < a < b$ with $a \equiv \ell \pmod{b}$. As a consequence,

$$\mathcal{S}_{(r,n)}(\ell) \notin \mathbb{Z} \quad (1 < \ell < b) \quad \textit{implies} \quad \mathcal{S}_{(r,n)}(\ell) \notin \mathbb{Z} \quad (\ell \in \mathbb{Z}).$$

Proof. Assume that $\mathcal{S}_{(r,n)}(\ell) \in \mathbb{Z}$. Let $b = \binom{r+n}{r}$. By Proposition 3.1, congruence (7.1) holds for ℓ, so also for the values

$$\ell = a + b\lambda \quad (\lambda \in \mathbb{Z}) \tag{7.2}$$

with some integer $a \equiv \ell \pmod{b}$, where $0 \leq a < b$. By Theorem 3.3(v), the case $a \in \{0, 1\}$ cannot occur, so we have that $1 < a < b$. Conversely, if $\mathcal{S}_{(r,n)}(\ell) \notin \mathbb{Z}$ for $1 < \ell < b$, then from (7.1) and (7.2), it follows that $\mathcal{S}_{(r,n)}(\ell) \notin \mathbb{Z}$ for all $\ell \in \mathbb{Z}$. $\square$

Extending the computations of Table 1.1 for the case $r = 2$ shows that different values of a can occur for a given modulus b.

Parameters (r, n, a, b)			
$(2, 4, 11, 15)$	$(2, 8, 28, 45)$	$(2, 12, 53, 91)$	$(2, 16, 35, 153)$
$(2, 16, 86, 153)$	$(2, 16, 137, 153)$	$(2, 20, 127, 231)$	$(2, 20, 160, 231)$
$(2, 20, 226, 231)$	$(2, 24, 176, 325)$	$(2, 28, 233, 435)$	$(2, 32, 298, 561)$

Table 7.1: Exceptions where $\mathcal{S}_{(r,n)}(a) \in \mathbb{Z}$ for $r = 2$, $1 \leq n \leq 32$, and $1 < a < b = \binom{r+n}{r}$.

The case $r = 3$ shows a different and more complex pattern.

Result 7.2. Let $1 \leq n < 200$. The exceptions $\mathcal{S}_{(3,n)}(\ell) \in \mathbb{Z}$, for some suitable $\ell \in \mathbb{Z}$, occur for

$$n \in \{7, 18, 23, 31, 36, 39, 54, 55, 71, 87, 90, 95, 103, 108, 119,$$
$$126, 127, 135, 144, 151, 159, 167, 180, 183, 198, 199\}.$$

See Figure 7.1. Any element n of the above sequence has the property that $3 + n$ has at least two different prime factors by Theorem 3.3(iii). Checking the exceptions $\mathcal{S}_{(3,18)}(\ell) \in \mathbb{Z}$ for $1 < \ell < \binom{21}{3} = 1330$ provides the values

$$\ell \in \{153, 191, 419, 457, 723, 951, 989, 1217\},$$

as displayed in Figure 7.2.

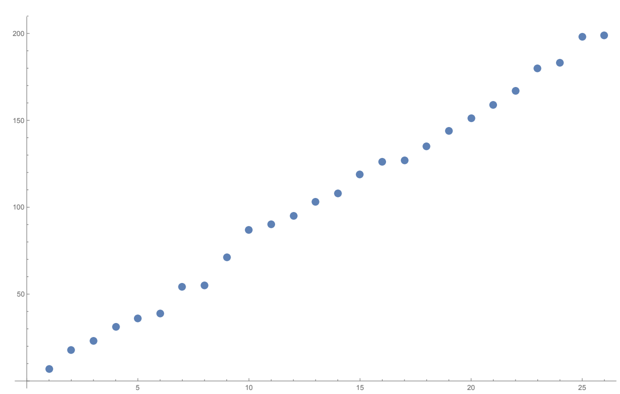

Figure 7.1: Exceptions where $\mathcal{S}_{(3,n)}(\ell) \in \mathbb{Z}$ for $1 \leq n < 200$ and suitable ℓ. Displayed values of n.

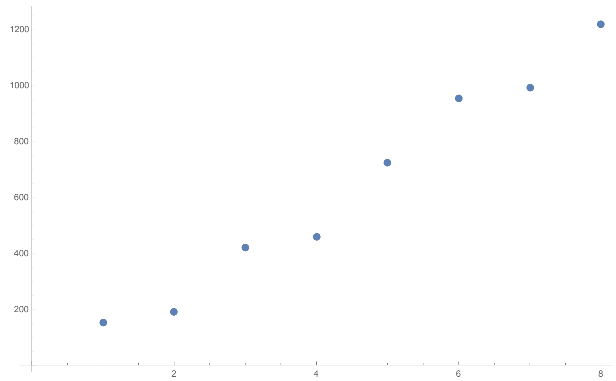

Figure 7.2: Exceptions where $\mathcal{S}_{(3,18)}(\ell) \in \mathbb{Z}$ for $1 < \ell < 1330$. Displayed values of ℓ.

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