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ODD DEFICIENT-PERFECT NUMBERS WITH FOUR DISTINCT PRIME FACTORS

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Abstract

For a positive integer n, let $\sigma(n)$ denote the sum of the positive divisors of n, and let d be a proper divisor of n. We call n a deficient-perfect number if $\sigma(n) = 2n - d$. We show that the only deficient-perfect number with four distinct prime divisors is $3^2 \cdot 7^2 \cdot 11^2 \cdot 13^2$.

1. Introduction

For a positive integer n, let $\sigma(n)$ denote the sum of the positive divisors of n and $\omega(n)$ denote the number of distinct prime factors. An integer n is a perfect number if $\sigma(n) = 2n$. We do not know whether or not there exists an odd perfect number (see [3, 4, 7-10]). Ochem and Rao [19] proved that if N is an odd perfect number, then $N > 10^{1500}$. Recently, Nielsen [17, 18] obtained the lower bound $\omega(N) \geq 10$. On the other hand, scholars began studying other similar numbers, such as pseudo-perfect numbers, near-perfect numbers, deficient-perfect numbers, which are closely related to perfect numbers, and a series of results were obtained; see [1, 7-13]. For more details, one can also see [4], [16], and [20].

We call n a *near-perfect number* if

$$\sigma(n) = 2n + d,\ d \mid n,\ \ d < n,$$

and a *deficient-perfect number* if

$$\sigma(n) = 2n - d,\ \ d \mid n,\ \ d < n. \tag{1}$$

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In 2012, Pollack and Shevelev [20] obtained three classes of even near-perfect numbers of the form $2^\alpha p^\beta$, where $\alpha, \beta > 1$ and p is an odd prime. Ren and Chen [21] improved the above result. In 2010, Shevelev conjectured that all near-perfect numbers are even (see [23]). However, at the beginning of 2012 Donovan Johnson found (up to $2 \cdot 10^{19}$) the only odd near-perfect number which is $3^4 \cdot 7^2 \cdot 11^2 \cdot 19^2$.

In 2013, Tang, Ren, and Li [27] showed that there is no odd near-perfect number with $\omega(n) = 3$ and determined all deficient-perfect numbers with at most two distinct prime factors. In [25], they showed that there are no odd deficient-perfect numbers with $\omega(n) = 3$. Li and Lao [14] gave two equivalent conditions of all even near-perfect numbers of the form $2^\alpha p_1 p_2$ and $2^\alpha p_1^2 p_2$. Recently, Tang, Ma, and Feng [26] proved that the only odd near-perfect number with four distinct prime factors is $3^4 \cdot 7^2 \cdot 11^2 \cdot 19^2$.

In this paper, we consider the odd deficient-perfect number with four distinct prime factors. We assume that

$$n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdot p_3^{\alpha_3} \cdot p_4^{\alpha_4}, \quad 3 \le p_1 < p_2 < p_3 < p_4$$

is an odd deficient-perfect number, where p_1, p_2, p_3, p_4 are distinct primes, and α_1, α_2, α_3, $\alpha_4 \in \mathbb{N}$. Using some results on Diophantine equations and some elementary properties of deficient-perfect numbers, we find all odd deficient-perfect numbers with $\omega(n) = 4$. Our main result is the following theorem.

Theorem 1. *If n is an odd deficient-perfect number with four distinct prime factors and d is a proper divisor of n, then*

$$n = 3^2 \cdot 7^2 \cdot 11^2 \cdot 13^2, \;\; d = 3^2 \cdot 7 \cdot 13.$$

We give the following remark.

Remark In [24], Sun and He claimed to prove the same result ($3^2 \cdot 7^2 \cdot 11^2 \cdot 13^2$ is the only odd deficient-perfect number with four distinct prime factors). Nevertheless, our paper gives a proof which is much shorter than the preprint [24]. Accordingly, the authors of the papers [5], [6] and [22] also give a partial conclusion about the part of the odd deficient-perfect number with four distinct prime factors.

We organize this paper as follows. In Section 2, we recall some auxiliary results. Second, we prove Theorem 1 in Section 3 - Section 5. In Section 6, we use some numerical computations to make a conjecture on odd deficient-perfect numbers. For the sake of completeness, we will give as much details as we can.

2. Auxiliary Results and Lemmas

In this section, we recall some auxiliary results and lemmas that we will need for the proof of Theorem 1.

Let d be a proper divisor of n. We assume that n is an odd deficient-perfect number with four distinct prime factors. If $p_1 \geq 5$, from Equation (1), we know that $\frac{\sigma(n)}{n} < \frac{5}{4} \cdot \frac{7}{6} \cdot \frac{11}{10} \cdot \frac{13}{12} - \frac{1}{5} < 2$, so we get $p_1 = 3$. From Equation (1), we get

$$\sigma(3^{\alpha_3}) \cdot \sigma(p_2^{\alpha_2}) \cdot \sigma(p_3^{\alpha_3}) \cdot \sigma(p_4^{\alpha_4}) = \left(2 - \frac{1}{D}\right) \cdot 3^{\alpha_1} \cdot p_2^{\alpha_2} \cdot p_3^{\alpha_3} \cdot p_4^{\alpha_4}, \tag{2}$$

where

$$D = \frac{d}{n} = 3^{\beta_1} \cdot p_2^{\beta_2} \cdot p_3^{\beta_3} \cdot p_4^{\beta_4}, \;\; 0 \leq \beta_i \leq \alpha_i, \; i = 1, 2, 3, 4.$$

Since $\sigma(p_i^{\alpha_i}) = \frac{p_i^{\alpha_i+1}-1}{p_i-1}$ for every $i = 1, 2, 3, 4$, from Equation (2) we have

$$\begin{aligned} G := & \left(1 - \frac{1}{3^{\alpha_1+1}}\right)\left(1 - \frac{1}{p_2^{\alpha_2+1}}\right)\left(1 - \frac{1}{p_3^{\alpha_3+1}}\right)\left(1 - \frac{1}{p_4^{\alpha_4+1}}\right) \\ & - \frac{2(2D-1)(p_2-1)(p_3-1)(p_4-1)}{3\,Dp_2\,p_3\,p_4} = 0 \end{aligned} \tag{3}$$

and

$$2 - \frac{1}{D} = \frac{\sigma(n)}{n} = \frac{\sigma(3^{\alpha_1})}{3^{\alpha_1}} \cdot \frac{\sigma(p_2^{\alpha_2})}{p_2^{\alpha_2}} \cdot \frac{\sigma(p_3^{\alpha_3})}{p_3^{\alpha_3}} \cdot \frac{\sigma(p_4^{\alpha_4})}{p_4^{\alpha_4}}. \tag{4}$$

Let p be a prime and $\alpha \geq 2$. We know that $\frac{p^2+p+1}{p^2} \leq \frac{\sigma(p^\alpha)}{p^\alpha} < \frac{p}{p-1}$. Hence, from Equation (1) we get

$$\frac{1}{2 - \frac{13}{9} \cdot \frac{p_2^2+p_2+1}{p_2^2} \cdot \frac{p_3^2+p_3+1}{p_3^2} \cdot \frac{p_4^2+p_4+1}{p_4^2}} \leq D < \frac{1}{2 - \frac{3}{2} \cdot \frac{p_2}{p_2-1} \cdot \frac{p_3}{p_3-1} \cdot \frac{p_4}{p_4-1}}. \tag{5}$$

Now, we recall the following lemmas.

Lemma 1. *If $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ is an odd deficient-perfect number with $\omega(n) = k$, where $k \geq 2,\; 3 \leq p_1 < \cdots < p_k$, then $2 \mid \alpha_i \, (i = 1, 2, \cdots, k)$.*

Proof. See Lemma 1 of [27] or Lemma 1 of [15]. □

Lemma 2. *Let S_1 be the set of all positive integers greater than 1 and composed only by 2 and by the primes of the form $2^a + 1$, for $a \geq 1$. Let p be a prime number of the form $2^a 3^b + 1$, with integers $a \geq 0$, $b > 0$ and $p \not\equiv 55 \pmod{63}$, let S_2 be the set of all numbers of the form $2^f p$, where f is any nonnegative integer satisfying $f \not\equiv 1, 4 \pmod 6$ if either $p \equiv 1 \pmod 9$, $p \equiv 3, 4 \pmod 7$ or $p \equiv 4 \pmod 9$, and $f \not\equiv 2, 5 \pmod 6$ if either $p \equiv 1 \pmod 9$, $p \equiv 2, 5 \pmod 7$ or $p \equiv 7 \pmod 9$. Put $S_3 = S_1 \cup S_2$. Then, the equation*

$$\frac{x^n - 1}{x - 1} = y^q, \;\; x > 1, \; y > 1, \; n > 2, \; q \geq 2$$

has no solution (x, y, n, q), where $x = h^t$, $h \in S_3$, and $t \geq 1$, other than $(h, t, y, n, q) = (3, 1, 11, 5, 2)$, $(7, 1, 20, 4, 2)$, and $(18, 1, 7, 3, 3)$.

Proof. See Corollary 1 of [2]. □

Notice that for our work, we will consider $f = 0$ in Lemma 2. In the following three sections, we will use the auxiliary results and Lemmas 1 and 2 to study the odd near-perfect numbers with four distinct prime factors. Since by Equation (4) we get $p_2 \le 23$, we discuss three kinds of cases: $p_2 = 5$, $p_2 = 7$, and $p_2 = 11$, 13, 17, $19 \le p_2 \le 23$. In [15], Ma and Wang discussed the case $19 \le p_2 \le 23$; here we give a more concise discussion.

3. The Odd Deficient-Perfect Numbers of the Form $3^{\alpha_1} \cdot 5^{\alpha_2} \cdot p_3^{\alpha_3} \cdot p_4^{\alpha_4}$

In this section, we will discuss the cases for $p_2 = 5$. We will prove that there is no deficient perfect number n with $\omega(n) = 4$ such that $p_2 = 5$.

If $p_3 = 7$, then from (4) we have $2 > \frac{\sigma(n)}{n} = \frac{\sigma(3^2)}{3^2} \cdot \frac{\sigma(5^2)}{5^2} \cdot \frac{\sigma(7^2)}{7^2} > 2.0835$, which is absurd. Therefore, $p_3 \ge 11$.

First, we suppose that $D = 3$, 5. If $\alpha_1 \ge 4$, from the expression of G, we have

$$G > \Big(1 - \frac{1}{p_3}\Big)\Big(1 - \frac{1}{p_4}\Big)\Big(0.9879\Big(1 + \frac{1}{p_3} + \frac{1}{p_3^2}\Big)\Big(1 + \frac{1}{p_4} + \frac{1}{p_4^2}\Big) - 0.9600\Big) > 0,$$

which contradicts (3). If $\alpha_1 = 2$, then by Equation (1) we have $\sigma(3^2) = 13 \mid (2 - \frac{1}{D})n$. We deduce that $13 \mid p_3 p_4$, and thus $p_3 = 13$ or $p_4 = 13$. Using a similar calculation, we get $G > 0$, which also contradicts (3). Therefore, we assume that $D \ge 7$.

If $D = 7$, then $7 \mid p_3 p_4$, which is impossible as $p_3 \ge 11$.

When $D = 9$, as $p_1 = 3$, $p_2 = 5$, and $p_3 < p_4$, if $p_3 < 23$, or $p_3 \ge 271$, then by Equation (4) this is impossible. Thus, we deduce that $23 \le p_3 \le 269$. If $\alpha_1 = 2$ or $\alpha_1 = 4$, then we have $\sigma(3^2) = 13 \mid p_3 p_4$, or $\sigma(3^4) = 11^2 \mid p_3 p_4$, which is impossible. Therefore, $\alpha_1 \ge 6$. If $23 \le p_3 \le 61$, then we get $G > 0$. So, we will only consider $67 \le p_3 \le 269$.

As $\alpha_1 \ge 6$, from (1) we get

$$\sigma(3^{\alpha_1})\,\sigma(5^{\alpha_2})\,\sigma(p_3^{\alpha_3})\,\sigma(p_4^{\alpha_4}) = 17 \cdot 3^{\alpha_1 - 2} \cdot 5^{\alpha_2} \cdot p_3^{\alpha_3} \cdot p_4^{\alpha_4},$$

which must be divisible by $3 \cdot 5 \cdot 17$. Knowing that all 3 (mod 5), 5 (mod 3), 3 (mod 17), 5 (mod 17) have even orders (we note that $\mathrm{ord}_q p$ usually denotes the multiplicative order of p modulo q, instead of q modulo p), one can see that $3 \cdot 5 \cdot 17$ divides $\sigma(p_3^{\alpha_3})\,\sigma(p_4^{\alpha_4})$. Hence, for $i = 3$ or $i = 4$, $p_i \pmod{q_j}$ must have order one and q_j divides $\alpha_i + 1$ for two primes q_1 and q_2 among $\{3, 5, 17\}$. Thus, we see that $\Phi_{q_1}(p_i)\,\Phi_{q_2}(p_i)$ divides $\sigma(p_i^{\alpha_i})$, where $\Phi_d(x)$ denotes the d-th cyclotomic polynomial. By Zsigmondy's theorem, each $\Phi_{q_j}(p_i)$ $(j = 1, 2)$ has a prime factor $q_j' \equiv 1 \pmod{q_j}$ and $q_1' \ne q_2'$. We cannot have $q_j \in \{3, 5, 17\}$ since $q_j' \equiv 1 \pmod{q_j}$. Clearly, $q_j \ne p_i$ and therefore $q_1' = q_2' = p_{7-i}$, which is a contradiction.

If $D = 11$, then $p_3 = 11$ or $p_4 = 11$. If $D = 13$, then $p_3 = 13$ or $p_4 = 13$. In both cases, we have $G > 0$, which is inconsistent with (3).

If $D \geq 15$, then from (4) we get $p_3 \leq 61$. Now, we will have seven cases to consider.

3.1. The Case $11 \leq p_3 \leq 13$

In this case, first we assume that $\alpha_1 \geq 4$ and we see that $G > 0$, so we take $\alpha_1 = 2$.

We consider $p_3 = 11$. If $D \leq 31$ or $p_4 \leq 61$, then from (3) we get $G > 0$. So, we suppose $D \geq 33$ and $p_4 \geq 67$. We have two cases: $D \leq 603$ and $D \geq 605$.

When $D \leq 603$, since $\alpha_1 = 2$, and p_4^2 does not divide D, from Equation (2) we see that p_4 must divide $\sigma(5^{\alpha_2})$ or $\sigma(11^{\alpha_3})$. Also, $5 \pmod{p_4}$ is of odd order for none of the values of p_4. Moreover, $11 \pmod{p_4}$ has odd order only for $p_4 = 167, 397$. If $p_4 = 167$, then $83 \,|\, (\alpha_3 + 1)$ and $691 \,|\, \sigma(11^{\alpha_3})$, which is a contradiction. If $p_4 = 397$, then $99 \,|\, (\alpha_3 + 1)$ and $12119 \,|\, \sigma(11^{\alpha_3})$, which is again a contradiction.

When $D \geq 605$, from (5) we deduce that $p_4 \leq 163$. If $67 \leq p_4 \leq 113$, when $\alpha_2 \geq 4$, we see that $G > 0$. When $\alpha_2 = 2$ and $p_4 \geq 73$, from (4) we have $\frac{\sigma(n)}{n} + \frac{1}{D} < 1.9993$. When $\alpha_2 = 2$ and $p_4 = 71$, if 71^2 does not divide D, then $71 | \, \sigma(n)$. Since 71 does not divide $\sigma(3^2 \cdot 5^2 \cdot 71^{\alpha_4})$, 71 must divide $\sigma(11^{\alpha_3})$. However, this is impossible since $11 \pmod{71}$ has even order. If $71^2 | \, D$, then $D \geq 5041$, and $\frac{\sigma(n)}{n} + \frac{1}{D} < 1.9986$. When $\alpha_2 = 2$ and $p_4 = 67$, if $\alpha_3 \geq 4$, we find $G > 0$. When $\alpha_3 = 2$, from (2) we find that $\sigma(3^2)\sigma(5^2)\sigma(11^2) \,|\, (2D - 1)$, then $D \geq 26801$, and by (4) we have $\frac{\sigma(n)}{n} + \frac{1}{D} < 1.9978$. All these cases give a contradiction.

Let us study the case $D \geq 605$ and $127 \leq p_4 \leq 163$. We consider $p_4 = 127$. When $\alpha_2 \geq 6$, one sees that $G > 0$. When $\alpha_2 = 2$, then from (4) we have $\frac{\sigma(n)}{n} + \frac{1}{D} < 2$. When $\alpha_2 = 4$ and $\alpha_3 \geq 4$, we have $G > 0$. When $\alpha_2 = 4$ and $\alpha_3 = 2$, from (2) we get $\sigma(3^2)\sigma(5^4)\sigma(11^2) \,|\, (2D - 1) \cdot 11^{\alpha_3}$, which implies that $13 \cdot 71 \cdot 133 | \; (2D - 1)$. Therefore, $D \geq 61381$, from (4), we have $\frac{\sigma(n)}{n} + \frac{1}{D} < 1.9998$. We also come to a contradiction. If $p_4 = 131, 137, 139, 149, 151, 157, 163$, one can use a similar method to obtain an impossibility.

Now, we consider $p_3 = 13$. Since $\gcd(5, \, \sigma(3^2) \, \sigma(5^{\alpha_2}) \, \sigma(13^{\alpha_3})) = 1$, from Equation (2) we observe that $p_4 \equiv 1 \pmod 5$ unless $5^{\alpha_2} | \, D$. If $p_4 \leq 31$, one sees that $G > 0$. Therefore, we suppose that $p_4 \geq 37$. If $D < 45$, from the expression of D we have $D = 15, 25, 27, 37, 39, 41, 43$. When $D = 15$, we obtain that

$$G \geq \Big(1 - \frac{1}{3^3}\Big)\Big(1 - \frac{1}{5^3}\Big)\Big(1 - \frac{1}{13^3}\Big)\Big(1 - \frac{1}{p_4^3}\Big) - \frac{928(p_4 - 1)}{975 \, p_4} > 0,$$

which contradicts (3). When $D = 25$, we first consider the range of p_4. If $p_4 \geq 499$, then we have $\frac{\sigma(n)}{n} + \frac{1}{D} < 2$. This contradicts (4). Thus, $p_4 \leq 491$. If $\alpha_2 = 2, \, 4, \, 6$, we see that $\sigma(5^2) = 31 | \, p_4$, or $\sigma(5^4) = 11 \cdot 71 \,|\, p_4$, or $\sigma(5^6) = 19531 \,|\, p_4$, which is impossible. So, $\alpha_2 \geq 8$. If $p_4 \leq 401$, we obtain $G > 0$ and then $409 \leq p_4 \leq 499$. As $5^{\alpha_2} \nmid D$, we have $p_4 \equiv 1 \pmod 5$ and then $p_4 = 421, 431, 461, 491$. If $\alpha_3 = 2, \, 4,$

hence we have $\sigma(13^2) = 3 \cdot 61 | n$ or $\sigma(13^4) = 30941 \,|\, n$, which is impossible. So, $\alpha_3 \geq 6$, and we see that $G > 0$, which contradicts (3). Similarly, when $D = 27$, 39, from (3) and (4) we get $83 < p_4 < 283$ or $53 < p_4 < 109$. Since $5^{\alpha_2} \nmid D$, then $p_4 \equiv 1 \pmod 5$, and then $p_4 = 101, 131, 151, 181, 191, 211, 241, 251, 271, 281$ or $p_4 = 61, 71, 101$, respectively. If $\alpha_2 \leq 4$ or $\alpha_3 = 2$, thus we have $\sigma(5^2) = 31 \,|\, n$, $\sigma(5^4) = 11 \cdot 71 \,|\, n$ or $\sigma(13^2) = 3 \cdot 61 | n$, which is impossible. Therefore, $\alpha_2 \geq 6$ and $\alpha_3 \geq 4$. This implies that $G > 0$, which also contradicts (3). When $D = 37$, 41, 43 and the corresponding values of p_4 are $p_4 = 37, 41, 43$, we obtain that $G > 0$, which contradicts (3).

When $D \geq 45$, from (4) we get $p_4 \leq 89$. When $D \leq 213$, if $\alpha_2 = 2$, from Equation (2) we have $\sigma(5^2) = 31 | \,(2D - 1)$, then $D = 47, 109, 171$ and the only corresponding value of p_4 is 47. We see that $G > 0$, which contradicts (3). If $\alpha_2 = 4$, from (2) we have $\sigma(5^4) = 11 \cdot 71 \,|\, (2D - 1)p_4$. So, $p_4 = 71$, and $11 | (2D - 1)$. However, from (4) we get $D \leq 59$, which is impossible. Thus, $\alpha_2 \geq 6$. So, $5^{\alpha_2} \nmid D$ and then $p_4 \equiv 1 \pmod 5$. We deduce that $p_4 = 41, 61, 71$. The cases $p_4 = 41$, or $p_4 = 61$ and $D \geq 81$, or $p_4 = 71$ and $D \geq 57$, imply that $G > 0$, which contradicts (3). If $p_4 = 61$ and $83 \leq D \leq 213$, or $p_4 = 71$ and $57 \leq D \leq 213$, from the expression of D, we have $D \geq 117$ or $D \geq 65$ respectively. Thus $\frac{\sigma(n)}{n} + \frac{1}{D} < 1.9994$. Both cases lead to a contradiction.

Finally, we take $D \geq 215$. If $p_4 \geq 53$, from (4) we have $\frac{\sigma(n)}{n} + \frac{1}{D} < 1.9983$, a contradiction. Thus, we only consider $p_4 = 37$, 41, 43, 47. If $p_4 = 37$, 41, 43, when $\alpha_2 \geq 4$, we find $G > 0$, and when $\alpha_2 = 2$, we have $\frac{\sigma(n)}{n} + \frac{1}{D} < 1.99893$, a contradiction. If $p_4 = 47$, from (5) and (2), we get $D \leq 685$ and $5^{\alpha_2} \,|\, D$. If $\alpha_2 \leq 4$, from (4) we have $\frac{\sigma(n)}{n} + \frac{1}{D} < 1.99994$. So, $\alpha_2 \geq 6$, and we deduce that $5^6 \,|\, D$, which is impossible.

3.2. The Case $p_3 = 17$

In this case, from Equation (1) we get

$$\sigma(3^{\alpha_1})\,\sigma(5^{\alpha_2})\,\sigma(17^{\alpha_3})\,\sigma(p_4^{\alpha_4}) = \left(2 - \frac{1}{D}\right) \cdot 3^{\alpha_1} \cdot 5^{\alpha_2} \cdot 17^{\alpha_3} \cdot p_4^{\alpha_4}. \tag{6}$$

If $p_4 = 19$, we see that $G > 3.841 \cdot 10^{-3}$. So, we suppose that $p_4 \geq 23$.

First, we assume that $D \leq 101$. We calculate the values of G as follows: when $15 \leq D \leq 27$, we have $G > 0$ ($\alpha_1 \geq 4$), when $29 \leq D \leq 57$, we have $G > 0$ ($\alpha_1 \geq 4$, $\alpha_2 \geq 4$), and when $59 \leq D \leq 101$, we have $G > 0$ ($\alpha_1 \geq 6$, $\alpha_2 \geq 4$). Each of the above cases gives a contradiction, so we will discuss the remaining cases. When $\alpha_1 = 2$, $\alpha_2 = 2$, or $\alpha_1 = 4$, from Equation (6) we have $\sigma(3^2) = 13 \,|\, (2D - 1)$, $\sigma(5^2) = 31 \,|\, (2D - 1)\, p_4$, or $\sigma(3^4) = 121 \,|\, (2D - 1)$. Thus, we deduce that $D =$ 59, 85 ($\alpha_1 = 2$), $D = 47$ or $p_4 = 31$ ($\alpha_2 = 2$), $D = 61$ or $p_4 = 61$ ($\alpha_1 = 4$), respectively. If $D = 59$, 47, 61, then $D = p_4$. We know that $\mathrm{ord}_{p_i} 17$ is even for $i = 1, 2, 4$, therefore $\gcd(\sigma(p_i^{\alpha_i}), 17) = 1$ for $i = 1, 2, 3, 4$, which is impossible. If

$D = 85$, $\alpha_1 \geq 6$ and $\alpha_2 \geq 4$, then we have $G > 1.003 \cdot 10^{-3}$, which contradicts (3). When $\alpha_1 \leq 4$ or $\alpha_2 = 2$, by (4) we have $p_4 \leq 467$. As $2D - 1 = 13^2$, $\mathrm{ord}_{p_i} 5\,(i = 1, 3)$ and $\mathrm{ord}_{p_i} 17\,(i = 1, 2)$ are all even, $\mathrm{ord}_{13} 5 = 4$, $\mathrm{ord}_{13} 17 = 6$, we have $\gcd(\sigma(5^{\alpha_2})\sigma(17^{\alpha_3}),\, 3 \cdot 5 \cdot 17 \cdot (2D - 1)) = 1$. By Equation (6) we see that

$$\sigma(5^{\alpha_2}) = p_4^{\beta}\ (1 \leq \beta \leq \alpha_4), \quad \sigma(17^{\alpha_3}) = p_4^{\beta'}(\ 1 \leq \beta' \leq \alpha_4).$$

Using Lemma 2, we get $\beta = \beta' = 1$, and then $p_4 = \sigma(5^{\alpha_2}) = \sigma(17^{\alpha_3})$. Note that for $p_4 \leq 467$, one can verify that this is impossible. If $p_4 = 31$, in this case we have $\alpha_2 \geq 4$ and $\alpha_2 = 2$. We see that $G > 0.02148$. This also contradicts (3).

Second, we assume that $103 \leq D \leq 353$. By the expression of D, one can check that $D = 125, 135, 153, 225, 243, 255, 289$, or $D = p_4\,(103 \leq p_4 \leq 353)$, or $D = 3p_4\,(37 \leq p_4 \leq 113)$, or $D = 5p_4\,(23 \leq p_4 \leq 67)$, or $D = 9p_4\,(23 \leq p_4 \leq 37)$, or $D = 15p_4\,(p_4 = 23)$. If $D = p_4$, $3p_4$, $5p_4$, $9p_4$, $15p_4$, we calculate the values of G and obtain: $G > 0\,(\alpha_1 \geq 8, \alpha_2 \geq 6)$, $G > 0\,(\alpha_1 \geq 4, \alpha_2 \geq 4)$, $G > 0\,(\alpha_1 \geq 4)$, $G > 0\,(\alpha_1 \geq 4)$, $G > 0\,(\alpha_1 \geq 4)$, respectively. When $\alpha_1 = 2$ and $p_4 \geq 29$, inequality (4) implies $\frac{\sigma(n)}{n} + \frac{1}{D} < 1.997$, which is a contradiction. When $\alpha_1 = 2$ and $p_4 = 23$, from Equation (6) we have $\sigma(3^2) = 13 \mid (2D - 1)$. We deduce that $D = 3 \cdot 5 \cdot 23$ and from (6) we get $5 \mid n$. However, $\mathrm{ord}_{p_i} 5$ is even for $i = 1, 3, 4$. Thus, $\gcd(\sigma(p_i^{\alpha_i}), 5) = 1$ for $i = 1, 2, 3, 4$, which is impossible. When $\alpha_1 = 4$, or $\alpha_1 = 6$, or $\alpha_2 = 4$, by (6), we have $\sigma(3^4) = 11^2 \mid (2D - 1)n$, or $\sigma(3^6) = 1093 \mid (2D - 1)n$, or $\sigma(5^4) = 11 \cdot 71 \mid (2D - 1)n$. We directly verify that it is impossible. When $\alpha_2 = 2$, by (6) and the expression of D we get $D = 109, 223$, so $p_4 = D$. Similarly, we deduce that $5 \mid n$ and $\gcd(\sigma(p_i^{\alpha_i}), 5) = 1$ for $i = 1, 2, 3, 4$, which is impossible.

If $D = 125$, when $\alpha_1 \geq 8$, $\alpha_2 \geq 6$, and $\alpha_3 \geq 4$, then we have $G > 0$. When $\alpha_1 \leq 6$, or $\alpha_2 \leq 4$, or $\alpha_3 = 2$, by (4) we have $p_4 \leq 9137$. Since $2D - 1 = 3 \cdot 83$, and $\mathrm{ord}_5 p_i$ is even for $i = 1, 3$, then $\mathrm{ord}_5 83 = 82$ and $\gcd(\sigma(5^{\alpha_2}), 3 \cdot 5 \cdot 17 \cdot (2D - 1)) = 1$. Using (6), we see that $\sigma(5^{\alpha_2}) = p_4^{\beta}$, $1 \leq \beta \leq \alpha_4$. By Lemma 2, we get $\beta = 1$, then $p_4 = \sigma(5^{\alpha_2})$. As $p_4 \leq 9137$, we verify that $p_4 = \sigma(5^2) = 31$ and $\alpha_2 = 2$. The fact that $D = 5^3$ implies that $\alpha_2 \geq 4$ giving a contradiction.

If $D = 135$, 153, 225, 243, 255, 289, then $2D - 1 = 269$, $5 \cdot 61$, 449, $5 \cdot 97$, 509, 577, respectively, and from (4) we see that $p_4 \leq 4919$. We verify that $\mathrm{ord}_3 p$ is even for $p = 5, 17, 269, 61, 97, 509, 577$ and $\gcd(\sigma(3^{\alpha_1}), 3 \cdot 5 \cdot 17 \cdot (2D - 1)) = 1$. By (6), we see that $\sigma(3^{\alpha_1}) = p_4^{\beta}\ (1 \leq \beta \leq \alpha_4)$. Using Lemma 2.2, we get $\beta = 1$, i.e. $p_4 = \sigma(3^{\alpha_1})$. As $p_4 \leq 4919$, we have $p_4 = \sigma(3^6) = 1093$. If $D \neq 225$, one sees that $5 \mid \sigma(n)$ and $\gcd(\sigma(p_i^{\alpha_i}), 5) = 1$ for $i = 1, 2, 3, 4$, which is absurd. If $D = 225$, we see that $17 \mid \sigma(n)$ and $\gcd(\sigma(p_i^{\alpha_i}), 17) = 1$ for $i = 1, 2, 3, 4$. This also leads to a contradiction.

Now, we consider $D \geq 355$. If $p_4 \geq 401$, from (4) we have $\frac{\sigma(n)}{n} + \frac{1}{D} < 1.99999$, a contradiction, then $p_4 \leq 397$. If $p_4 \geq 23$ and $p_4 \neq 103$, 137, 239, 307, we see that $\mathrm{ord}_{p_4} 17$ are all even. Using $\mathrm{ord}_3 17 = \mathrm{ord}_5 17 = 16$, we get $\gcd(\sigma(p_i^{\alpha_i}), 17) = 1$ for $i = 1, 2, 3, 4$. Thus, from (6) we have $17^{\alpha_3} \mid D$. As $\alpha_3 \geq 2$ and $D \geq 355$, we obtain that $D \geq 3 \cdot 289 = 867$ ($p_4 \neq 103$, 137, 239, 307). Furthermore, using a similar method, from (4) we get $p_4 \le 293$.

First, we consider $23 \le p_4 \le 251$ and calculate the values of G. We have $G > 0$ under the condition of α_i as outlined in Table 1.

p_4	α_i	p_4	α_i
$23 \le p_4 \le 61$	$\alpha_1 \ge 4$	$211 \le p_4 \le 223$	$\alpha_1 \ge 8,\ \alpha_2 \ge 4$
$67 \le p_4 \le 113$	$\alpha_1 \ge 4,\ \alpha_2 \ge 4$	$227 \le p_4 \le 239$	$\alpha_1 \ge 8,\ \alpha_2 \ge 6$
$127 \le p_4 \le 199$	$\alpha_1 \ge 6,\ \alpha_2 \ge 4$	$241 \le p_4 \le 251$	$\alpha_1 \ge 8,\ \alpha_2 \ge 6,\ \alpha_3 \ge 4$

Table 1: The condition of α_i, when $p_2 = 5$, $p_3 = 17$.

Each of the cases in Table 1 gives an impossibility. For the remaining cases not in the above table, we use a case-by-case study to obtain a contradiction.

Secondly, we consider $257 \le p_4 \le 293$. We use (4) to determine the values of D corresponding to each p_4 as follows: $p_4 = 257$, $D \le 32767$; $p_4 = 263$, $D \le 4289$; $p_4 = 269$, $D \le 2637$; $p_4 = 271$, $D \le 2303$; $p_4 = 277$, $D \le 1681$; $p_4 = 281$, $D \le 1433$; $p_4 = 283$, $D \le 1335$; $p_4 = 293$, $D \le 1009$. Through similar discussions, we can also draw contradictory results.

Finally, we discuss the cases $p_4 = 103, 137, 239, 307$. We verify that all the $\operatorname{ord}_{p_i} 5$ are even for $i = 1, 3, 4$ and if $p_4 = 137, 239$, then we see that $\operatorname{ord}_{p_i} 3$ are all even for $i = 2, 3, 4$. Thus, from Equation (6) we have $3^{\alpha_1} 5^{\alpha_2} \mid D$ when $p_4 = 137, 239$, and $5^{\alpha_2} \mid D$ when $p_4 = 103, 307$. If $p_4 = 103$, 137, 239, we see that $G > 0$ when $\alpha_1 \ge 4$ and $\alpha_2 \ge 4$, when $\alpha_1 \ge 6$ and $\alpha_2 \ge 4$, when $\alpha_1 \ge 8$, $\alpha_2 \ge 6$ and $\alpha_2 \ge 4$, respectively. When $\alpha_1 = 2, 4, 6$, or when $\alpha_2 = 2, 4$, or when $\alpha_3 = 2$, for $p_4 = 103, 137, 239, 307$, we use (4) to verify that $\frac{\sigma(n)}{n} + \frac{1}{D} < 2$, giving a contradiction.

3.3. The Case $p_3 = 19$

In this case, Equation (1) becomes

$$\sigma(3^{\alpha_1})\,\sigma(5^{\alpha_2})\,\sigma(19^{\alpha_3})\,\sigma(p_4^{\alpha_4}) = \left(2 - \frac{1}{D}\right) \cdot 3^{\alpha_1} \cdot 5^{\alpha_2} \cdot 19^{\alpha_3} \cdot p_4^{\alpha_4}. \tag{7}$$

First, we assume $D \le 223$. If $\alpha_1 = 2$, then Equation (7) implies $13 \mid 2D - 1$ and $D = 59, 111, 137, 163, 215$. From (4), we have $\frac{\sigma(n)}{n} + \frac{1}{D} < 1.9679$, which is a contradiction. If $\alpha_1 = 4$, thus Equation (7) gives $121 \mid 2D - 1$ and then $D = 61$, $p_4 = 61$. One can see that $G > 1.956 \cdot 10^{-3}$, which is also a contradiction. If $\alpha_1 = 6$, from (7) we have $1093 \mid (2D - 1)p_4$ and then $p_4 = 1093$. By (4) we have $D \le 49$, so $D = 15, 25, 27, 45$. Taking $D = 15, 25, 27$ gives $G > 4.964 \cdot 10^{-4}$, a contradiction. When $D = 45$, then $2D - 1 = 89$ and all $\operatorname{ord}_{p_i} 89$ are even for $i = 1, 2, 3, 4$. So, $\gcd(\sigma(p_i^{\alpha_i}), 89) = 1$ for $i = 1, 2, 3, 4$, which is impossible. Thus, we deduce that $\alpha_1 \ge 8$. If $\alpha_2 = 2$, or $\alpha_2 = 4$, or $\alpha_3 = 2$, using the same method, we get contradictions. So, we have $\alpha_2 \ge 6$ and $\alpha_3 \ge 4$.

If $D \leq 47$, as $\alpha_1 \geq 8$, $\alpha_2 \geq 6$, and $\alpha_3 \geq 4$, from (3) one sees that $G > 0$. If $49 \leq D \leq 223$, then by the expression of D, we get $D = 57, 75, 81, 95, 125, 135$, or $D = p_4$, where $23 \leq p_4 \leq 223$, or $D = 3p_4$, where $23 \leq p_4 \leq 71$, or $D = 5p_4$, where $23 \leq p_4 \leq 53$. If $D = p_4 \geq 147$, from (4) we have $\frac{\sigma(n)}{n} + \frac{1}{D} < 1.9997$, a contradiction. If $D = p_4 \leq 141$, or $D = 3p_4$, or $D = 5p_4$, we get $G > 1.341 \cdot 10^{-4}$, which contradicts (3). For $D = 57, 75, 81, 95, 125, 135, 171$, using a study of the p_i and α_i as in the previous subsections, we come to a contradiction.

Secondly, we consider $D \geq 225$. If $p_4 \geq 127$, from (4) we have $\frac{\sigma(n)}{n} + \frac{1}{D} < 1.9994$, giving a contradiction. Therefore, we consider $p_4 \leq 113$. We calculate G and see that each of the cases in Table 2 gives a contradiction.

p_4	α_i	p_4	α_i
$23 \leq p_4 \leq 47$	$\alpha_1 \geq 4$	$79 \leq p_4 \leq 109$	$\alpha_1 \geq 6$, $\alpha_2 \geq 4$
$53 \leq p_4 \leq 73$	$\alpha_1 \geq 4$, $\alpha_2 \geq 4$	$p_4 = 113$	$\alpha_1 \geq 8$, $\alpha_2 \geq 4$

Table 2: The condition of α_i, when $p_2 = 5$, $p_3 = 19$.

So, we will discuss the remaining cases. If $\alpha_1 = 2$ and $D \geq 225$, then from (4) we have $\frac{\sigma(n)}{n} + \frac{1}{D} < 1.9970$, which implies a contradiction. If $\alpha_1 = 4$, or $\alpha_2 = 2$, we only need to consider $79 \leq p_4 \leq 113$ or $53 \leq p_4 \leq 113$, respectively. Since $\sigma(3^4) \mid (2D-1)n$ or $\sigma(5^2) \mid (2D-1)n$, we get $D \geq 303$ or $D \geq 419$, respectively. By (4), we have $\frac{\sigma(n)}{n} + \frac{1}{D} < 1.9996$, which is a contradiction. If $\alpha_1 = 6$, then we only need to consider $p_4 = 113$ and then $D \leq 315$, since $\sigma(3^6) = 1093 \mid (2D-1)n$, which is also impossible. This completes the case $p_3 = 19$.

3.4. The Case $p_3 = 23$

In this case, we have

$$\sigma(3^{\alpha_1})\,\sigma(5^{\alpha_2})\,\sigma(23^{\alpha_3})\,\sigma(p_4^{\alpha_4}) = \left(2 - \frac{1}{D}\right) \cdot 3^{\alpha_1} \cdot 5^{\alpha_2} \cdot 23^{\alpha_3} \cdot p_4^{\alpha_4}. \quad (8)$$

First, we suppose $D \leq 109$. If $\alpha_1 = 2, 4, 6$, or $\alpha_2 = 2, 4$, or $\alpha_3 = 2$, using the same method, we get a contradiction. Hence, $\alpha_1 \geq 8$, $\alpha_2 \geq 6$, and $\alpha_3 \geq 4$.

If $15 \leq D \leq 25$, we have $G > 0$, which contradicts (3). If $27 \leq D \leq 109$, then we get $D = 27, 45, 69, 75, 81$ or $D = p_4$, where $29 \leq p_4 \leq 109$ or $D = 3p_4$, where $29 \leq p_4 \leq 31$. If $D = p_4 \geq 79$, from (4) we have $\frac{\sigma(n)}{n} + \frac{1}{D} < 1.9981$. If $D = p_4 \leq 73$, or $D = 3p_4$, then $G > 0$, which contradicts (3). If $D = 27, 45, 69, 75, 81$, then $2D - 1 = 53, 89, 137, 149, 7 \cdot 23$ and from (4) we get $p_4 \leq 709$, $p_4 \leq 109$, $p_4 \leq 73$, $p_4 \leq 73$, $p_4 \leq 71$, respectively. We verify that $\mathrm{ord}_5 p$ is even for $p = 3, 7, 23, 53, 89, 137$. Hence, if $D \neq 75$, we see that $\gcd(\sigma(5^{\alpha_2}), (2D-1) \cdot 3 \cdot 5 \cdot 23) = 1$. From Equation (8), we have $\sigma(5^{\alpha_2}) = p_4^{\beta}$ $(1 \leq \beta \leq \alpha_4)$. By Lemma 2, we get $\beta = 1$, then $p_4 = \sigma(5^{\alpha_2})$, and one directly verifies that this is impossible. If $D = 75$, then $p_4 \leq 73$ and $G > 0$, which contradicts (3).

Second, we suppose that $D \geq 111$. If $p_4 \geq 67$, from (4) we have $\frac{\sigma(n)}{n} + \frac{1}{D} < 1.99894$, so we get $p_4 \leq 61$. When $29 \leq p_4 \leq 47$, we calculate the values of G as follows: when $29 \leq p_4 \leq 31$, $G > 0$ ($\alpha_1 \geq 4$), when $37 \leq p_4 \leq 41$, $G > 0$ ($\alpha_1 \geq 4$, $\alpha_2 \geq 4$), when $p_4 = 47$, $G > 0$ ($\alpha_1 \geq 6$, $\alpha_2 \geq 4$). Each of the above cases gives a contradiction. Thus, we will discuss the remaining cases. If $\alpha_1 = 2$, $p_4 \geq 29$, and as $D \geq 111$, using (4) we have $\frac{\sigma(n)}{n} + \frac{1}{D} < 1.9641$, a contradiction. If $\alpha_2 = 2$, we only need to consider $37 \leq p_4 \leq 47$. By Equation (8), we deduce that $D \geq 729$. From (4), we have $\frac{\sigma(n)}{n} + \frac{1}{D} < 1.99994$, a contradiction. If $\alpha_1 = 4$, we only need to consider $p_4 = 47$. Using (8), we get $D \geq 20001$. From (4), we have $\frac{\sigma(n)}{n} + \frac{1}{D} < 1.9947$, which is a contradiction.

When $p_4 = 53$, 59, 61, by Equation (8) we deduce that $D \leq 405$ or $D = 477$ ($p_4 = 53$), $D \leq 135$ ($p_4 = 59$), $D \leq 135$ ($p_4 = 61$). We calculate the values of G as follows. If $p_4 = 53$, we have $G > 0$ when $D \leq 405$, $\alpha_1 \geq 8$, $\alpha_2 \geq 6$, if $p_4 = 59$, we have $G > 0$ when $D \leq 135$, $\alpha_1 \geq 8$, $\alpha_2 \geq 4$, and if $p_4 = 47$, we have $G > 0$ when $D \leq 135$, $\alpha_1 \geq 8$, $\alpha_2 \geq 4$. They all give a contradiction. So, we will discuss the other cases. If $\alpha_1 \leq 4$, or $\alpha_2 = 2$, then (4) implies $\frac{\sigma(n)}{n} + \frac{1}{D} < 1.9988$, a contradiction. If $\alpha_1 = 6$, or $\alpha_2 = 4$, from (8) we have $\sigma(5^6) = 11093 \mid (2D-1)n$, or $\sigma(5^4) = 11 \cdot 71 \mid (2D-1)n$. Thus, we see that it is impossible. If $p_4 = 53$, $D = 477$, since all $\mathrm{ord}_{p_i} 5$ are even for $i = 1, 3, 4$, then $\gcd(\sigma(p_i^{\alpha_i}), 5) = 1$ for $i = 1, 2, 3, 4$. From (8), this is also impossible. This completes the case $p_3 = 23$.

3.5. The Case $p_3 = 29$

In this case, we have

$$\sigma(3^{\alpha_1})\,\sigma(5^{\alpha_2})\,\sigma(29^{\alpha_3})\,\sigma(p_4^{\alpha_4}) = \left(2 - \frac{1}{D}\right) \cdot 3^{\alpha_1} \cdot 5^{\alpha_2} \cdot 29^{\alpha_3} \cdot p_4^{\alpha_4}. \tag{9}$$

First, we consider $D \leq 73$. When $\alpha_1 = 2$, or $\alpha_1 = 4$, or $\alpha_2 = 2$, we see that it is impossible. Thus, we assume that $\alpha_1 \geq 6$ and $\alpha_2 \geq 4$.

As $15 \leq D \leq 73$, we get $D = 15, 25, 27, 45$, or $D = p_4$ ($31 \leq p_4 \leq 73$). If $D = p_4 \geq 53$, from (4) we have $\frac{\sigma(n)}{n} + \frac{1}{D} < 1.9982$, a contradiction. If $D = p_4 \leq 47$, then $G > 0$, which also contradicts (3). If $D = 15$, since $\alpha_1 \geq 6$ and $\alpha_2 \geq 4$, we see that $G > 0$, which also contradicts (3). If $D = 25$, 27, 45, from (4) we get $p_4 \leq 107$, 89, 53, $G > 0$ ($\alpha_1 \geq 8, \alpha_2 \geq 6$), $G > 0$ ($\alpha_1 \geq 8, \alpha_2 \geq 4$), and $G > 0$ ($\alpha_1 \geq 8, \alpha_2 \geq 4$), respectively. This contradicts (3). When $\alpha_1 = 6$ or $\alpha_2 = 4$, by Equation (9) we have $\sigma(3^6) \mid (2D-1)n$ or $\sigma(5^4) \mid (2D-1)n$, respectively, it is impossible.

Second, we assume that $D \geq 75$. If $p_4 \geq 47$, from (4) we have $\frac{\sigma(n)}{n} + \frac{1}{D} < 1.9976$, so we take $p_4 \leq 43$. If $p_4 = 31$ and $\alpha_1 \geq 6$, $\alpha_1 \geq 4$, we get $G > 0$, a contradiction. When $\alpha_1 = 2$, or $\alpha_1 = 4$ and $D \geq 641$, or $\alpha_2 = 2$ and $D \geq 107$, by (4) we have $\frac{\sigma(n)}{n} + \frac{1}{D} < 2$, another contradiction. When $\alpha_1 = 4$ and $D \leq 639$, or $\alpha_2 = 2$ and $D \leq 105$, by Equation (9) we get $\sigma(3^4) \mid (2D-1)n$, or $\sigma(5^2) \mid (2D-1)n$, respectively.

This is impossible.

If $p_4 = 37$, 41, 43, by (4) we have $D \le 243$, 105, 85, and using the expression of D, we see that $D \le 225$ or $D = 243$, $D \le 87$, $D \le 81$, respectively. We compute the corresponding values of G and get $G > 0$ ($D \le 225$, $\alpha_1 \ge 8, \alpha_2 \ge 6$), $G > 0$ ($\alpha_1 \ge 6, \alpha_2 \ge 4$), and $G > 0$ ($\alpha_1 \ge 8, \alpha_2 \ge 6$), respectively. When $\alpha_1 = 2$, or $\alpha_2 = 2$, by (4) we have $\frac{\sigma(n)}{n} + \frac{1}{D} < 1.9933$, a contradiction. When $\alpha_1 = 4$, or $\alpha_1 = 6$, or $\alpha_2 = 4$, by (9) we have $\sigma(3^4) \mid (2D-1)n$, or $\sigma(3^6) \mid (2D-1)n$, or $\sigma(5^4) \mid (2D-1)n$, respectively. This is impossible. If $p_4 = 37$ and $D = 243$, then $5 \mid \sigma(n)$. However, all $\text{ord}_{p_i}5$ are even for $i = 1, 3, 4$, thus $\gcd(\sigma(p_i^{\alpha_i}), 5) = 1$ for $i = 1, 2, 3, 4$, a contradiction. This completes the case $p_3 = 29$.

3.6. The Case $p_3 = 31$

In this case, using (5) and the ideas of the previous discussion, we get $15 \le D \le 115$.

If $D = 15$, then $2D - 1 = 29$. As $\text{ord}_3 5 = 2$, $\text{ord}_3 29 = 28$, and $\text{ord}_3 31 = 30$, from Equation (1) we have $\sigma(3^{\alpha_1}) = p_4^{\beta}$, $1 \le \beta \le \alpha_4$. By Lemma 2, we have $\beta = 1$ and $p_4 = \sigma(3^{\alpha_1})$. If $\alpha_1 = 2$ or $\alpha_1 = 4$, then $p_4 = 13$ or $p_4 = 11$. This is impossible and we deduce that $\alpha_1 \ge 6$. If $\alpha_2 = 2$, from (4) we get $p_4 \le 167$. However, $p_4 = \sigma(3^{\alpha_1}) \le 167$, which is also impossible and $\alpha_2 \ge 4$. From (3), we get $G > 0$, a contradiction.

Now, we assume that $D \ge 25$. From (4), we get $p_4 \le 83$. By the expression of D and (4), we find $D = 25$, 27, 31, 45, 75, 81, 93, 111, or $D = p_4$ ($37 \le p_4 \le 47$). If $D = 25$, 27, 31, 41, 43, 45, 47, 75, then $2D-1 = 7^2$, 53, 61, 3^4, $5 \cdot 17$, 89, $3 \cdot 31$, 149. Since all $\text{ord}_3 p$ are even for $p = 5, 7, 17, 31, 53, 61, 89, 149$, then from Equation (2) we get

$$\sigma(3^{\alpha_1}) = p_4^{\beta}, \;\; 1 \le \beta \le \alpha_4.$$

Using Lemma 2, we have $\beta_4 = 1$ and $p_4 = \sigma(3^{\alpha_1})$. Note that $p_4 \le 83$, so we directly verify that this is impossible. If $D = 81$, 93, 111, from (2) we have $p_4 = 37$. As $\text{ord}_3 37 = 18$, from (2) we have $\sigma(3^{\alpha_1}) \mid (2D-1)$. We directly verify that this is also impossible. If $D = p_4 = 37, 41, 43$, we verify that $3 \pmod{p_i}$ $(i = 1, 3, 4)$ has an even order, from (2) we see that this is impossible. If $D = p_4 = 47$, from (2) we verify that $\alpha_1 \ne 2, 4, 6$ and $\alpha_2 \ne 2$, then $\alpha_1 \ge 8$ and $\alpha_2 \ge 4$. We deduce that $G > 0$, a contradiction. This completes the case $p_3 = 31$.

3.7. The Case $37 \le p_3 \le 61$

From (4) and the expression of D, we get: if $p_3 = 37$ then $D \le 39$ so $D = 15$, 25, 27, 37; if $p_3 = 41$ then $D \le 29$ so $D = 15$, 25, 27; if $p_3 = 43$ then $D \le 25$ so $D = 15$, 25; and if $p_3 = 47$, 53, 59, 61, then $D = 15$.

If $p_3 \ne 47$, 59, we verify that all $\text{ord}_3 p_3$ are even and all $\text{ord}_3 p$ are also even, where p is any prime satisfying $p \mid (2D-1)$. Thus, from Equation (1) we have $\sigma(3^{\alpha_1}) = p_4^{\beta}$, $1 \le \beta \le \alpha_4$. Lemma 2.2 implies that $\beta_4 = 1$ and then $p_4 = \sigma(3^{\alpha_1})$.

From (4), we have $p_4 \leq 307$, so we directly verify that Equation (1) has no integer solutions.

If $p_3 = 47$ and since $\text{ord}_5 3 = 2$, $\text{ord}_5 47 = 46$, $\text{ord}_5 2D - 1 = \text{ord}_5 29 = 14$, then from (1) and Lemma 2.2, we obtain $p_4 = \sigma(5^{\alpha_2})$. Using (4), we have $p_4 \leq 109$, which is impossible.

If $p_3 = 59$ and as $D = 15$, from (4) we get $61 \leq p_4 \leq 73$. If $p_4 = 61, 67, 73$, we verify that all $\text{ord}_3 p_4$ are even. Thus, using Equation (1) and Lemma 2.2, we obtain $p_3 = \sigma(3^{\alpha_2})$, which is impossible. If $p_4 = 71$ and $\alpha_1 \leq 4$ or $\alpha_2 \leq 4$, then by (1) we verify that it is also impossible. Therefore, $\alpha_1 \geq 6$ and $\alpha_2 \geq 6$, and we get $G > 0$. This contradicts (3).

4. The Odd Deficient-Perfect Numbers of the Form $3^{\alpha_1} \cdot 7^{\alpha_2} \cdot p_3^{\alpha_3} \cdot p_4^{\alpha_4}$

In this section, we will discuss the case $p_2 = 7$. We will show that the only odd near-perfect number of the form $3^{\alpha_1} \cdot 7^{\alpha_2} \cdot p_3^{\alpha_3} \cdot p_4^{\alpha_4}$ is $n = 3^2 \cdot 7^2 \cdot 11^2 \cdot 13^2$.

If $D = 3$, then from (5) we have $\frac{\sigma(n)}{n} + \frac{1}{3} > \frac{\sigma(3^2)}{3^2} \cdot \frac{\sigma(7^2)}{7^2} + \frac{1}{3} > 2.1244$, which is a contradiction. Therefore, $D \geq 7$. Furthermore, from (4), we deduce that $p_3 \leq 31$. We will consider four cases.

4.1. The Case $p_3 = 11$

In this case, from Equation (1) we have

$$\sigma(3^{\alpha_1})\,\sigma(7^{\alpha_2})\,\sigma(11^{\alpha_3})\,\sigma(p_4^{\alpha_4}) = \left(2 - \frac{1}{D}\right) \cdot 3^{\alpha_1} \cdot 7^{\alpha_2} \cdot 11^{\alpha_3} \cdot p_4^{\alpha_4}. \tag{10}$$

First, we consider $D \leq 11$. If $\alpha_1 \geq 4$, we obtain $G > 0$, which contradicts (3). If $\alpha_1 = 2$, then from Equation (10) we have $\sigma(3^2) = 13 \mid (2D-1)p_4$, so $p_4 = 13$ or $D = 7$. If $p_4 = 13$ and as $D \leq 11$, to give $G > 0$. When $D = 7$, if $p_4 \geq 541$, by (4) we have $\frac{\sigma(n)}{n} + \frac{1}{D} < 2$, thus $p_4 \leq 523$. In this case, when $\alpha_2 \geq 6$ and $\alpha_3 \geq 4$, we get $G > 0$, a contradiction. When $\alpha_2 = 2$ or $\alpha_3 = 2$, from (10) we have $\sigma(7^2) = 57 \mid n$ or $\sigma(11^2) = 119 \mid n$, So $p_4 = 19$ and $G > 0$. When $\alpha_2 = 4$, from (10) we have $\sigma(7^4) = 2801 \mid n$, then $p_4 = 2801$, which is impossible.

If $D = 13, 17, 19$, then from Equation (10) we have $p_4 = 13, 17, 19$ respectively and $G > 0$. If $D = 15, 25$, then $5 \mid n$, which is impossible. If $D = 21$, from (10) we get $41 \mid \sigma(n)$. When $p_4 \geq 73$, by (4) we have $\frac{\sigma(n)}{n} + \frac{1}{D} < 1.9994$. When $13 \leq p_4 \leq 71$, we verify that all $\text{ord}_{p_4} 41$ are even, except for $p_4 = 37, 59$. Note that $\text{ord}_3 41, \text{ord}_7 41, \text{ord}_{11} 41$ are also even. Thus, if $p_4 \neq 37, 59$, we have $\gcd(\sigma(p_i^{\alpha_i}), 41) = 1$ for $i = 1, 2, 3, 4$, which is impossible. When $p_4 = 37, 59$, if $\alpha_1 = 2$ or $\alpha_2 = 2$, from (10) we have $\sigma(3^2) \mid 41n$ or $\sigma(5^2) \mid 41n$, which is impossible. If $\alpha_1 = 4$, we see that $\frac{\sigma(n)}{n} + \frac{1}{D} < 1.9978$ or $G > 0$, respectively giving a contradiction. Thus, we take $\alpha_1 \geq 6$ and $\alpha_1 \geq 4$ to obtain $G > 0$, which contradicts

(3). If $D = 23$, then from (10) we deduce that $p_4 = 23$. If $\alpha_1 = 2$, by (10) we have $\sigma(3^2) \,|\, (2D-1) = 45$, a contradiction. If $\alpha_1 \geq 4$, we get $G > 0$, which also contradicts (3).

Now, we consider $D \geq 27$. If $p_4 \geq 53$, from (4) we have $\frac{\sigma(n)}{n} + \frac{1}{D} < 1.9991$, a contradiction. If $p_4 = 29, 31, 37, 41, 43, 47$, from (4) we have $D \leq 159, 91, 45, 37, 33, 29$, and by the expression of D, we see that $D \leq 147$, $D \leq 81$, $D = 27, 33, 37$, $D = 27, 33$, $D = 27, 33$, $D = 27$, respectively. Similarly, we get contradictions.

If $p_4 = 23$, when $\alpha_1 \geq 6$ or $\alpha_1 \geq 4$ and $\alpha_2 \geq 4$, we obtain $G > 0$. When $\alpha_1 = 2$, by (4) we have $\frac{\sigma(n)}{n} + \frac{1}{D} < 1.9751$. When $\alpha_1 = 4$ and $\alpha_2 = 2$, by (10) we have $19 \,|\, 2D - 1$. Using the expression of D one sees that $D \geq 1587$, so from (4) we have $\frac{\sigma(n)}{n} + \frac{1}{D} < 1.9991$. All these cases give a contradiction.

If $13 \leq p_4 \leq 19$ and when $\alpha_1 \geq 4$, we get $G > 0$, which contradicts (3). When $\alpha_1 = 2$ and $p_4 = 19$, from (4) we have $D \leq 23$, a contradiction. When $\alpha_1 = 2$ and $p_4 = 17$, since $\mathrm{ord}_3 17$, $\mathrm{ord}_7 17$ and $\mathrm{ord}_{11} 17$ are all even, then $\gcd(\sigma(p_i^{\alpha_i}), 17) = 1$ for $i = 1, 2, 3, 4$. From (10), we have $17^{\alpha_3} \,|\, D$, so $D \geq 17^2 = 289$. By (4), we have $\frac{\sigma(n)}{n} + \frac{1}{D} < 1.974$, a contradiction. When $\alpha_1 = 2$ and $p_4 = 13$, if $\alpha_2 \geq 4$, or $\alpha_3 \geq 4$, or $\alpha_4 \geq 4$, we see that $G > 0$. Thus, we obtain that $\alpha_2 = \alpha_3 = \alpha_4 = 2$, i.e. So

$$n = 3^2 \cdot 7^2 \cdot 11^2 \cdot 13^2, \tag{11}$$

and $d = 2n - \sigma(n) = 2 \cdot 3^2 \cdot 7^2 \cdot 11^2 \cdot 13^2 - \sigma(3^2 \cdot 7^2 \cdot 11^2 \cdot 13^2) = 3^2 \cdot 7 \cdot 13$. Therefore, $n = 3^2 \cdot 7^2 \cdot 11^2 \cdot 13^2$ is an odd deficient-perfect number.

4.2. The Case $p_3 = 13$

In this case, Equation (1) implies

$$\sigma(3^{\alpha_1})\,\sigma(7^{\alpha_2})\,\sigma(13^{\alpha_3})\,\sigma(p_4^{\alpha_4}) = \left(2 - \frac{1}{D}\right) \cdot 3^{\alpha_1} \cdot 7^{\alpha_2} \cdot 13^{\alpha_3} \cdot p_4^{\alpha_4}. \tag{12}$$

We first take $D \leq 9$. If $\alpha_1 \geq 6$ and $\alpha_2 \geq 4$, we see that $G > 0$. If $\alpha_1 = 2$, then from (4) we have $p_4 \leq 53$ when $D = 7$, and $p_4 \leq 29$ when $D = 9$. When $D = 7$ and $p_4 \leq 53$, we verify that $2 \,|\, \mathrm{ord}_7 p_4$ for $p_4 = 17, 23, 41, 43, 53$, and $2 \,|\, \mathrm{ord}_7 p_4$ for $p_4 = 19, 29, 31, 37, 47$. Since $\mathrm{ord}_7 13 = 12$ and $\mathrm{ord}_{13} 7 = 2$, then we have $\gcd(\sigma(7^{\alpha_2}), 7 \cdot 13 \cdot p_4) = 1$, or $\gcd(\sigma(13^{\alpha_2}), 7 \cdot 13 \cdot p_4) = 1$. By Equation (12), we obtain $\sigma(7^{\alpha_2}) = 3^2$, or $\sigma(13^{\alpha_2}) = 3^2$, which is impossible. When $D = 9$ and $p_4 \leq 29$, then $2D - 1 = 17$, and we verify that $\mathrm{ord}_{p_i} 17$ are all even for $i = 2, 3, 4$, so $\gcd(\sigma(7^{\alpha_2})\,\sigma(13^{\alpha_3})\,\sigma(31^{\alpha_4}),\, 17) = 1$. Thus, Equation (12) leads to an absurdity. If $\alpha_1 = 4$, from (12) we have $\sigma(3^4) = 11^2 \,|\, (2D-1)p_4$, note that $7 \leq D \leq 9$, which is impossible. If $\alpha_2 = 2$, from (12) we have $p_4 = 31$. When $D = 7$, the calculation gives that $G > 0.01169$, a contradiction. When $D = 9$, we get $2D - 1 = 17$ and similarly we see that this is impossible.

If $D = 11$, 15, 25, then $11 \,|\, n$, or $5 \,|\, n$, which is a contradiction.

If $D = 13$, then $2D - 1 = 25$. As $\mathrm{ord}_3 5 = \mathrm{ord}_7 5 = \mathrm{ord}_{13} 5 = 4$, from Equation (12) we have $5 \mid \sigma(p_4^{\alpha_4})$ and then $p_4 \equiv 1 \pmod{10}$. In fact, if $p_4 \equiv 3, 7, 9 \pmod{10}$, thus all $\mathrm{ord}_{p_4} 5$ are even and $\gcd(p_4^{\alpha_4}, 5) = 1$. Using (4), we get $p_4 \leq 67$, i.e. $p_4 = 31$, 41, 61. When $\alpha_1 = 2$, from (4) we have $\frac{\sigma(n)}{n} + \frac{1}{13} < 1.9634$. When $\alpha_1 = 4$, then $\sigma(3^4) = 11^2 \mid 5n$, contradicting (12). When $\alpha_1 \geq 6$ and $p_4 = 31$ or $p_4 = 41$, we see that $G > 0$ respectively, a contradiction. When $\alpha_1 \geq 6$ and $p_4 = 61$, if $\alpha_2 = 2$, then $\sigma(7^2) = 57 \mid 5n$, which is impossible, so $\alpha_2 \geq 4$. In this case, we get $G > 0$. This also leads to a contradiction to (3).

If $D = 17, 19, 23$, then $p_4 = 17, 19, 23$, respectively. When $\alpha_1 \geq 4$, then $G > 0$, giving a contradiction. When $\alpha_1 = 2$, from (4) we have $\frac{\sigma(n)}{n} + \frac{1}{D} < 1.9986$, which is also a contradiction.

If $D = 21$, then $2D - 1 = 41$ and $41 \mid \sigma(n)$. From (4) we get $p_4 \leq 31$. Since we verify that $\mathrm{ord}_{p_i} 41$ is even for $i = 1, 2, 3, 4$, we have $\gcd(\sigma(p_i^{\alpha_i}), 41) = 1$, where $i = 1, 2, 3, 4$. Therefore, we get a contradiction.

Now, we suppose that $D \geq 27$; hence, from (4) we get $p_4 \leq 29$. We use the same method as above for $p_4 = 17, 19, 23, 29$ and come to an impossible conclusion. This ends the study for $p_3 = 13$.

4.3. The Case $p_3 = 17$

If $p_3 = 17$, from Equation (1) we have

$$\sigma(3^{\alpha_1})\,\sigma(7^{\alpha_2})\,\sigma(17^{\alpha_3})\,\sigma(p_4^{\alpha_4}) = \left(2 - \frac{1}{D}\right) \cdot 3^{\alpha_1} \cdot 7^{\alpha_2} \cdot 17^{\alpha_3} \cdot p_4^{\alpha_4}. \tag{13}$$

If $D \geq 27$, then, by (4), we have $\frac{\sigma(n)}{n} + \frac{1}{D} < \frac{3}{2} \cdot \frac{7}{6} \cdot \frac{17}{16} \cdot \frac{19}{18} + \frac{1}{27} < 1.9992$, which is a contradiction. So $D \leq 25$. By the expression of D, we see that $D = 7$, 9, 17, 19, 21, 23.

Take $D = 7$. When $\alpha_1 \geq 6$ and $\alpha_2 \geq 4$, then $G > 4.803 \cdot 10^{-4}$, which is inconsistent with (3). If $\alpha_1 = 2$ or $\alpha_2 = 2$, from (4) we have $p_4 = 19$, 23 or by Equation (13) we have $\sigma(7^{\alpha_2}) = 3 \cdot 19 \mid 3p_4$, then $p_4 = 19$. Since $\mathrm{ord}_3 17 = \mathrm{ord}_7 17 = 16$ and $\mathrm{ord}_{p_4} 17$ are even for $p_4 = 19$, 23, then $\gcd(\sigma(p_i^{\alpha_i}), 17) = 1$, for $i = 1, 2, 3, 4$. Thus, we get a contradiction. If $\alpha_1 = 4$, from (13) we get $11 \mid n$, which is impossible.

If $D = 9$, 17, 19, 21, 23, from (4) and (13), we get $p_4 \leq 61$, $p_4 \leq 23$, $p_4 = 19$, $p_4 = 19$, and $p_4 = 19$, respectively. For the last value, it is obviously impossible. For the first four values of D, we verify that all $\mathrm{ord}_{p_i} 17$ are even, where $i = 1, 2, 4$, then $\gcd(\sigma(p_i^{\alpha_i}), 17) = 1$ holds for $i = 1, 2, 3, 4$. Therefore, from (13) we have $17^{\alpha_3} \mid D$, which is impossible.

4.4. The Case $19 \leq p_3 \leq 31$

In this case, using (4) and the expression of D, we determine the values of D: if $p_3 = 19, 23$, we have $D = 7$, 9 and if $p_3 = 29, 31$, we have $D = 7$. Taking $D = 9$, we have $2D-1=17$ and from (4) we get $p_4 \le 47$. We verify that all $\mathrm{ord}_{p_i}17$ are even for $i=1,2,3,4$ and then $\gcd(\sigma(p_i^{\alpha_i}),\, 17)=1$, which is impossible.

Now, we suppose that $D=7$. From (4), we determine the values of p_4 according to the values of p_3. If $p_3 = 19, 23, 29, 31$, we have $p_4 \le 181$, $p_4 \le 67$, $p_4 \le 41$, $p_4 \le 37$, respectively. If $p_3 = 19$, we see that $G > 0$ according to α_i as outlined in Table 3. Each of the cases in Table 3 leads to a contradiction. When $\alpha_1 = 2$

p_4	α_i	p_4	α_i
$23 \le p_4 \le 79$	$\alpha_1 \ge 4$	$167 \le p_4 \le 173$	$\alpha_1 \ge 8,\ \alpha_2 \ge 4$
$83 \le p_4 \le 113$	$\alpha_1 \ge 6$	$179 \le p_4 \le 181$	$\alpha_1 \ge 8,\ \alpha_2 \ge 4,\ \alpha_3 \ge 4$
$127 \le p_4 \le 163$	$\alpha_1 \ge 6,\ \alpha_2 \ge 4$		

Table 3: The condition of α_i, when $p_2 = 7$, $p_3 = 19$.

and $p_4 \ge 29$, from (4) we have $\frac{\sigma(n)}{n} + \frac{1}{7} < 1.9852$, a contradiction. When $\alpha_1 = 2$ and $p_4 = 23$, since $\mathrm{ord}_{19}7 = 6$ and $\mathrm{ord}_{19}23 = 22$, from Equation (1) we have $\sigma(19^{\alpha_3}) = 9$, which is impossible. When $\alpha_2 = 2$ and as $p_4 \ge 127$, from (4) we have $\frac{\sigma(n)}{n} + \frac{1}{7} < 1.9994$, a contradiction. When $\alpha_1 = 4$, or $\alpha_1 = 6$, or $\alpha_3 = 2$, from Equation (1) we have $\sigma(3^4) \mid 13n$, or $\sigma(3^6) \mid 13n$, or $\sigma(19^2) \mid 13n$. This is impossible. If $p_3 = 23$, we determine the values of G that satisfy $G > 0$, the condition of α_i in Table 4. All of the cases in Table 4 give a contradiction. Using the same method,

p_4	α_i	p_4	α_i
$29 \le p_4 \le 43$	$\alpha_1 \ge 4$	$59 \le p_4 \le 61$	$\alpha_1 \ge 6,\ \alpha_2 \ge 4$
$47 \le p_4 \le 53$	$\alpha_1 \ge 6$	$p_4 = 67$	$\alpha_1 \ge 8,\ \alpha_2 \ge 6,\ \alpha_3 \ge 4$

Table 4: The condition of α_i, when $p_2 = 7$, $p_3 = 23$.

we see that the rest of the situation is impossible. If $p_3 = 29$, we compute the values of G as follows. When $p_4 = 31$, we have $G > 0$ ($\alpha_1 \ge 4$), when $p_4 = 37$, we have $G > 0$ ($\alpha_1 \ge 8$), and when $p_4 = 41$ we have $G > 0$ ($\alpha_1 \ge 8$, $\alpha_2 \ge 4$). We get a contradiction. Similarly, using the same method, we can prove that the rest of the situation is also impossible. This completes Section 4 and the study of the case $p_2 = 7$.

5. The Odd Deficient-Perfect Numbers of the Form $3^{\alpha_1} \cdot p_2^{\alpha_2} \cdot p_3^{\alpha_3} \cdot p_4^{\alpha_4}$ with $p_2 = 11$, 13, 17 and $19 \le p_2 \le 23$

5.1. The Case $p_2 = 11$

In this case, we first assume that $D \ge 9$. From (5), we get $p_3 = 13$, $p_4 = 17$, and $D = 9$. As $\mathrm{ord}_{17}3 = 2$, $\mathrm{ord}_{17}11 = 10$, and $\mathrm{ord}_{17}13 = 6$, we obtain $\gcd\big(\sigma(17^{\alpha_4}), 3 \cdot$

$11 \cdot 13 \cdot 17\big) = 1$, which is impossible by Equation (1).

Now, we consider $D = 3$. From (1), we have

$$\sigma(3^{\alpha_1})\,\sigma(11^{\alpha_2})\,\sigma(p_3^{\alpha_3})\,\sigma(p_4^{\alpha_4}) = 5 \cdot 3^{\alpha_2 - 1} \cdot 11^{\alpha_2} \cdot p_3^{\alpha_3} \cdot p_4^{\alpha_4}. \tag{14}$$

If $p_3 \geq 199$, (4) implies $\frac{\sigma(n)}{n} + \frac{1}{3} < 2$, a contradiction; thus, $p_3 \leq 197$. We will discuss the following cases: $13 \leq p_3 \leq 97$ and $101 \leq p_3 \leq 197$.

When $13 \leq p_3 \leq 97$, we determine the condition of α_i when $G > 0$ as outlined in Table 5. All these results in Table 5 are inconsistent with (3). If $\alpha_1 = 2$, or $\alpha_1 = 6$,

p_3	α_i	p_3	α_i
$13 \leq p_3 \leq 19$	$\alpha_1 \geq 2$	$71 \leq p_3 \leq 89$	$\alpha_1 \geq 6,\ \alpha_2 \geq 4$
$23 \leq p_3 \leq 67$	$\alpha_1 \geq 4$	$p_3 = 97$	$\alpha_1 \geq 8,\ \alpha_2 \geq 4$

Table 5: The condition of α_i when $p_2 = 11$, $13 \leq p_3 \leq 97$.

or $\alpha_2 = 2$, or $\alpha_2 = 4$, from Equation (14) we have $\sigma(3^2) \mid n$, or $\sigma(3^6) \mid n$, or $\sigma(11^2) \mid n$, or $\sigma(11^4) \mid n$. A study of the values of p_3 gives $p_3 = 97$ and $p_4 = 1093\,(\alpha_1 = 6)$, $3221\,(\alpha_2 = 4)$. When $p_3 = 97$ and $p_4 = 1093$, we have $G > 0$, a contradiction. When $p_3 = 97$ and $p_4 = 3221$, since $\mathrm{ord}_3 p_i$ is even for $i = 3, 4$ and $\mathrm{ord}_3 5 = 4$, then $\gcd(\sigma(3^{\alpha_1}), 5 \cdot 3 \cdot 97 \cdot p_4) = 1$, by (14) we get $\sigma(3^{\alpha_1}) = 11^{\beta} (1 \leq \beta \leq \alpha_2)$. Using Lemma 2.2, we have $\alpha_1 = 4$ and (4) implies $\frac{\sigma(n)}{n} + \frac{1}{3} < 1.9942$, a contradiction. Finally, it remains to consider $\alpha_1 = 4$ and $71 \leq p_3 \leq 97$. We use (3) and (4) to obtain the condition of α_i when $G > 0$ as outlined in Table 6. All these results in

p_3	p_4	α_i	p_3	p_4	α_i
71		$\alpha_2 \geq 4$	83	$89 \leq p_4 \leq 487$	$\alpha_2 \geq 6$
73	$79 \leq p_4 \leq 2621$	$\alpha_2 \geq 6,\ \alpha_3 \geq 4)$	89	$97 \leq p_4 \leq 347$	$\alpha_2 \geq 4$
79	$83 \leq p_4 \leq 691$	$\alpha_2 \geq 4$	97	$101 \leq p_4 \leq 257$	$\alpha_2 \geq 4$

Table 6: The condition of α_i, when $p_2 = 11$, $71 \leq p_3 \leq 97$.

Table 6 are impossible. When $\alpha_2 = 2$ or $\alpha_2 = 4$, by (14) we have $\sigma(11^2) = 7 \cdot 19 \mid n$ or $\sigma(11^4) = 5 \cdot 3221 \mid n$. When $\alpha_3 = 2$ and $p_3 = 73$, as $\sigma(73^2) = 3 \cdot 1801 \mid n$ we get $p_4 = 1801$ and then $G > 0$. This leads to a contradiction.

When $101 \leq p_3 \leq 197$, for each p_3 fixed, we use (4) to find the range of p_4 as outlined in Table 7. One can see that $G > 0$ for the following two conditions: $101 \leq p_3 \leq 107$ $(\alpha_1 \geq 14,\ \alpha_2 \geq 6,\ \alpha_3 \geq 4)$; $109 \leq p_3 \leq 197$ $(\alpha_1 \geq 12,\ \alpha_2 \geq 6)$.

Now, we discuss the remaining cases. If $\alpha_1 = 2, 6, 8, 10, 12$, or $\alpha_2 = 2, 4$, or $\alpha_3 = 2$, by Equation (14) we have $\sigma(3^2) = 13 \mid 5n$, or $\sigma(3^6) = 1093 \mid 5n$, or $\sigma(3^8) = 13 \cdot 757 \mid 5n$, or $\sigma(3^{10}) = 23 \cdot 3851 \mid 5n$, or $\sigma(3^{12}) = 797671 \mid 5n$, or $\sigma(11^2) = 7 \cdot 19 \mid n$, or $\sigma(11^4) = 5 \cdot 3221 \mid n$, or $\sigma(p_3^2) \mid n$. Thus, we obtain $101 \leq p_3 \leq 109$ and $p_4 = 1093\,(\alpha_1 = 6)$, or $101 \leq p_3 \leq 103$ and $p_4 = 3221\,(\alpha_2 = 4)$. When

p_3	p_4	p_3	p_4	p_3	p_4	p_3	p_4
101	$p_4 \leq 9973$	127	$p_4 \leq 463$	151	$p_4 \leq 293$	179	$p_4 \leq 223$
103	$p_4 \leq 3391$	131	$p_4 \leq 419$	157	$p_4 \leq 271$	181	$p_4 \leq 211$
107	$p_4 \leq 1511$	137	$p_4 \leq 367$	163	$p_4 \leq 257$	191	$p_4 = 199$
109	$p_4 \leq 1193$	139	$p_4 \leq 353$	167	$p_4 \leq 241$	193	$p_4 = 199$
113	$p_4 \leq 859$	149	$p_4 \leq 293$	173	$p_4 \leq 233$	197	$p_4 = 199$

Table 7: The range of p_4, when $p_2 = 11$, $101 \leq p_3 \leq 197$.

$101 \leq p_3 \leq 103$ and $p_4 = 3221$, $\alpha_2 = 4$. Since $\text{ord}_3 p_i$ is even, for $i = 3, 4$, we have $\gcd(\sigma(3^{\alpha_1}), 3 \cdot p_3 \cdot 3221) = 1$. From (14), we get $\sigma(3^{\alpha_1}) = 11^\beta (1 \leq \beta \leq 4)$. Then, we have $\alpha_1 = 4$, so from (4) we obtain $\frac{\sigma(n)}{n} + \frac{1}{3} < 2$, a contradiction. When $p_3 = 101$, 103 and $p_4 = 1093$, $\alpha_1 = 6$, as $\alpha_2 \geq 6$, we get $G > 0$, which is inconsistent with (3). When $p_3 = 107$, 109 and $p_4 = 1093$, since all $\text{ord}_{p_3} p$ are even for $p = 5$, 11, 1093, then $\gcd(\sigma(p_3^{\alpha_3}), 5 \cdot 11 \cdot p_3 \cdot 1093) = 1$. Using (14), we have $\sigma(p_3^{\alpha_3}) = 3^\beta$, where $1 \leq \beta \leq 5$, which is impossible.

Finally, we consider $\alpha_1 = 4$. If $p_3 \geq 139$, from (4) we have $\frac{\sigma(n)}{n} + \frac{1}{3} < 2$, a contradiction, and thus $101 \leq p_3 \leq 137$. By (4), we find the ranges of p_4 as follows: $p_4 \leq 223$, $p_4 \leq 199$, $p_4 \leq 199$, $p_4 \leq 181$, $p_4 \leq 157$, $p_4 \leq 151$, $p_4 = 139$, respectively. As $\alpha_1 = 4$ and $\alpha_2 \geq 6$, we obtain $G > 0$ in all cases. This contradicts (3).

5.2. The Case $p_2 = 13$

In this case, if $D \geq 9$, from (4) we have $\frac{\sigma(n)}{n} + \frac{1}{D} < 1.9336$. We deduce that $D = 3$. If $p_3 \geq 79$, then from (4) we get $\frac{\sigma(n)}{n} + \frac{1}{3} < 1.9993$. So, we have $p_3 \leq 73$.

If $17 \leq p_3 \leq 31$ and $\alpha_1 \geq 4$, one can see that $G > 0$. When $\alpha_1 = 2$, if $p_3 = 17$, 19, from (4) we have $p_4 \leq 409$, or $p_4 \leq 109$, respectively. If $p_3 = 17$ and $\alpha_2 = 2$, from (2) we have $\sigma(13^2) = 3 \cdot 61 \mid 5n$, then $p_4 = 61$. We get $G > 0$. Thus, $\alpha_2 \geq 4$. If $p_3 = 17$ and $\alpha_3 = 2$, from (2) we have $\sigma(17^2) = 307 \mid 5n$, then $p_4 = 307$. We get $G > 0$ also, so $\alpha_4 \geq 4$. In this case, we obtain $G > 0$, a contradiction. If $p_3 = 19$ and $\alpha_2 = 2$, from (2) we have $61 \mid 5n$, then $p_4 = 61$. We get $G > 0$, then $\alpha_2 \geq 4$, we see that $G > 0$, another contradiction. If $\alpha_1 = 2$ and $p_3 = 23$, from (4) we have $p_4 \leq 53$. In this case, if $\alpha_2 = 2$, from (2) we have $\sigma(13^2) = 3 \cdot 61 \mid 5n$, a contradiction, then $\alpha_2 \geq 4$. We deduce that $G > 0$, another contradiction. If $\alpha_1 = 2$ and $p_3 = 29$, from (4) we have $p_4 = 31$, and we get $G > 0$, a contradiction. If $\alpha_1 = 2$ and $p_3 = 31$, then from (4) we get $\frac{\sigma(n)}{n} + \frac{1}{3} < 1.9953$, a contradiction. If $p_3 = 37$ and $\alpha_1 \geq 6$, we have $G > 0$, and when $\alpha_1 = 4$, from (2) we have $\sigma(3^4) = 11^2 \mid 5n$, when $\alpha_1 = 2$, from (4) we get $\frac{\sigma(n)}{n} + \frac{1}{3} < 1.9819$, a contradiction.

If $41 \leq p_3 \leq 73$, for p_3 fixed, by (4), we get the range of p_4 as outlined in Table 8. In each case of Table 8, we use the values of $\alpha_1, \alpha_2, \alpha_3$ and the above method to prove that Equation (1) has no solution.

p_3	p_4	p_3	p_4	p_3	p_4
41	$43 \le p_4 \le 1597$	53	$59 \le p_4 \le 157$	67	$71 \le p_4 \le 97$
43	$47 \le p_4 \le 557$	59	$61 \le p_4 \le 113$	71	$73 \le p_4 \le 89$
47	$53 \le p_4 \le 257$	61	$67 \le p_4 \le 113$	73	$79 \le p_4 \le 83$

Table 8: The range of p_4, when $p_2 = 13$, $41 \le p_3 \le 73$.

5.3. The Case $p_2 = 17$

In this case, (4) and (5) imply that $19 \le p_3 \le 43$ and $D = 3$. If $p_3 = 19$, then $p_4 \ge 23$. When $\alpha_1 \ge 4$, we obtain $G > 0$. When $\alpha_1 = 2$, we have $\sigma(3^2) = 13 \mid n$, by (2), which is impossible.

If $p_3 = 23$, from Equation (1), we have

$$\sigma(3^{\alpha_1})\,\sigma(17^{\alpha_2})\,\sigma(23^{\alpha_3})\,\sigma(p_4^{\alpha_4}) = 5 \cdot 3^{\alpha_2 - 1} \cdot 17^{\alpha_2} \cdot 23^{\alpha_3} \cdot p_4^{\alpha_4}. \tag{15}$$

By Equation (15) we have $\alpha_1 \ge 6$, and from (3) and (4) we get $977 \le p_4 \le 3517$. Since $\mathrm{ord}_5 3 = \mathrm{ord}_5 23 = \mathrm{ord}_5 17 = 4$, then from (15) we have $5 \mid \sigma(p_4^{\alpha_4})$, and then $p_4 \equiv 1 \pmod 5$. Similarly, because $\mathrm{ord}_3 17 = \mathrm{ord}_3 23 = 2$, $\mathrm{ord}_{17} 3 = \mathrm{ord}_{17} 23 = 16$, then from (15) we have $3 \mid \sigma(p_4^{\alpha_4})$ and $17 \mid \sigma(p_4^{\alpha_4})$, then $p_4 \equiv 1 \pmod 3$ and $p_4 \equiv 1 \pmod{17}$. Thus, we obtain that $p_4 \equiv 1 \pmod{510}$. Since $977 \le p_4 \le 3517$, we get $p_4 = 1021, 1531, 2551$. If $p_4 = 1021, 1531$, since $\mathrm{ord}_{1021} 17 = 510$, $\mathrm{ord}_{1531} 23 = 510$, from (15) we get a contradiction, respectively. If $p_4 = 2551$, when $\alpha_1 \ge 10$ and $\alpha_2 \ge 4$, we get $G > 0$, a contradiction. When $\alpha_1 = 2$, 4, 6, 8, or $\alpha_2 = 2$, we have $13 \mid n$, $11^2 \mid n$, $1093 \mid n$, $13 \cdot 757 \mid n$, or $307 \mid n$, respectively, which is also impossible.

If $p_3 = 29$, 31, 37, 41, 43, then from (4) we have $p_4 \le 103$, $p_4 \le 83$, $p_4 \le 53$, $p_4 \le 47$, $p_4 = 47$ respectively. When $\alpha_1 \ge 8$ and $\alpha_2 \ge 4$, we see that $G > 0$. This is inconsistent with (3). When $\alpha_1 = 2$, 4, 6, or $\alpha_2 = 2$, by the same method, we get $13 \mid n$, $11^2 \mid n$, $1093 \mid n$, or $307 \mid n$, respectively. All this is impossible.

5.4. The Case $19 \le p_2 \le 23$

In this case, Equation (4) and (5) imply that $D = 3$. If $p_2 = 19$, then by (4) we have $p_3 \le 37$. And then we have $p_4 \le 139$ when $p_3 = 23$, $p_4 \le 61$ when $p_3 = 29$, $p_4 \le 53$ when $p_3 = 31$, $p_4 = 41$ when $p_3 = 37$, respectively. Then by Equation (1), we have $\sigma(3^2) \nmid n$, $\sigma(3^4) \nmid n$, and $\sigma(3^6) \nmid n$, so one can see that $\alpha_1 \ge 8$, and one sees that $G > 0$ in all cases, giving contradictions. If $p_2 = 23$, by (4) we have $p_3 = 29$, and $p_4 = 31$, 37. Similarly, we get $\alpha_1 \ge 8$, and we obtain $G > 0.001386$, which is also a contradiction. This completes the proof of Theorem 1.

6. Final Remark

During the review of the paper, the reviewer pointed out that Lemma 2 can be replaced by more elementary fact: $\sigma(3^\alpha)$ has a prime factor ≥ 34511 for any even $\alpha \geq 12$ with $\alpha + 1$ prime. $\sigma(5^\alpha)$ has a prime factor ≥ 19531 for any even $\alpha \geq 6$ with $\alpha + 1$ prime. Moreover, $\sigma(17^\alpha)$ has a prime factor ≥ 88741 for any even $\alpha \geq 4$. This can be confirmed from [11] or the Table 1 of [9]. Hence, if $\sigma(p^\alpha)$ with $p \in \{3, 5, 17\}$ and α even is a power of prime < 19531, then $(p, \alpha) = (3, 2), (3, 4), (3, 6), (5, 2), (5, 4)$, or $(17, 2)$ (we note that if $\sigma(p^\alpha)$ is a prime power, then $\alpha + 1$ must be a prime by Zsigmondy's theorem, see [12]). Using the above conclusion, we find that we can make some proofs more concise in the process of proving theorems. So that, the reviewer's method can be more accessible.

On the other hand, for $\omega(n) = 5$, we used a computer search and found no odd deficient-perfect number n. Then, we are sure that there is no odd deficient-perfect number with five distinct prime factors.

Moreover, for near-perfect numbers with at least three distinct prime divisors, there is the following conjecture in [21]: for any $k \geq 3$, there exist only finitely many near-perfect numbers n with $\omega(n) = k$. So we make the following similar conjecture.

Conjecture 1. *For any $k \geq 5$, there exist only finitely many deficient-perfect numbers with exactly k distinct prime divisors.*

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References

[1] A. Bege, K. Fogarasi, Generalized perfect numbers, *Acta Univ. Sapientiae, Mathematica*, **1** (2009), 73-82.

[2] Y. Bugeaud, M. Mignotte, Y. Roy, On the diophantine equation $\frac{x^n-1}{x-1} = y^q$, *Pacific J. Math.* **193** (2000), 257-268.

[3] G. L. Cohen, On odd perfect numbers. II. Multiperfect numbers and quasiperfect numbers, *J. Austral. Math. Soc. Ser. A.* **29** (1980), 369-384.

[4] G. L. Cohen, The nonexistence of quasiperfect numbers of certain forms, *Fibonacci Quart.* **20** (1982), 81-84.

[5] L. Cui, C. Zhang, Y. Li, On odd deficient-perfect numbers with four distinct prime divisors, *Pure Math.* **6** (2015), 411-417.

[6] P. Dutta, M. Saikia, On deficient perfect numbers with four distinct prime factors, *Asian-European J. Math.* **13**(2020), 2050126.

[7] P. Erdös, Remarks on number theory II: some problems on the σ function, *Acta Arith.* **5** (1959), 171-177.

[8] P. Hagis, Cohen G. L., Some results concerning quasiperfect numbers, *J. Austral. Math. Soc. Ser. A.* **33** (1982), 275-286.

[9] P. Hagis, G. L. Cohen, Every odd perfect number has a prime factor which exceeds 10^6, *Math. Comp.* **67** (1998), 1323šC1330.

[10] B. E. Hardy, M. V. Subbarao, On hyperperfect numbers, *Congr. Number* **42** (1984), 183-198.

[11] Takashi Goto, Yasuo Ohno, Odd perfect numbers have a prime factor exceeding 10^8, *Math. Comp.* **77** (2008), 1859šC1868.

[12] H.-J. Kanold, Sätze über kreisteilungspolynome und ihre andwendungen auf einige zahlentheoretische probleme. I., *J. Reine Angew. Math.* **187** (1950), 169–182.

[13] M. Kishore, On odd perfect, quasiperfect, and odd almost perfect numbers, *Math. Comp.* **36** (1981), 583-586.

[14] Y. Li, Q. Liao, A class of new near-perfect numbers, *J. Korean Math. Soc.* **52** (2015), 751-763.

[15] X. Ma, Y. Wang, On deficient-perfect numbers with four distinct prime divisors, *Pure Appl. Math.* **31** (2015), 643-649.

[16] J. C. M. Nash, Hyperperfect numbers, *Period. Math. Hungar.* **45** (2002), 121-122.

[17] P. P. Nielsen, Odd perfect numbers have at least nine different prime factors, *Math. Comp.* **76** (2007), no. 260, 2109-2126.

[18] P. P. Nielsen, Odd perfect numbers, Diophantine equations, and upper bounds, *Math. Comp.* **84** (2015), no. 295, 2549-2567.

[19] P. Ochem, M. Rao, Odd perfect numbers are greater than 10^{1500}, *Math. Comp.* **81** (2012), no. 279, 1869-1877

[20] P. Pollack, V. Shevelev, On perfect and near-perfect numbers, *J. Number Theory* **132** (2012), 3037-3046.

[21] X. Z. Ren, Y. G. Chen, On near perfect numbers with two distinct prime factors, *Bull. Austral. Math. Soc.* **88** (2013), 520-524.

[22] M. Saikia, On Deficient Perfect Numbers with Four Distinct Prime Factors, II, preprint.

[23] N. J. A. Sloane, The On-Line Encyclopedia of Integer Sequences, accessible at http: //oeis. org/.

[24] C. F. Sun, Z. C. He, On odd deficient-perfect numbers with four distinct prime divisors, preprint, https://arxiv.org/abs/1908.04932

[25] M. Tang, M. Feng, On deficient-perfect numbers, *Bull. Aust. Math. Soc.* **90** (2014), 186-194.

[26] M. Tang, X. Ma, M. Feng, On near-perfect numbers, *Colloq. Math.* **144** (2016), 157-188.

[27] M. Tang, X. Z. Ren, M. Li, On near-perfect and deficient-perfect numbers, *Colloq. Math.* **133** (2013), 221-226.