LIMITING BEHAVIOR IN GROWTH OF BULGARIAN SOLITAIRE ORBITS

Nhung Pham[1]
Department of Mathematics, Yale University, New Haven, Connecticut
nhung.pham@yale.edu

Received: 3/20/23, Accepted: 7/28/23, Published: 8/4/23

Abstract

The Bulgarian Solitaire rule induces a finite dynamical system on the set of integer partitions of n. Brandt characterized and counted all cycles in its recurrent set for any given n, with orbits parametrized by necklaces of black and white beads. However, the transient behavior within each orbit has been almost completely unknown. The only known case is when $n = \binom{k}{2}$ is a triangular number, in which case there is only one orbit. Eriksson and Jonsson gave an analysis for convergence of the structure as k grows, and to what extent the limit coincides with the finite case for each k. In this article, we generalize the convergent structure for orbits of the Bulgarian Solitaire system for any n. For necklaces of the form $(BW)^k = BWBW\cdots$, we give the precise limit of the generating functions as k grows. For other necklaces, we prove that the generating functions are rational and provide a bound for their denominator and numerator degrees.

1. Introduction to the Bulgarian Solitaire System

The game of Bulgarian Solitaire (BS) was introduced by Martin Gardner in 1983 (see Hopkins [8] for the full story). The original game starts with 45 cards divided into a number of piles. Now keep repeating the *Bulgarian Solitaire moves*: in each turn, take one card from each pile and form a new pile. The game ends when the sizes of the piles are not changed by performing the moves. Surprisingly, it turns out that regardless of the initial state of the game, it must end in a finite number of moves at the state with one pile of one card, one pile of two cards, ..., and one pile of nine cards. The rule was then generalized for any n as the BS operation β on the set of partition $\mathcal{P}(n) \coloneqq \{\lambda = (\lambda_1, \lambda_2, \dots, \lambda_m) : \text{length } l(\lambda) = m, \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_m > 0 \text{ are integers}, \lambda_1 + \dots + \lambda_m = n\}$. The operation $\lambda \to \beta(\lambda)$ is described as follows: in each step, take one from each part, form a new part, and put the

DOI: 10.5281/zenodo.8214839

[1] This work was done when the author was an undergraduate at the University of Minnesota - Twin Cities

parts in weakly decreasing order. In addition, the game can be described in terms of Young diagrams. That is, in each turn, remove the longest column and reinsert it as a new row into the diagram. An example is shown in Figure 1.

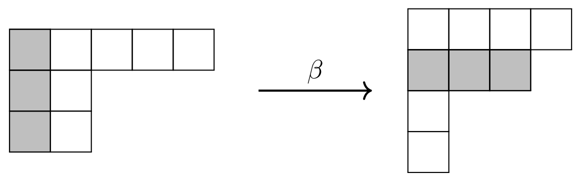

Figure 1: The BS operation on Young diagram $\beta((5,2,2)) = (4,3,1,1)$.

The BS has been an interesting structure for mathematicians. There have been many variations of the system, such as Carolina Solitaire, in which compositions are considered instead of partitions [4]. Other variations and generalizations in 3D and stochastic settings were also described by Drensky in [4]. Recent advances in the topic include the limiting shape of the stochastic BS and birth-death processes (see [6]), the enumeration of the *Garden of Eden states*, which are the configurations with no predecessors (see [9]), and its connection to the sum of the distinct parts congruent to r modulo m of partitions of n (see [12]). Moreover, the promotion sorting (see [3]) concerned an analogue to a classical question about the furthest states of the BS system from the recurrent cycles, which was answered by Igusa [10] and Etienne [7]. However, what happens between the *Garden of Eden states* and the recurrent cycles has been almost completely unknown.

The *game graph* of the BS system is a directed graph whose vertices are partitions of n with directed edges connecting λ to $\beta(\lambda)$. Some examples of BS game graphs are given in Figure 2 and Figure 3.

As in any finite dynamical system, that is, any self-map $\beta : X \to X$ on a finite set X, the elements in any orbit $\mathcal{O}$ under repeated application of β eventually lead to a *recurrent cycle* $\boldsymbol{C} \subset \mathcal{O}$, consisting of the elements λ in $\mathcal{O}$ having $\lambda = \beta^m(\lambda)$ for some $m \geq 1$. When β is the BS map, these recurrent cycles were analyzed by Brandt [2] in terms of objects called necklaces. A *necklace* N of black and white beads is an equivalence class of sequences of letters $\{B, W\}$ under cyclic rotation. We call N a *primitive necklace* if it cannot be written as a concatenation $N = P^k = PP \cdots P$ of copies of another necklace P. We will reserve P for *primitive necklaces*. We also say a binary sequence encodes a necklace class N if the sequence represents an element of the class N when we assign 1 for B and 0 for W. Brandt [2] showed there is a bijection

$$\begin{aligned} \mathcal{O} : \ \{\text{necklaces}\} &\longrightarrow \{\text{BS orbits}\} \\ N &\longmapsto \mathcal{O}_N \end{aligned}$$

that maps a necklace to the orbit of the BS system which has the unique recurrent

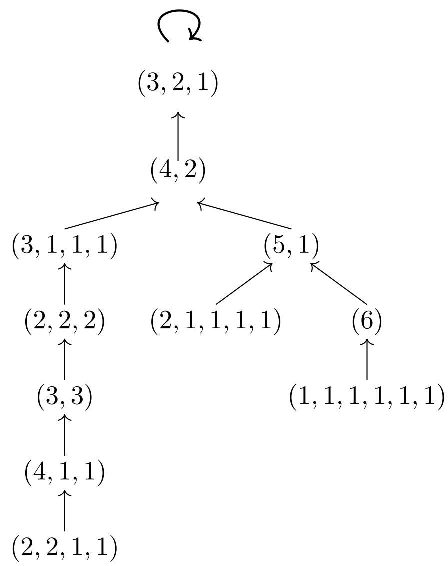

Figure 2: BS game graph for $n = 6$.

cycle $\boldsymbol{C}_N$ represented by the necklace. Specifically, if the necklace N is of length m, a partition λ is in the corresponding recurrent cycle $\boldsymbol{C}_N$ if its *difference labelling* from the staircase $\Delta_{m-1} = (m-1, \ldots, 1, 0)$, defined as

$$\lambda^{-} = \lambda - \Delta_{m-1}, \tag{1}$$

encodes the necklace class N. Figure 4 and Figure 5 illustrate the bijection.

We will also be interested in the *non-recurrent* elements λ in $\mathcal{O}_N$, and the distribution of the following level statistic on the orbit:

$$\text{level}(\lambda) := \min\{m \in \{0, 1, 2, \ldots\} : \beta^m(\lambda) \in \boldsymbol{C}_N\}.$$

Our main results concern its generating function, which is defined as the following polynomial in $\mathbb{Z}[x]$:

$$\mathcal{D}_N(x) := \sum_{\lambda \in \mathcal{O}_N} x^{\text{level}(\lambda)}.$$

Note that Brandt's bijection implies that the BS system on $\mathcal{P}(n)$ for $n = \binom{k+1}{2}$ has only one orbit $\mathcal{O}_{W^{k+1}}$ with the recurrent set $\boldsymbol{C}_{W^{k+1}} = \{\Delta_k\}$. Thus, the game graph $\mathcal{O}_{W^{k+1}}$ turns out to be a tree, rooted at the vertex Δ_k, and for any partition $\lambda \in \mathcal{O}_{W^{k+1}}$, the statistic level$(\lambda)$ is the distance in the tree from λ to the staircase Δ_k. Eriksson and Jonsson prove [5] that, in the limit as k grows, the sequence

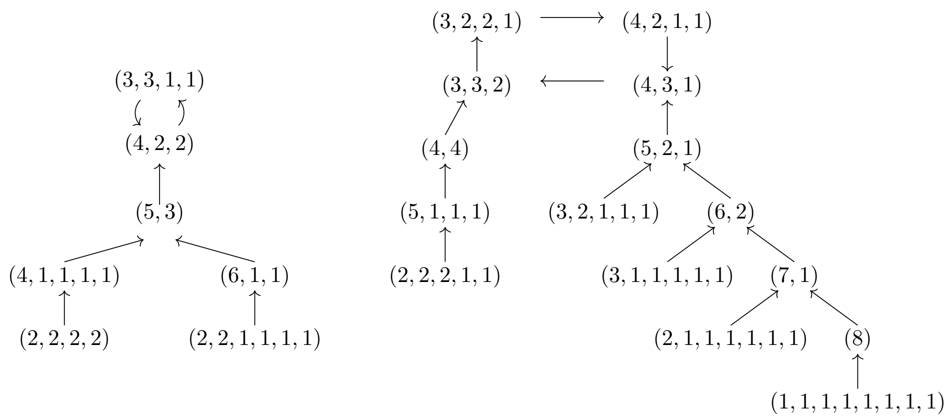

Figure 3: The BS game graph for $n = 8$.

of level sizes of $\mathcal{O}_{W^k}$ converges to the subsequence of evenly-indexed Fibonacci numbers $(F_{2d})_{d=0}^{\infty}$, with the generating function

$$H_W(x) \coloneqq \lim_{k\to\infty} \mathcal{D}_{W^k}(x) = \frac{(1-x)^2}{1-3x+x^2}.$$

Eriksson and Jonsson also showed that for $\mathcal{O}_{W^{k+1}}$, the sizes of levels $0, 1, \ldots, \lfloor \frac{k}{2} \rfloor$ in the BS game tree coincide with those of an object that they called the *quasi-infinite game tree* after pruning an appropriate branch (described later in the next section). We generalize this, describing the limit of the level sizes for arbitrary n, with the following two main results.

Theorem 1.1. *There is a power series $H_{BW}(x)$ in $\mathbb{Z}[[x]]$ such that*

$$\lim_{k\to\infty} \mathcal{D}_{(BW)^k} = H_{BW}(x).$$

Moreover, $H_{BW}(x)$ is a rational function, given by

$$\begin{aligned} H_{BW}(x) &= \frac{(x-1)^2(3x+2)}{x^3-3x^2-x+1} \\ &= 2 + x + 3x^2 + 7x^3 + 15x^4 + 33x^5 + 71x^6 + 155x^7 + 335x^8 + \cdots . \end{aligned}$$

Theorem 1.2. *For primitive necklaces P with $|P| \geq 3$, there is a power series $H_P(x)$ in $\mathbb{Z}[[x]]$ such that the sequence of generating functions $(\mathcal{D}_{P^k})_{k=0}^{\infty}$ converges to $H_P(x)$ coefficient-wise. Moreover, $H_P(x)$ is a rational function, whose denominator is a polynomial of degree at most $|P|$, and whose numerator is a polynomial of degree at most $2|P|$.*

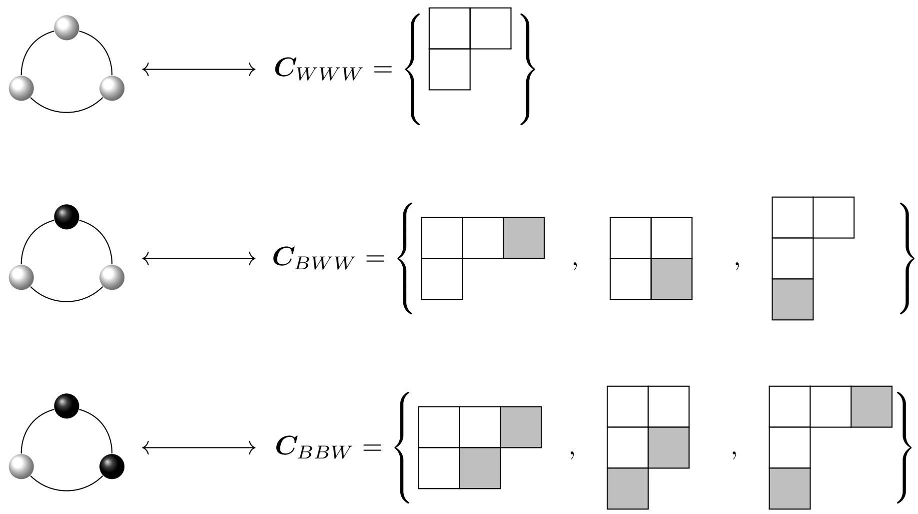

Figure 4: The map $\mathcal{O}$ for necklaces of length 3, which are WWW (non-primitive) and BWW, BBW (primitive).

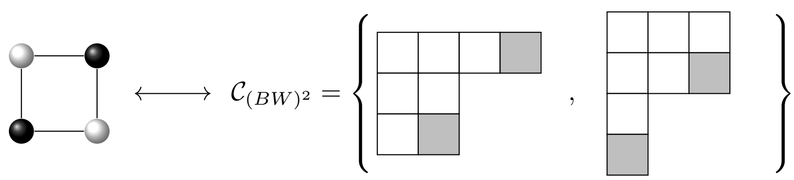

Figure 5: The map $\mathcal{O}$ for non-primitive necklaces of length 4. The recurrent set in $\mathcal{O}_{(BW)^2}$ has only 2 elements, shown above.

2. The Quasi-infinite BS Tree and Forest

Definition 1 (reversed BS rule)**.** The forward BS map $\lambda \mapsto \beta(\lambda)$ has (partially defined) reverse maps, which we will denote by $\lambda \xrightarrow{j} R_j(\lambda)$, for which $\beta(R_j(\lambda)) = \lambda$. They are described as follows:

- For Young diagrams: take out the jth row if it is no shorter than the number of rows minus 1 and insert it again as the leftmost column.
- For a partition λ: take a part $\lambda_j \geq l(\lambda) - 1$, then distribute it into other parts, one for each.

For example, $(4, 2, 2) \xrightarrow{1} (3, 3, 1, 1)$ and $(4, 2, 2) \xrightarrow{2} (5, 3)$. Figure 6 illustrates the reverse rule.

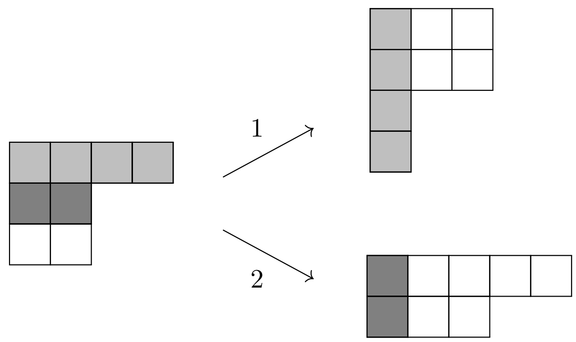

Figure 6: Reverse BS on a Young diagram.

We also use **0** as a result of an invalid move, that is, the removed row is too short to be the leftmost (longest) column. Obviously, if λ is a partition, $R_j(\lambda)$ is valid (or legal) if and only if $\lambda_j \geq l(\lambda) - 1$ and we display λ_j in brackets, as $\langle \lambda_j \rangle$. If $\lambda \in \mathcal{O}_N$ where N is a necklace of length m, then its *difference labelling* λ^- is defined the same as in Equation (1). Also, if λ_j is bracketed, so is $(\lambda^-)_j$.

Example 1 (bracketing rule)**.** In $\mathcal{O}_{W^4}$ in Figure 2 we have

$$\begin{pmatrix} \langle 3\rangle \\ \langle 2\rangle \\ 1 \end{pmatrix}^- = \begin{matrix} \langle 0\rangle \\ \langle 0\rangle \\ 0 \\ 0 \end{matrix} \text{ and } \begin{pmatrix} \langle 5\rangle \\ \langle 1\rangle \end{pmatrix}^- = \begin{matrix} \langle 2\rangle \\ \langle -1\rangle \\ -1 \\ 0 \end{matrix}.$$

In the left orbit $\mathcal{O}_{(BW)^2}$ of Figure 3, we have

$$\begin{pmatrix} \langle 4\rangle \\ \langle 2\rangle \\ \langle 2\rangle \end{pmatrix}^- = \begin{matrix} \langle 1\rangle \\ \langle 0\rangle \\ \langle 1\rangle \\ 0 \end{matrix} \text{ and } \begin{pmatrix} \langle 4\rangle \\ 1 \\ 1 \\ 1 \end{pmatrix}^- = \begin{matrix} \langle 1\rangle \\ -1 \\ 0 \\ 1 \end{matrix}.$$

Note that a staircase has 0 as its last part while a partition requires all parts to be positive. Also, a *difference labelling* does not necessarily have positive parts. The inverse is denoted by

$$\mu^+ = \mu + \Delta_{m-1},$$

where μ is a difference labelling in the orbit $\mathcal{O}_N$, and μ^+ must be an ordinary partition. For example, the inverses for difference labelings in Example 1 are

$$\begin{pmatrix} \langle 0\rangle \\ \langle 0\rangle \\ 0 \\ 0 \end{pmatrix}^+ = \begin{matrix} \langle 3\rangle \\ \langle 2\rangle \\ 1 \end{matrix} \text{ and } \begin{pmatrix} \langle 2\rangle \\ \langle -1\rangle \\ -1 \\ 0 \end{pmatrix}^+ = \begin{matrix} \langle 5\rangle \\ \langle 1\rangle \end{matrix}.$$

The reversed BS rule reverses all the arrows in $\mathcal{O}_N$ for some necklace N to get the digraph $\mathcal{O}_N^{\mathrm{op}}$. A *(reverse BS) playing sequence* σ from λ is a sequence of parts that are consecutively played legally starting from λ. We use $R_\sigma(\lambda)$ to represent the result of performing σ from λ.

Example 2 ((reverse BS) playing sequence)**.** When reversing all the arrows in Figure 2 and Figure 3, we get $\mathcal{O}_{W^4}^{op}$, $\mathcal{O}_{(BW)^2}^{op}$ and $\mathcal{O}_{BBWW}^{op}$, respectively. A playing sequence from $(3,2,1)$ in $\mathcal{O}_{W^4}^{op}$ is $[121]$ and $R_{[121]}((3,2,1)) = (2,1,1,1,1)$. A playing sequence from $(3,3,1,1)$ in $\mathcal{O}_{(BW)^2}^{op}$ can be $[1111\ldots]$, where $R_{[1^{2t}]}((3,3,1,1)) = (3,3,1,1)$ and $R_{[1^{2t+1}]}((3,3,1,1)) = (4,2,2)$ for any nonnegative integer t.

For any primitive necklace P of length p, there are p elements in the recurrent set $\boldsymbol{C}_{P^k}$ for any k. We consider the difference labellings of the elements in the recurrent set $\boldsymbol{C}_P$, which are $\{0,1\}$-vectors of length p that encode necklace class P. Their playable parts are bracketed following specific rules given in Section 4. Let that set be

$$\mathcal{C}_P = \{\gamma^{(t)} \in \boldsymbol{C}_P : R_1(\gamma^{(t)}) = \gamma^{(t+1)}, 1 \le t \le p \text{ and } \gamma^{(p+1)} \equiv \gamma^{(1)}\}.$$

Example 3. The recurrent set for the necklace BWW is

$$\mathcal{C}_{BWW} = \{BWW, WWB, WBW\} = \left\{ \begin{matrix} \langle 1 \rangle \\ \langle 0 \rangle \\ 0 \end{matrix}, \begin{matrix} \langle 0 \rangle \\ 0 \\ 1 \end{matrix}, \begin{matrix} \langle 0 \rangle \\ \langle 1 \rangle \\ 0 \end{matrix} \right\}.$$

For some vector α, we will use α^k for the concatenation $\begin{matrix} \alpha \\ \vdots \\ \alpha \end{matrix}$ of k blocks α. Brandt's bijection [2] maps necklace P^k to the orbit whose recurrent cycle $\mathcal{C}_{P^k} = \{(\gamma^{(t)})^k : \gamma^{(t)} \in \mathcal{C}_P\}$. We also use $\mathcal{C}_{P^k}^+ := \{\alpha^+ : \alpha \in \mathcal{C}_{P^k}\}$ for the ordinary recurrent set (whose elements are the same as those of $\boldsymbol{C}_{P^k}$ but their playable parts are bracketed). The bracketing rule for this set will be discussed in Section 4.

With the definition of the reversed rule in hand, we now discuss Eriksson and Jonsson's *quasi-infinite game tree* $\mathcal{F}_W$ for orbits $\mathcal{O}_{W^k}^{op}$. Figure 7 displays some initial difference reversed BS game graphs up to some levels and Figure 8 is the quasi-infinite game tree $\mathcal{F}_W$. After pruning the branch formed by playing sequences $[1\ldots]$ and adding a self-cycle to the root, we can see the coincidence between the quasi-infinite tree and the finite trees up to some levels.

Recall that $\mathcal{C}_{W^k} = \{(0)^k\}$. $\mathcal{F}_W$ starts at $\begin{matrix} \langle 0 \rangle \\ \langle 0 \rangle \end{matrix}$ which represents all bracketed parts in the roots of any trees $\mathcal{O}_{W^k}^{op}$ for $k \ge 2$. The rules for $\xrightarrow{i}$ in the quasi-infinite game tree [5, Section 3] are described below:

1. Delete the bracketed ith part.

2. Increase all parts above it by 1 and make them bracketed.

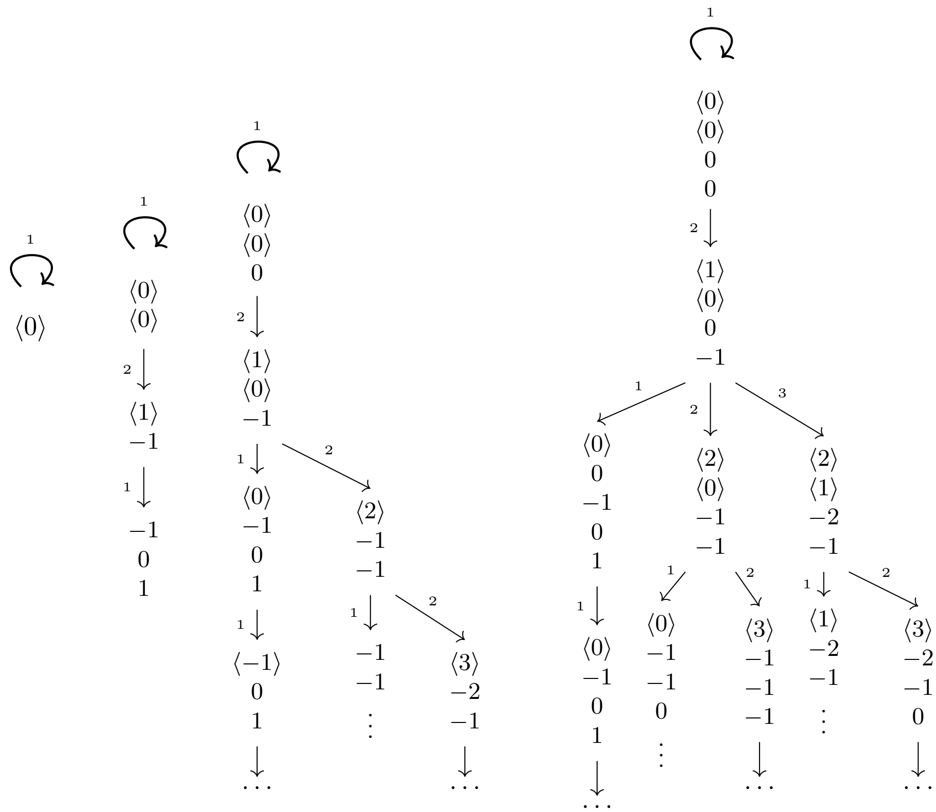

Figure 7: Difference reversed BS game graph for $n = \binom{k+1}{2}$ with $k = 1, 2, 3, 4$ up to level $\lfloor \frac{k+1}{2} \rfloor + 1$.

3. Bracket the new ith part (if there is one) if it differs by at most 1 from the old one.

4. If a zero was played, add zeros at the end so that there are two, and make them bracketed.

Eriksson and Jonsson [5] showed $\mathcal{O}^{\text{op}}_{W^\infty} = \lim_{m\to\infty} \mathcal{O}^{\text{op}}_{W^m}$. Generalizing that idea, we will build a *P-quasi-infinite forest* $\mathcal{F}_P$ consisting of t trees denoted $G_1, \ldots, G_p$, with G_t rooted at $\gamma^{(t)} \in \mathcal{C}_P$. The remaining vertices of G_t are generated from $\gamma^{(t)}$ by applying the reverse BS operations R_i in all possible ways, and modifying the parts according to rules described in the next section. An example is the $BWWW$-quasi-infinite forest in Figure 9.

Let $d(\lambda, \gamma^{(t)})$ be the length of the reversed playing sequence from $\gamma^{(t)}$ to λ. Let

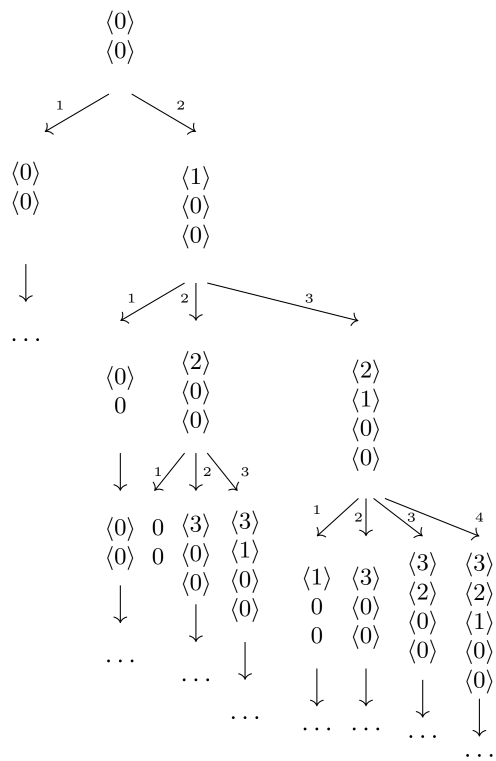

Figure 8: The quasi-infinite game tree $\mathcal{F}_W$.

$g_t(x)$ be the generating functions by level sizes (the growth function) of G_t, that is,

$$g_t(x) := \sum_{\lambda \in G_t} x^{d(\lambda, \gamma^{(t)})}.$$

We will show later that $\mathcal{O}^{op}_{P\infty} = \lim_{k\to\infty} \mathcal{O}^{op}_{Pk}$ is the quasi-infinite forest $\mathcal{F}_P$ after pruning the branches formed by playing sequences $[1\ldots]$ and adding arrows $\gamma^{(t)} \xrightarrow{1} \gamma^{(t+1)}$ to make the roots a recurrent cycle. In fact, up to level k, the finite digraph $\mathcal{O}^{op}_{Pk}$ coincides with the limit digraph $\mathcal{O}^{op}_{P\infty}$.

3. The Rules for Producing the Forest $\mathcal{F}_P$

The idea for the P-quasi-infinite forest is that each root $\gamma^{(t)}$ is actually an infinite periodic binary vector whose consecutive segments $(\gamma^{(t)}_{mp+1}, \ldots, \gamma^{(t)}_{(m+1)p})$ of length p each form a copy of $\gamma^{(t)}$. The reason for this beginning will be explained in the

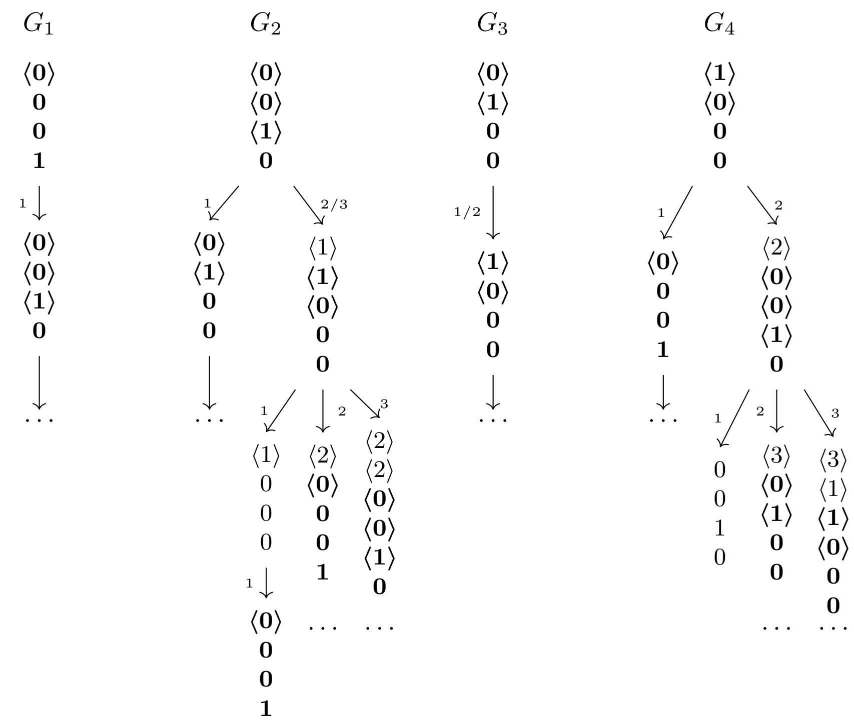

Figure 9: The BWWW quasi-infinite forest. The bold entries are $\gamma^{(t)}$ for some t.

next section. This infinite vector guarantees that we will not encounter negative parts in difference labellings when performing the R_j operations as in the finite cases. However, we cannot write out infinite-length vectors for the elements in the forest. Instead, we start with only one block and add more blocks as one goes further down the tree G_t and applies the operators R_j. In fact, each segment $(\gamma^{(t)}_{mp+s}, \ldots, \gamma^{(t)}_{(m+1)p+s-1})$ is a copy of the root $\gamma^{(t+s)}$ for any nonegative integers m, s.

In this section, for a vector $\lambda = (\lambda_1, \ldots, \lambda_m)$ and $j \le k$, we write $\lambda[j : k] = (\lambda_j, \ldots, \lambda_k)$. Also, we use $\lambda[: j] = \lambda[1 : j]$ and $\lambda[j :] = (\lambda_j, \ldots, \lambda_m)$.

As part of the rules, we will want to maintain the property that every λ in the forest $\mathcal{F}_P$ has a *tail* (the final segment of length p) exactly matching one of the roots $\gamma^{(t)}$, although the subscript t may not match the subscript t' of the tree $G_{t'}$ which contains λ. Assume in the rules below that λ's tail matches the root $\gamma^{(t)}$.

Below are the three rules for $\lambda \xrightarrow{i} R_i(\lambda)$ (or $\lambda \xrightarrow{i/i+1} R_i(\lambda)$) in the tree G_t in the general quasi-infinite forest — that is, λ_i (respectively, both λ_i and λ_{i+1}) is bracketed and R_i (respectively, either R_i or R_{i+1}) is played.

1. If λ_i is in the tail (among the lowest p parts), append to λ so that $\lambda[i:] = \gamma^{(s)}$ for some (unique) s. Replace $\lambda[i:]$ by $\gamma^{(s+1)}$ and bracket the tail as in the root $\gamma^{(s+1)}$.
2. Otherwise, delete λ_i. If $\begin{matrix}\lambda_i \\ \lambda_{i+1}\end{matrix} = \begin{matrix}s \\ s+1\end{matrix}$, playing either R_i or R_{i+1} has the same result, so we label the play by $\xrightarrow{i/i+1}$ and apply R_{i+1}. Bracket the new i^{th} part (if there is one) if it differs by at most 1 from λ_i. If there are two consecutive entries $\begin{matrix}s \\ s+1\end{matrix}$ and s is bracketed, so is $s+1$.
3. Increase all entries $\lambda[:(i-1)]$ by 1 each, and bracket them.

Here is an example of these rules in the $BWWW$ quasi-infinite forest from Figure 9.

Example 4. Let $\lambda = \begin{matrix}\langle 2\rangle \\ \langle \mathbf{0}\rangle \\ \langle \mathbf{0}\rangle \\ \langle \mathbf{1}\rangle \\ \mathbf{0}\end{matrix}$. The bold parts are $\gamma^{(2)}$.

- If we play $\lambda \xrightarrow{3} R_3(\lambda)$, Rule 1 is applied as follows:

Step 1: Note that λ_3 is among the last 4 parts. We append to $\lambda \longrightarrow \begin{matrix}\langle 2\rangle \\ \langle 0\rangle \\ \langle \mathbf{0}\rangle \\ \langle \mathbf{1}\rangle \\ \mathbf{0} \\ \mathbf{0}\end{matrix}$, so that $\lambda[3:] = \gamma^{(3)}$, the next root.

Step 2: Replace the tail $\gamma^{(3)}$ by $\gamma^{(4)}$ and bracket as in $\gamma^{(4)}$: $\begin{matrix}\langle 2\rangle \\ \langle 0\rangle \\ \langle \mathbf{0}\rangle \\ \langle \mathbf{1}\rangle \\ \mathbf{0} \\ \mathbf{0}\end{matrix} \longrightarrow \begin{matrix}\langle 2\rangle \\ \langle 0\rangle \\ \langle \mathbf{1}\rangle \\ \langle \mathbf{0}\rangle \\ \mathbf{0} \\ \mathbf{0}\end{matrix}$.

Step 3: Add 1 to each part in $\lambda[:2]$ to get $R_3(\lambda) = \begin{matrix}\langle 3\rangle \\ \langle 1\rangle \\ \langle \mathbf{1}\rangle \\ \langle \mathbf{0}\rangle \\ \mathbf{0} \\ \mathbf{0}\end{matrix}$.

- If we play $\lambda \xrightarrow{1} R_1(\lambda)$, Rule 2 is applied. Note that λ_1 is not among the last 4 parts, so Rule 2 applies to obtain $R_1(\lambda) = \begin{matrix}0 \\ 0 \\ 1 \\ 0\end{matrix}$, because the new first part, 0, differs from λ_1 by 2.

4. Explanations for the Rules of the General Forest for Primitive Necklaces $|P| \geq 3$

We wish to explain why the rules in Section 3 for constructing the (quasi-infinite) forest $\mathcal{F}_P$ from the recurrent set $\mathcal{C}_P$ actually produce a forest which is the limiting digraph $\lim_{k\to\infty} \mathcal{O}^{\text{op}}_{P^k}$ (after pruning the branches whose playing sequences are $[1\ldots]$ and adding arrows $\gamma^{(t)} \xrightarrow{1} \gamma^{(t+1)}$ to form the recurrent cycle $\mathcal{C}_P$). To this end, we answer some questions below about the construction. In this section, if no further information is given, the statements relate to the forest. Also, the notion $\lambda = \begin{matrix}\mu\\ \nu\end{matrix}$ means the concatenation of either two partitions or two difference labelings μ and ν. For a partition or a difference labeling λ and an integer h, we write $\lambda + h = (\lambda_1 + h, \ldots, \lambda_m + h)$. Obviously, if λ is a difference labeling then $(\lambda+h)^+ = \lambda^+ + h$.

The first question is how to bracket the roots $\gamma^{(t)}$ of the tree G_t, even though they are infinite vectors. The following proposition shows that bracketing them as in the recurrent set of the orbit $\mathcal{O}_P$ still makes sense.

Proposition 1. *For any primitive necklace P of length at least 3 and any k, elements in the recurrent set $\mathcal{C}_{P^k}$ restricted to the first p parts are identical (with brackets). Moreover, only the first 3 parts can be bracketed.*

Proof. Recall that for a partition λ in the reverse BS system, λ_j is playable if and only if $\lambda_j \geq l(\lambda) - 1$. If $\lambda \in \mathcal{C}^+_{P^k}$, it must have the form $\lambda = \Delta_{kp-1} + (\gamma^{(t)})^k$ for some t. Then $l(\lambda) \geq kp - 1$, and thus, λ_j is playable if and only if $\lambda_j = kp - j + (\gamma^{(t)})^k_j \geq kp - 2$. The latter is equivalent to $1 \geq (\gamma^{(t)})^k_j \geq j - 2$. Hence, $j \leq 3$. Therefore, the bracketed parts in elements of the recurrent set $\mathcal{C}_{P^k}$ are all among the first $3 \leq p$ parts, and our proposition is verified. $\square$

Example 5. The recurrent sets in the finite reverse BS graphs $\mathcal{O}^{\text{op}}_{BWW}$ and $\mathcal{O}^{\text{op}}_{(BWW)^2}$ are

$$\mathcal{C}^+_{BWW} = \left\{ \begin{matrix}\langle 3\rangle\\ \langle 1\rangle\end{matrix}, \begin{matrix}\langle 2\rangle\\ 1\\ 1\end{matrix}, \begin{matrix}\langle 2\rangle\\ \langle 2\rangle\end{matrix} \right\} \text{ and } \mathcal{C}_{BWW} = \left\{ \begin{matrix}\langle 1\rangle\\ \langle 0\rangle\\ 0\end{matrix}, \begin{matrix}\langle 0\rangle\\ 0\\ 1\end{matrix}, \begin{matrix}\langle 0\rangle\\ \langle 1\rangle\\ 0\end{matrix} \right\},$$

$$\mathcal{C}^+_{(BWW)^2} = \left\{ \begin{matrix}\langle 6\rangle\\ \langle 4\rangle\\ 3\\ 3\\ 1\end{matrix}, \begin{matrix}\langle 5\rangle\\ 4\\ 4\\ 2\\ 1\\ 1\end{matrix}, \begin{matrix}\langle 5\rangle\\ \langle 5\rangle\\ 3\\ 2\\ 2\end{matrix} \right\} \text{ and } \mathcal{C}_{(BWW)^2} = \left\{ \begin{matrix}\langle 1\rangle\\ \langle 0\rangle\\ 0\\ 1\\ 0\\ 0\end{matrix}, \begin{matrix}\langle 0\rangle\\ 0\\ 1\\ 0\\ 0\\ 1\end{matrix}, \begin{matrix}\langle 0\rangle\\ \langle 1\rangle\\ 0\\ 0\\ 1\\ 0\end{matrix} \right\}.$$

Their difference labellings, including their brackets, are identical if we restrict them to the first 3 parts.

Remark 1. Moreover, as we restrict the roots to their first p parts, we only need to know how to bracket the recurrent cycle $\mathcal{C}_P$. A partition $\lambda \in \mathcal{C}_P$ (that is, λ^- is one of the roots) possesses the following properties for their first three parts:

- The part λ_1 is always bracketed, because $\lambda_1 \geq p-1 \geq l(\lambda)-1$.
- There are two cases for λ_2:
  - If $\lambda_2^- = 1$ then it is bracketed, because then $\lambda_2 = (p-2)+1 = p-1 \geq l(\lambda)-1$.
  - If $\lambda_2^- = 0$ then it is bracketed only if $\lambda_p^- = 0$, since then $\lambda_2 = p-2 = l(\lambda)-1$.
- The part λ_3 can only be bracketed if $\lambda_3^- = 1$ and $\lambda_p^- = 0$, since then $\lambda_3 = (p-3)+1 = p-2 = l(\lambda)-1$.

The next question is about the bracketing rules after playing a part from an element in the forest. Recall that due to the rules of the forest, each element has its length p tail matching $\gamma^{(t)}$ for some t. Hence, we treat the cases where we play a part in the tail or above the tail separately as Rule 1 and Rule 2 in Section 3.

Before discussing the bracketing rules, we prove a general property:

Claim 4.1. In a difference labelling λ, there are no triples $j_1 < j_2 < j_3$ such that $\lambda_{j_1} < \lambda_{j_2} < \lambda_{j_3}$. This assertion is still true for any finite $\mathcal{O}^{op}_{Pk}$ where $k \geq j_3$.

Proof. Suppose there is such triple, then $\lambda_{j_3} \geq 2$, which means some part lower than λ_{j_3} has already been played. But during any play, the amounts added to λ_{j_1} and λ_{j_2} are at least the amount added to λ_{j_3}. Since λ_{j_3} was either 0 or 1 in the root of the tree containing λ, the part λ_{j_1} was negative at the start. We obtain a contradiction. □

Corollary 1. *A similar argument to the proof above shows that if $\lambda_j < \lambda_{j+1}$ then $\lambda_{j+1} - \lambda_j = 1$.*

Now we explain how to change the bracketing in λ in the forest after we play a part above the tail $\gamma^{(t)}$.

Proposition 2. *Let λ be a difference labelling in $\mathcal{O}^{op}_N$. If a part λ_j is playable, let $\psi = R_j(\lambda)$. Then ψ_j is playable if and only if $|\lambda_j - \lambda_{j+1}| \leq 1$. Moreover, if $\lambda_j \geq \lambda_{j+1}$, any parts ψ_s where $s \geq j+3$ are not playable. Otherwise $\lambda_j + 1 = \lambda_{j+1}$ and any parts ψ_s where $s \geq j+4$ are not playable.*

Proof. For the first statement, we consider $\Lambda = \lambda^+$. Since playing parts other than Λ_j and Λ_{j+1} does not affect the gap between them, we only consider the moment when performing R_j. The result is a partition $\overline{\Lambda}$ with Λ_j parts. Thus, a part is playable in the state $R_j(\lambda)$ if and only if its size is at least $\Lambda_j - 1$. Because R_j adds one to every other part, and the difference labeling is obtained by subtracting a staircase, which yields $\lambda_j - \lambda_{j+1} = (\Lambda_j - \Lambda_{j+1}) - 1$, we obtain the first statement.

Now if $\lambda_j \geq \lambda_{j+1}$, let $|N| = m$ where we subtract Δ_{m-1} to get the difference labellings. Then $\psi = R_j(\lambda)$ has $m - j + \lambda_j$ parts. Note that $\psi_{j+3} = \lambda_{j+4}$. By Claim 4.1, we see that $\lambda_{j+4} \leq \max\{\lambda_s : j \leq s \leq j+3\}$. By Corollary 1, that maximum is at most $\lambda_j + 1$. Thus,

$$\psi^+_{j+3} = m - j - 3 + \lambda_{j+4} \leq m - 2 - j + \lambda_j = l(\psi) - 2.$$

Thus, ψ_{j+3} is not playable and neither are the later parts. Moreover, $\psi^+_{j+1} = m - j - 1 + \lambda_{j+2} \geq m - j + \lambda_j - 1$ if and only if $\lambda_{j+2} \geq \lambda_j$. Similarly, ψ_{j+2} is playable if and only if $\lambda_{j+3} \geq \lambda_j + 1$. Hence, $\lambda_{j+3} = \lambda_j + 1$.

Otherwise $\lambda_{j+1} - \lambda_j = 1$, adding a staircase we get $\lambda^+_j = \lambda^+_{j+1}$, and thus we can play R_{j+1} instead of R_j. □

Proposition 2 explains the Rule 2 of the forest. Moving on, we explain how a part in the "tail" γ^t is played, provided that the multiple k (in the necklace $N = P^k$) is large enough.

Proposition 3. *Assume that* $\lambda = \begin{matrix} \alpha \\ \gamma^{(t)} \\ \gamma^{(t)} \\ \beta \end{matrix}$ *is a difference labelling in* $\mathcal{O}^{op}_{P^k}$*, where* α, β *are some difference labellings, that* $l(\alpha) = a$ *and that* λ_{a+1} *is playable. Then*

$$R_{a+1}(\lambda) = \begin{matrix} \alpha + 1 \\ \gamma^{(t+1)} \\ \gamma^{(t)}[2:] \\ \beta' \end{matrix} \tag{2}$$

and the $\gamma^{(t+1)}$ *segment is bracketed similarly to the root.*

Proof. We have $l(\gamma^{(t)}) = p$ and note that β is not necessarily non-negative. Let $n = pk - 1$. The partition corresponding to λ is

$$\Lambda = \lambda^+ = \lambda + \Delta_n$$

of length $l \geq a + 2p$. So λ_{a+1} is playable if and only if $\Lambda_{a+1} = \gamma^{(t)}_1 + n - a \geq l - 1$. The latter condition is equivalent to $1 \geq \gamma^{(t)}_1 \geq l - n + a - 1$. Playing λ_{a+1} adds 1 to each of the entries of both α^+ and α, erases λ_{a+1} and keeps the rest of λ, and thus we obtain Equation (2). Let $\Psi = R_{a+1}(\Lambda)$ and $\psi = R_{a+1}(\lambda)$, so $l(\Psi) = \gamma^{(t)}_1 + n - a$. Now, in the root $\gamma^{(t+1)}$, the part $\gamma^{(t+1)}_j$ is bracketed if and only if

$$\gamma^{(t+1)}_j + (p - j) \geq \begin{cases} p - 1 & \text{if } \gamma^{(t)}_1 = 1 \\ p - 2 & \text{if } \gamma^{(t)}_1 = 0 \end{cases},$$

and thus $\gamma_j^{(t+1)} \geq j-2+\gamma_1^{(t)}$. We show that the same j^{th} part of the $\gamma^{(t+1)}$ segment is bracketed in ψ. Actually, that part is ψ_{a+j}, and

$$\Psi_{a+j} = \gamma_j^{(t+1)} + n - a - j + 1 \geq \gamma_1^{(t)} + n - a - 1 = l(\Psi) - 1.$$

This concludes the proof. □

Proposition 4. *Let $\lambda = R_\sigma((\gamma^{(t)})^k)$, for some $(\gamma^{(t)})^k$ in $\mathcal{C}_{P^k}$ and some arbitrary playing sequence σ of length l. If k is large enough, then λ is of the form*

$$\lambda = \begin{matrix} \alpha \\ \gamma^{(s)} \\ \gamma^{(s)} \\ \beta \end{matrix}$$

for some $1 \leq s \leq p$ and $l(\alpha) + 1 \geq \sigma_l$. Moreover, α is determined uniquely for all large k including the brackets, and if Rule 1 applies to R_{σ_l} then the first $\gamma^{(s)}$ is bracketed similarly to the root $\gamma^{(s)}$ as in Remark 1.

Proof. Take $k \geq l+2$, and we start with

$$(\gamma^{(t)})^k = \begin{matrix} \gamma^{(t)} \\ \gamma^{(t)} \\ \beta \end{matrix} = \begin{matrix} \gamma_1^{(t)} \\ \gamma^{(t+1)} \\ \gamma^{(t+1)} \\ \beta' \end{matrix} = \begin{matrix} \gamma_1^{(t)} \\ \gamma_2^{(t)} \\ \gamma^{(t+2)} \\ \gamma^{(t+2)} \\ \beta'' \end{matrix},$$

in which at most 3 first parts are playable by Proposition 1. In each play, if Rule 1 applies, then Proposition 3 guarantees the next state has the same form as in the proposition, because in each play we only lose at most 1 block of $\gamma^{(s)}$ for some s. Otherwise, Rule 2 applies which does not change the tail (consisting of blocks of $\gamma^{(s)}$). Proposition 2 shows that Rule 2 also determines the playable parts in the next state. The conclusion follows for all trees rooted at $(\gamma^{(t)})^h$ for $h \geq k$, since σ only takes place in the first k blocks. □

The last question is why we can represent the infinite-vector elements of the forest with finite prefixes. In other words, we can restrict the infinite vector to some finite prefix without losing any playable parts.

We defined earlier that $\mathcal{O}_{P^\infty}^{op} = \lim_{k\to\infty} \mathcal{O}_{P^k}^{op}$. Because of Proposition 4, we see that for a playing sequence σ and k large enough, the elements $\lambda_h = R_\sigma((\gamma^{(t)})^h)$ for $h \geq k$ share the same prefix $\lambda = \begin{matrix} \alpha \\ \gamma^{(s)} \end{matrix}$ as in the statement of the proposition. Thus, we can represent all those λ_h's by that prefix λ in $\mathcal{O}_{P^\infty}^{op}$. Actually, λ is in the form

$$\lambda = \begin{matrix} \alpha \\ \gamma^{(s)} \\ \gamma^{(s)} \\ \vdots \end{matrix},$$

so that Rule 1 works for playing a part in the "tail" of λ. Rule 2 also works since each element λ has a tail of length $p \geq 3$, and by Proposition 2, the number of new bracketed parts is at most 3, and thus the lower parts are not affected.

The conclusion of this section is that the finite reverse BS forest $\mathcal{O}^{op}_{P^k}$ coincides with the infinite digraph $\mathcal{O}^{op}_{P^\infty}$ obtained from the forest $\mathcal{F}_P$ (by modifying any branch of $\mathcal{F}_P$ as in Section 1) up to at least level k. Therefore, we confirm that $\mathcal{O}^{op}_{P^\infty} = \lim_{k\to\infty} \mathcal{O}^{op}_{P^k}$.

5. Limiting Level Sizes of $\mathcal{O}^{op}_{(BW)^k}$ as k Grows

5.1. The Quasi-infinite Forest $\mathcal{F}_{BW}$

We first analyze $P = BW$. The exception in this case is that the root $\begin{smallmatrix}\langle 1\rangle\\ \langle 0\rangle\end{smallmatrix} \in \mathcal{C}_{BW}$ we defined in Section 1 does not reflect all the playable parts in $\mathcal{C}^+_{(BW)^k}$ for $k \geq 2$ (e.g., $\begin{smallmatrix}\langle 3\rangle\\ \langle 2\rangle\\ \langle 1\rangle\\ 0\end{smallmatrix}$). However, $\begin{smallmatrix}\langle 1\rangle\\ \langle 0\rangle\\ \langle 1\rangle\end{smallmatrix}$ does, by the same argument as in Proposition 1 in Section 4. Thus, we build the forest $\mathcal{F}_{BW}$ as in Figure 10 following the rules given in Section 3, except that the roots are $\gamma^{(1)} = \begin{smallmatrix}\langle 0\rangle\\ \langle 1\rangle\end{smallmatrix}$ and $\gamma^{(2)} = \begin{smallmatrix}\langle 1\rangle\\ \langle 0\rangle\\ \langle 1\rangle\end{smallmatrix}$.

5.2. Proof of Theorem 1.1

Let $g(x)$ be the generating function for the level sizes of $\mathcal{F}_{BW}$. Then $g(x) = g_1(x) + g_2(x)$. Starting from $\gamma^{(1)}$, we play $R_{1/2}$ (the only playable part) and get back to the root $\gamma^{(2)}$. Thus, the level generating functions of the two trees satisfy

$$g_1(x) = 1 + xg_2(x). \tag{3}$$

As a reminder, if we have $\begin{smallmatrix}\langle 0\rangle\\ \langle 1\rangle\end{smallmatrix}$ in a difference λ, adding a staircase makes their values the same. Thus, playing either of them results in the same element. To represent this move in a playing sequence, we will use the vertex with $\langle 0\rangle$ on it. Moreover, we use the notation R_σ for playing sequences in the quasi-infinite game graph similarly to the normal BS game graph, e.g., $R_{[222]}\begin{pmatrix}\langle 1\rangle\\ \langle 0\rangle\\ \langle 1\rangle\end{pmatrix} = \begin{smallmatrix}\langle 4\rangle\\ \langle 1\rangle\\ \langle 0\rangle\\ \langle 1\rangle\end{smallmatrix}$.

Similar to Jonsson and Eriksson's paper [5, Proposition 3.1, p. 4], we have the following proposition.

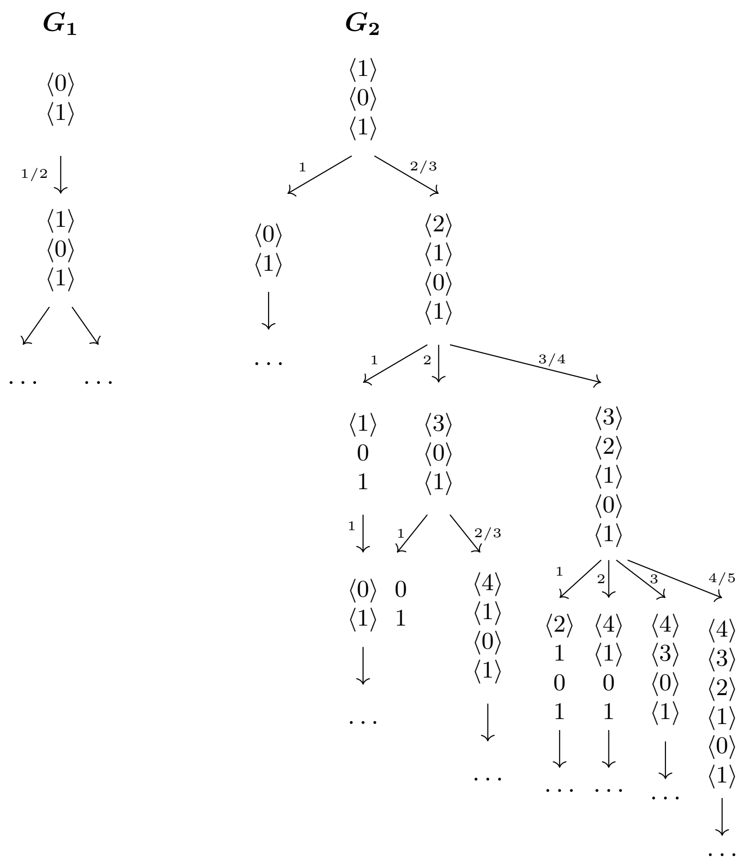

Figure 10: The BW-quasi-infinite $\mathcal{F}_{BW}$.

Proposition 5. *The tree G_2 rooted at $\gamma^{(2)} = \begin{matrix}\langle 1\rangle\\ \langle 0\rangle\\ \langle 1\rangle\end{matrix}$ has the following properties.*

(i) Once R_1 is played, only R_1 can be played (until $\gamma^{(1)} = \begin{matrix}\langle 0\rangle\\ \langle 1\rangle\end{matrix}$ or a leaf is reached).

(ii) For $r \geq 2$, the playing sequence $[234\dots r1^r]$ leads to $\gamma^{(1)}$.

Proof.

(i) When we play R_1, if the second part differs from the first part by at least 2, nothing is bracketed in the next state, and thus we reach the leaf. Otherwise, according to the Rule 3, either we reach the root $\begin{matrix}\langle 0\rangle\\ \langle 1\rangle\end{matrix}$ if the second and third parts are $\begin{matrix}0\\1\end{matrix}$ or the only bracketed part in the next state is the first part.

(ii) It is easy to see that

$$R_{[23\ldots r]}\begin{pmatrix}\langle 1\rangle\\ \langle 0\rangle\\ \langle 1\rangle\end{pmatrix} = \begin{matrix}\langle r\rangle\\ \langle r-1\rangle\\ \vdots\\ \langle 1\rangle\\ \langle 0\rangle\\ \langle 1\rangle\end{matrix}.$$

Thus, playing $[1^r]$ consecutively deletes the first r rows and reaches $\begin{matrix}\langle 0\rangle\\ \langle 1\rangle\end{matrix}$. From Property (i), we see that this is the only way to get back to the roots. □

Now we give the proof of Theorem 1.1.

Proof of Theorem 1.1. We begin by using Proposition 5 to construct the *growth function*, or the generating function of g_2 by its level sizes. Table 1 shows various playing sequences and their contributions to the generating function $g_2(x)$, explained below. We use $[\geq t]$ to denote the set of playing sequences with entries no less than t, including the empty sequence.

Sequence	**Contribution**	**Sequence**	**Contribution**
$[1\ldots]$	xg_1	$[2[\geq 2]]$	xg_2
$[21^2\ldots]$	x^3g_1	$[2[\geq 2]1]$	x^2g_2
$[231^3\ldots]$	x^5g_1	$[23[\geq 3]1^2]$	x^4g_2
$[2341^4\ldots]$	x^7g_1	$[234[\geq 4]1^3]$	x^6g_2

Table 1: Growth function for tree G_2.

By Proposition 5 (ii), the sequences $[23\ldots r1^r]$ lead to the root of G_1, so each of them contribute the whole G_1 tree at level $2r-1$, that is, $x^{2r-1}g_1(x)$. This is also true for playing sequence [1], since $R_1\begin{pmatrix}\langle 1\rangle\\ \langle 0\rangle\\ \langle 1\rangle\end{pmatrix} = \begin{matrix}\langle 0\rangle\\ \langle 1\rangle\end{matrix}$.

Next, if we play R_2, we reach $\begin{matrix}\langle 2\rangle\\ \langle 1\rangle\\ \langle 0\rangle\\ \langle 1\rangle\end{matrix} = \begin{matrix}\langle 2\rangle\\ \gamma_2\end{matrix}$. Thus, if we leave the top part untouched, which means we only play parts of indices greater or equal than 2, then we have a subtree that is isomorphic to G_2. The isomorphism is defined by excluding the top part. Thus, sequences $[2[\geq 2]]$ contribute xg_2. Similarly with $[23\ldots r[\geq r]1^{r-1}]$, each contributes $x^{2r-2}g_2$, since

$$R_{[23\dots r]}(\gamma_2) = \begin{matrix} \langle r \rangle \\ \langle r-1 \rangle \\ \vdots \\ \langle 1 \rangle \\ \langle 0 \rangle \\ \langle 1 \rangle \end{matrix} = \begin{matrix} \langle r \rangle \\ \langle r-1 \rangle \\ \vdots \\ \gamma_2 \end{matrix}.$$

If we play $[23\dots r[\geq r]]$, the top $r-1$ rows above $\begin{matrix} \langle 1 \rangle \\ \langle 0 \rangle \\ \langle 1 \rangle \end{matrix}$ are always playable due to Rule 2. Thus, the playing sequences $[23\dots r[\geq r]1^s]$ with any $s \leq r-1$ are legal. For each $r \geq 2$, we only count for $[23\dots r[\geq r]1^r]$, since if $s < r$, $[23\dots r[\geq r]1^s]$ are counted as $[23\dots s[\geq s]1^s]$. Hence, the type $[23\dots r[\geq r]1^{r-1}]$ contributes $x^{2r-2}g_2$. Hence, we obtain

$$\begin{aligned} g_2(x) &= 1 + (x + x^3 + \cdots)g_1(x) + (x + x^2 + x^4 + \cdots)g_2(x) \\ &= 1 + (x + x^3 + \cdots) + (x + 2x^2 + 2x^4 + \cdots)g_2(x) \\ &= \left(1 + \frac{x}{1-x^2}\right) + xg_2(x) + \frac{2x^2}{1-x^2}g_2(x), \end{aligned}$$

where in the second equality we substitute $g_1(x)$ using Equation (3). Therefore

$$g_2(x) = \frac{-x^2 + x + 1}{x^3 - 3x^2 - x + 1}.$$

From this one concludes, again using Equation (3), that

$$\begin{aligned} g(x) = g_1(x) + g_2(x) &= (1 + xg_2(x)) + g_2(x) = 1 + (1+x)g_2(x) \\ &= \frac{-3x^2 + x + 2}{x^3 - 3x^2 - x + 1} = \frac{(1-x)(3x+2)}{x^3 - 3x^2 - x + 1}. \end{aligned}$$

However, we desire the generating function for the level sizes of $\mathcal{O}^{op}_{(BW)^k}$ in the limit as $k \to \infty$. As constructed, our quasi-infinite forest has an entire copy of itself after playing R_1, giving rise to the left branch $[1\dots]$, which we wish to disregard. Letting $H_{BW}(x)$ denote the height generating function for the rest of the quasi-infinite forest, that is, the two roots and the elements in the branch $[2\dots]$, one then has

$$g(x) = xg(x) + H_{BW}(x).$$

Therefore,

$$H_{BW}(x) = (1-x)g(x) = \frac{(1-x)^2(3x+2)}{x^3 - 3x^2 - x + 1}.$$

Thus, to complete the proof of Theorem 1.1, it only remains to show that the level sizes of $\mathcal{O}^{op}_{(BW)^k}$ actually converge to the coefficients given by $H_{BW}(x)$. This is a consequence of Theorem 5.1. $\square$

Theorem 5.1. *The finite reverse BS graph $\mathcal{O}^{op}_{(BW)^{k+1}}$ coincides at least up to level k with the BW quasi-infinite forest after removing its branch* $[1\ldots]$.

Proof. For each k, to get to level k in the quasi-infinite forest, there are at most k times that we change the tail $\begin{matrix}\langle 0\rangle\\ \langle 1\rangle\end{matrix}$ to $\begin{matrix}\langle 1\rangle\\ \langle 0\rangle\\ \langle 1\rangle\end{matrix}$, which means we add at most k blocks $\begin{matrix}\langle 0\rangle\\ \langle 1\rangle\end{matrix}$. Thus, if we start with the root $\begin{matrix}\langle 1\rangle\\ \langle 0\rangle\\ \langle 1\rangle\\ \begin{pmatrix}0\\1\end{pmatrix}^{k-1}\\ 0\end{matrix}$, we do not need to add new parts until using up those k blocks of $\begin{matrix}0\\1\end{matrix}$. Moreover, it is easy to confirm that the rule of bracketing makes sense if k is large enough. This implies the coincidence of the quasi-infinite forest with the finite graph. $\square$

In the next section, similar arguments are used to prove the limiting behavior of BS orbits of primitive necklaces of length at least 3.

6. The Limiting Generating Function for General $\mathcal{O}^{op}_{P^k}$ by Level Sizes

In this section, we prove our second main theorem, Theorem 1.2, from the Introduction. We also recall the following claim that was mentioned in Section 4.

Claim 6.1. Each $\gamma^{(t)} \in \mathcal{C}_P$ has at most 3 playable parts. Moreover, if $\gamma^{(t)}_p = 1$, then $\gamma^{(t)}_2$ is playable if and only if $\gamma^{(t)}_2 = 1$.

The next claim points out a “special” element in the recurrent set.

Claim 6.2. For any primitive necklace P of length at least 3, there is at least one $\gamma^{(t)} \in \mathcal{C}_P$ of the form $\begin{matrix}\langle\sigma\rangle\\ 0\\ \vdots\end{matrix}$, where $\sigma \in \{0,1\}$.

Proof. There are two cases to consider.

1. The necklace P has two consecutive black beads, which means that any $\gamma^{(t)} \in \mathcal{C}_P$ has either two consecutive $1's$, or a 1 at the top and a 1 at the bottom. There exists a $\gamma^{(t)}$ of the form $\begin{matrix}1\\ 0\\ \vdots\\ 1\end{matrix}$, or else $P = B^p$ (all black beads) not primitive. Thus,

$\gamma_1^{(t)} + p - 1 = p > p - 1 = l(\gamma^{(t)}) - 1$, so it is playable. Moreover, Claim 6.1 admits that $\gamma_2^{(t)}$ is not playable.

2. The necklace P has two consecutive white beads. In this case we consider

$$\gamma^{(t)} = \begin{matrix} 0 \\ 0 \\ \vdots \\ 1 \end{matrix}.$$

Thus, $\gamma_1^{(t)}$ is playable and Claim 6.1 admits that $\gamma_2^{(t)}$ is not playable. □

The third claim regards a special situation that separates types of playing sequences.

Claim 6.3. Let S be the subtree S of the forest $\mathcal{F}_P$ which is rooted at the element

$$\lambda = \begin{matrix} \lambda[:j] \\ \langle \lambda_{j+1} \rangle \\ 0 \\ \vdots \end{matrix} = \begin{matrix} \alpha \\ \gamma^{(t)} \end{matrix} = \begin{matrix} \alpha \\ \langle \sigma \rangle \\ 0 \\ \vdots \\ 1 \end{matrix}$$

where $\alpha = \lambda[:j]$ has $\alpha_j \geq 1$, with $\sigma \in \{0,1\}$ and $\begin{matrix} \langle \sigma \rangle \\ 0 \\ \vdots \\ 1 \end{matrix} = \gamma^{(t)}$ chosen as in Claim 6.2.

Then S has growth function

$$h(x) = A(x) + B(x)g_t(x)$$

where $A, B \in \mathbb{Z}[x]$ are of degree at most j.

Proof. If $\langle \sigma \rangle$ is bracketed, any parts of α are bracketed, due to Rule 3. Since $\lambda_j = \alpha_j \geq 1 > 0 = \lambda_{j+2}$ and σ is the only playable part in the segment $\gamma^{(t)}$, by the proof of Proposition 2 and Rule 1, the part $\lambda_{j+2} = \gamma_2^{(t)} = 0$ cannot be playable any time before σ is played, and the same for any entries below it. Let δ be a playing sequence starting at λ. Let $ind_\sigma^\delta(t)$ be the index of σ in $R_{\delta[:(t-1)]}(\lambda)$. We have two cases to consider.

1. Play $\lambda \xrightarrow{\delta} R_\delta(\lambda)$ where $\delta_i \leq ind_\sigma^\delta(i)$ for all i. Clearly, $l(\delta) \leq j$ because we delete the played entry in each operation, by Rule 2. Let $A(x)$ be the growth function for this set of elements obtained from performing such playing sequences.

2. Part σ is played at some time; assume $\lambda \xrightarrow{\delta} R_\delta(\lambda)$ where $\delta_i = ind_\sigma^\delta(i)$. Because $\alpha_j \geq 1$, the Rule 3 confirms that in any states before playing R_{δ_i}, the immediate

part above the tail segment $\gamma^{(t)}$ is at least 1. Hence, $R_{\delta[:i]}(\lambda)$ results in $\overset{\beta}{\gamma^{(t+1)}}$, in which the lowest part of β is at least 2 and we claim that β has $j-i+1$ parts. The reason for this claim is because we erase one part of α each time we play $\delta[:(i-1)]$. Thus, playing any parts of β cannot make the 0 at the top of $\gamma^{(t+1)}$ playable due to Rule 2. From there, if any $\delta_{i+s} < \delta_i$ for some $s > 0$, that is, a part of the segment β is played, then any parts below β are not playable. Thus, the rest of the playing sequence δ only plays parts of β and leads to a leaf.

The playing sequences δ in this case take $i-1$ levels to reach $\gamma^{(t)}$. From then, playing $[\geq \delta_i]$ forms a copy of $\gamma^{(t)}$, while playing entries of β at any time reaches a leaf in at most $j-i+1$ plays. Finally, these playing sequences contribute $B(x)g_t(x)$ for $B \in \mathbb{Z}[x]$ of degree j. □

Proof of Theorem 1.2. We only need to connect the dots by proving that every branch of the tree G_s rooted at $\gamma^{(s)}$ will eventually hit the element of the form λ as in Claim 6.3. In $\mathcal{C}_P$, one cycles the positions of the 1's to obtain the successive elements $\gamma^{(s)}$, so in any G_s with $s \neq t$, the first part $\gamma_1^{(t)}$ is at some position $\gamma_{j+1}^{(s)}$ $(1 \leq j \leq p-1)$. We can append some entries at the end of $\gamma^{(s)}$ so that the tail is $\gamma^{(t)}$. Hence the Claim 6.3 applies, with j at most $p-1$.

Furthermore, $g_t(x) = 1 + xg_{t+1}(x)$ because $\gamma_1^{(t)}$ is the only playable part in $\gamma^{(t)}$. Now there are polynomials K, L of degree at most $p-1$ satisfying

$$g_{t+1}(x) = K(x)+L(x)g_t(x) = K(x)+L(x)(1+xg_{t+1}(x)) = K(x)+L(x)+xL(x)g_{t+1},$$

implying there are polynomials P, Q of degree at most p such that

$$g_{t+1}(x) = \frac{M(x)}{N(x)}.$$

Then $g_t(x)$ is rational with denominator of degree at most p and numerator of degree at most $p+1$. Specifically,

$$g_t(x) = \frac{N(x) + xM(x)}{N(x)}.$$

Any other $g_s(x)$ where $s \notin \{t, t+1\}$ is of the form

$$g_s(x) = A(x) + B(x)g_t(x) = \frac{AN + BN + xBM}{N},$$

where A, B are polynomials of integer coefficients of degree at most $p-2$. Thus, any such g_s is rational with denominator of degree at most p and numerator of degree at most $2p-1$. Let g be the growth function of the P quasi-infinite forest, then $g = \sum_{s=1}^{p} g_s$ is rational with denominator $Q(x)$ and numerator of degree at most $2p-1$.

Now we want to discard a copy of the forest coming from the branches $[1\ldots]$ in each tree G_s to get the limit generating function H_P of $\mathcal{O}^{op}_{P^\infty}$. Thus,

$$g(x) = H_P(x) + xg(x),$$

and then

$$H_P(x) = (1-x)g(x),$$

which is a rational function with denominator of degree at most p and numerator of degree at most $2p$. □

7. Further Discussion and Conjectures on the Finite BS Systems

The precise limiting generating functions H_{BWW} and H_{BBW} were computed in detail in the author's bachelor thesis [13, Section 5]. The limiting generating functions for primitive necklaces of greater length can be easily computed in that fashion. Below are some of them that were computed by hand:

$$\begin{aligned}
H_{BWW}(x) = H_{BBW}(x) &= (1-x)\frac{x^3-3x^2-4x-3}{2x^3+x^2-1},\\
H_{BWWW}(x) &= (1-x)\frac{x^5+8x^4-3x^3-8x^2-6x-4}{6x^4+4x^3+x^2-1},\\
H_{BBBW}(x) &= (1-x)\frac{2x^5+8x^4-5x^3-10x^2-7x-4}{6x^4+4x^3+x^2-1},\\
H_{BBWW}(x) &= (1-x)\frac{x^5+4x^4-3x^3-6x^2-6x-4}{3x^4+2x^3+x^2-1},\\
H_{BWWWW}(x) &= (1-x)\frac{2x^6+16x^5-12x^4-23x^3-16x^2-8x-5}{12x^5+8x^4+2x^3-1}.
\end{aligned}$$

They led us to conjecture in addition to Theorem 1.2 that the denominator degree of H_P is exactly $|P|$.

Another interesting question about the BS dynamical system is the sizes of the orbits, which are parametrized by necklaces as discussed in Section 1. Recall that if $N = P^k$ for some primitive necklaces P of length p, an element λ in the recurrent set $\boldsymbol{C}_{P^k}$ is of the form $\lambda = (\gamma^{(t)})^k + \Delta_{pk-1}$ for some $\gamma^{(t)} \in \mathcal{C}_P$. Since the number of 1's in $\gamma^{(t)}$ is equal to the number of black beads in N, we have that the size of the partition n that the BS operation acts on is

$$n = \binom{pk}{2} + k \cdot \#\text{black beads of } P.$$

As we know, when $N = W^k$, the orbit $\mathcal{O}_{W^k}$ is actually the whole BS system on the partition set $\mathcal{P}\left(\binom{k}{2}\right)$. The author's bachelor thesis [13] computed the sizes of orbits parametrized by necklaces of the form $(BW)^k, (BBW)^k$ and $(BWW)^k$. One of the results involves the sequence of *Chebyshev polynomials of the first kind* $\{T_k(x)\}_{k=0}^{\infty}$ evaluated at $x = 2$, which satisfies the recurrent formula

$$\begin{aligned} &T_0(2) = 1 \\ &T_1(2) = 2 \\ &T_k(2) = 4T_{k-1}(2) - T_{k-2}(2) \qquad \text{for } k \geq 2. \end{aligned}$$

The results are the theorem below.

Theorem 7.1 (Pham [13]). *For each $k = 1, 2, \ldots,$ one has*

$$\begin{aligned} \left|\mathcal{O}_{(BW)^k}\right| &= T_k(2) \\ \left|\mathcal{O}_{(BWW)^k}\right| &= 5^k \\ \left|\mathcal{O}_{(BBW)^k}\right| &= 7 \cdot 5^{k-1}. \end{aligned}$$

We also conjectured that orbits parametrized by P^k for primitive necklaces P of length greater than 3 grow geometrically as well. Some data for primitive necklaces of length 4 and 5 are given in [13, Section 3.1].

Conjecture 1. For any primitive necklace P with $|P| \geq 3$, there is an integer c_P such that for $k \geq 2$,

$$|\mathcal{O}_{P^k}| = (c_P)^{k-1} |\mathcal{O}_P|.$$

Moreover, when P' is obtained from P by reversing the letters of P swapping black beads to white beads and vice versa, then $c_P = c_{P'}$.

Acknowledgement. We thank Son Nguyen for his revision regarding the dual necklace P' by swapping the beads in Conjecture 1.

References

[1] E. Akin and M. Davis, Bulgarian solitaire, *Amer. Math. Monthly*, **92**(4) (1985), 237-250.

[2] J. Brandt, Cycles of partitions, *Proc. Amer. Math. Soc.* **85**(3) (1982), 483-486.

[3] C. Defant and N. Kravitz, Promotion sorting, *Order* **40** (2023), 199-216.

[4] V. Drensky, The Bulgarian solitaire and the mathematics around it, Math in the future, preprint, `arXiv:1503.00885` (2015).

[5] H. Eriksson and M. Jonsson, Level sizes of the Bulgarian solitaire game tree, *Fibonacci Quart.* **55**(3) (2017), 243-251.

[6] K. Eriksson and J. Sjöstrand, Limiting shapes of birth-and-death processes on Young diagrams, *Adv. in Appl. Math.* **48**(4) (2012), 575-602.

[7] G. Etienne, Tableaux de Young et solitaire Bulgare, *J. Combin. Theory Ser. A* **58**(2) (1991), 181-197.

[8] B. Hopkins, 30 years of Bulgarian solitaire, *College Math. J.* **43**(2) (2012), 135-140.

[9] B. Hopkins and J.A. Sellers, Exact enumeration of Garden of Eden partitions, *Integers* **7**(2) (2007), 299-303.

[10] K. Igusa, Solution of the Bulgarian solitaire conjecture, *Math. Mag.* **58**(5) (1985), 259-271.

[11] M. Jonsson, Processes on Integer Partitions and Their Limit Shapes, PhD thesis, Mälardalen University, Sweden, 2017.

[12] M. Merca, Linear inequalities concerning the sum of the distinct parts congruent to r modulo m in all the partitions of n, *Quaest. Math.*, to appear. DOI: 10.2989/16073606.2023.2174911.

[13] N. Pham, Combinatorics of Bulgarian Solitaire, Bachelor thesis, University of Minnesota - Twin Cities, Minnesota, 2022.