THETA-FUNCTION IDENTITIES, EXPLICIT VALUES FOR RAMANUJAN'S CONTINUED FRACTIONS OF ORDER SIXTEEN AND APPLICATIONS TO PARTITION THEORY

Shraddha Rajkhowa
Department of Mathematics, Rajiv Gandhi University, Rono Hills, Doimukh, Arunachal Pradesh, India
shraddha.rajkhowa@rgu.ac.in

Nipen Saikia[1]
Department of Mathematics, Rajiv Gandhi University, Rono Hills, Doimukh, Arunachal Pradesh, India
nipennak@yahoo.com

Received: 3/21/23, Revised: 5/21/23, Accepted: 7/31/23, Published: 8/4/23

Abstract

We derive two continued fractions $M(q)$ and $N(q)$ of order sixteen connected with the Ramanujan-Göllnitz-Gordon continued fraction. We obtain theta-function identities of $M(q)$ and $N(q)$ and prove general theorems for the explicit evaluations of $M(\pm q)$ and $N(\pm q)$. As applications, we show that color partition identities can be obtained from the theta-function identities of $M(q)$ and $N(q)$. Some matching coefficients arising from the continued fractions $M(q)$ and $N(q)$ are also offered.

1. Introduction

Throughout the paper, for any complex numbers a and q, define the q-product $(a;q)_\infty$ as

$$(a;q)_\infty := \prod_{t=0}^{\infty}\left(1-aq^t\right), \qquad |q|<1. \tag{1}$$

For brevity, we will write

$$(a_1;q)_\infty(a_2;q)_\infty(a_3;q)_\infty\cdots(a_m;q)_\infty = (a_1,a_2,a_3,\cdots,a_m;q)_\infty.$$

Ramanujan's general theta-function $\mathfrak{f}(a,b)$ [5, p. 34] is defined as

$$\mathfrak{f}(a,b) = \sum_{t=-\infty}^{\infty} a^{t(t+1)/2} b^{t(t-1)/2}, \qquad |ab|<1. \tag{2}$$

DOI: 10.5281/zenodo.8214849

[1]Corresponding author.

The Jacobi triple product identity [5, p. 35, Entry 19] can be expressed in terms of $\mathfrak{f}(a,b)$ as

$$\mathfrak{f}(a,b) = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty = (-a,-b,ab;ab)_\infty. \tag{3}$$

Three important special cases of $\mathfrak{f}(a,b)$ are the theta-functions $\phi(q)$, $\psi(q)$ and $f(-q)$ [5, p. 36, Entry 22 (i)-(iii)] given by

$$\phi(q) := \mathfrak{f}(q,q) = \sum_{t=-\infty}^{\infty} q^{t^2} = \frac{(q^2;q^2)_\infty^5}{(q;q)_\infty^2(q^4;q^4)_\infty^2}, \tag{4}$$

$$\psi(q) := \mathfrak{f}(q,q^3) = \sum_{t=0}^{\infty} q^{t(t+1)/2} = \frac{(q^2;q^2)_\infty^2}{(q;q)_\infty}, \tag{5}$$

$$f(-q) := \mathfrak{f}(-q,-q^2) = \sum_{t=-\infty}^{\infty} (-1)^t q^{t(3t-1)/2} = (q;q)_\infty, \tag{6}$$

respectively. Ramanujan also defined the function $\chi(q)$ [5, p. 36, Entry 22(iv)] as

$$\chi(q) = (-q;q^2)_\infty. \tag{7}$$

For convenience, we will write

$$f_n := f(-q^n) = (q^n;q^n)_\infty.$$

To prove our results, we need the following lemmas.

Lemma 1 ([1, p. 39, Entry 24]). *We have*

$$\phi(q) = \frac{f_2^5}{f_1^2 f_4^2}, \qquad \phi(-q) = \frac{f_1^2}{f_2}, \qquad \psi(q) = \frac{f_2^2}{f_1}, \qquad \psi(-q) = \frac{f_1 f_4}{f_2},$$

$$f(q) = \frac{f_2^3}{f_1 f_4}, \qquad \chi(q) = \frac{f_2^2}{f_1 f_4}, \qquad \chi(-q) = \frac{f_1}{f_2}.$$

Lemma 2 ([13, (1.9.4)]). *The following 2-dissection of $\phi(q)$ holds:*

$$\frac{1}{f_1^2} = \frac{f_8^5}{f_2^5 f_{16}^2} + 2q\frac{f_4^2 f_{16}^2}{f_2^5 f_8}.$$

Ramanujan made remarkable contributions in the field of q-continued fractions. One of the interesting continued fractions is the Ramanujan-Göllnitz-Gordon continued fraction $H(q)$, recorded by Ramanujan on page 299 of his second notebook [15] and given by

$$H(q) := q^{1/2}\frac{(q;q^8)_\infty(q^7;q^8)_\infty}{(q^3;q^8)_\infty(q^5;q^8)_\infty} = q^{1/2}\frac{\mathfrak{f}(-q,-q^7)}{\mathfrak{f}(-q^3,-q^5)}$$

$$= \cfrac{q^{1/2}}{1+q+\cfrac{q^2}{1+q^3+\cfrac{q^4}{1+q^5+\cdots}}} \quad |q|<1. \tag{8}$$

The second equality of (8) follows from the first equality and (3). Göllnitz [11] and Gordon [12], independently, rediscovered and proved (8). Andrews [2] proved (8) as a corollary of a more general result. An alternative proof of (8) was also given by Ramanathan [14]. Ramanujan offered two other identities [15, p. 299] for $H(q)$, namely,

$$\frac{1}{H(q)} - H(q) = \frac{\phi(q^2)}{q^{1/2}\psi(q^4)} \tag{9}$$

and

$$\frac{1}{H(q)} + H(q) = \frac{\phi(q)}{q^{1/2}\psi(q^4)}. \tag{10}$$

Proofs of (9) and (10) can be found in [5, p. 221]. The identities (9) and (10) also follow as special cases of identities established in [8]. Chan and Huang [7] found many identities involving the continued fraction $H(q)$ and evaluated explicitly $H(e^{-\pi\sqrt{n}/2})$ for several positive integers n. Vasuki and Srivatsa Kumar [18] also established new modular relations for $H(q)$. Baruah and Saikia [4] established some general theorems for the explicit evaluations of $H(q)$ and evaluated some values.

Closely related to the Ramanujan-Göllnitz-Gordon continued fraction $H(q)$ are the continued fractions $M(q)$ and $N(q)$ of order sixteen, which are defined, respectively, as

$$M(q) := q^{3/2}\frac{(q,q^{15};q^{16})_\infty}{(q^7,q^9;q^{16})_\infty} = q^{3/2}\frac{\mathfrak{f}(-q,-q^{15})}{\mathfrak{f}(-q^7,-q^9)}$$
$$= \cfrac{q^{3/2}(1-q)}{(1-q^4)+\cfrac{q^4(1-q^3)(1-q^5)}{(1-q^4)(1+q^8)+\cfrac{q^4(1-q^{11})(1-q^{13})}{(1-q^4)(1+q^{16})+\cdots}}} \tag{11}$$

and

$$N(q) := q^{1/2}\frac{(q^3,q^{13};q^{16})_\infty}{(q^5,q^{11};q^{16})_\infty} = q^{1/2}\frac{\mathfrak{f}(-q^3,-q^{13})}{\mathfrak{f}(-q^5,-q^{11})}$$
$$= \cfrac{q^{1/2}(1-q^3)}{(1-q^4)+\cfrac{q^4(1-q)(1-q^7)}{(1-q^4)(1+q^8)+\cfrac{q^4(1-q^9)(1-q^{15})}{(1-q^4)(1+q^{16})+\cdots}}}. \tag{12}$$

In fact, the continued fractions $M(q)$ and $N(q)$ are special cases of the following general continued fraction recorded by Ramanujan [5, p. 24, Entry 12] in his notebook: Suppose that a, b and q are complex numbers with $|ab|<1$ and $|q|<1$, or that $a = b^{2m+1}$ for some integer m. Then

$$\frac{(a^2q^3;q^4)_\infty(b^2q^3;q^4)_\infty}{(a^2q;q^4)_\infty(b^2q;q^4)_\infty} = \cfrac{1}{1-ab+\cfrac{(a-bq)(b-aq)}{(1-ab)(q^2+1)+\cfrac{(a-bq^3)(b-aq^3)}{(1-ab)(q^4+1)+\cdots}}}. \tag{13}$$

To obtain $M(q)$ and $N(q)$, we replace q by q^4 in (13), then set $\{a = q^{3/2}, b = q^{5/2}\}$ and $\{a = q^{1/2}, b = q^{7/2}\}$, simplify using (3) and employ the results $(q^{17};q^{16})_\infty = (q;q^{16})_\infty/(1-q)$ and $(q^{19};q^{16})_\infty = (q^3;q^{16})_\infty/(1-q^3)$, respectively. It is noteworthy that the continued fractions $H(q^2)$, $M(q)$ and $N(q)$ are continued fractions of order sixteen.

Vanitha [17] studied the 2-, 4-, 8-, and 16-dissections of the continued fraction $M(q)$ and its reciprocal, and also studied signs and the periodic nature of the coefficients in the power series expansion. Vanitha [17] also gave combinatorial interpretations of the coefficients in the power series expansion of $M(q)$ and its reciprocal. Surekha [16] obtained 2-, 4-, 8-, and 16-dissections of the continued fraction $N(q)$ and its reciprocal, and also studied signs and the periodic nature of the coefficients in the power series expansion.

In this paper, we are concerned with the theta-function identities, explicit values, partition-theoretic results, and some matching coefficients of the continued fractions $M(q)$ and $N(q)$. Even though Ramanujan's theta-function identities are mainly employed to prove our results, it is important to note that some of the identities of $M(q)$ and $N(q)$ may also be obtained from the more general identities and q-difference equations established in [8, 9, 10].

In Section 2, we prove some theta-function identities for $M(q)$ and $N(q)$. We also prove identities connecting the continued fraction $H(q)$ with $M(q)$ and $N(q)$. In Section 3, we establish general theorems for the explicit evaluations of $M(\pm q)$ and $N(\pm q)$ with examples. In Section 4, we demonstrate that color partition identities can be obtained from theta-function identities of $M(q)$ and $N(q)$ by deriving a color partition identity. Finally, in Section 5, we derive some matching coefficient results arising from the continued fractions $M(q)$ and $N(q)$.

2. Theta-Function Identities for $M(q)$ and $N(q)$

In this section, we prove some theta-function identities for the continued fractions $M(q)$ and $N(q)$, and identities connecting $M(q)$ and $N(q)$ with the continued fraction $H(q)$.

Theorem 3. *We have*

$$(i)\quad \frac{1}{M(q)}-M(q)=\frac{\phi(q^4)\big(\phi(q)+\phi(q^2)\big)}{2q^{3/2}\psi(q^8)\psi(q^4)},$$

$$(ii)\quad \frac{1}{M(q)}-M(q)=\frac{\phi(q^4)}{q\psi(q^8)}\frac{1}{H(q)},$$

$$(iii)\quad \frac{1}{N(q)}-N(q)=\frac{\phi(q^4)\big(\phi(q)-\phi(q^2)\big)}{2q^{3/2}\psi(q^8)\psi(q^4)},$$

$$(iv)\quad \frac{1}{N(q)}-N(q)=\frac{\phi(q^4)}{q\psi(q^8)}H(q),$$

$$(v)\quad \frac{1}{M(q)}+M(q)=\frac{\phi(-q^4)\mathfrak{f}(q^3,q^5)}{q^{3/2}\psi(q^8)\mathfrak{f}(-q,-q^7)}$$
$$=\frac{\phi(-q^4)\psi(q^4)\psi(q)}{q^{1/2}\phi(-q^8)\psi(q^8)}\left(\frac{2}{\psi(-q^2)(\phi(q^2)-\phi(q^4))}\right)^{1/2},$$

$$(vi)\quad \frac{1}{N(q)}+N(q)=\frac{\phi(-q^4)\mathfrak{f}(q,q^7)}{q^{1/2}\psi(q^8)\mathfrak{f}(-q^3,-q^5)}$$
$$=\frac{\phi(-q^4)\psi(q^4)\psi(q)}{q^{1/2}\phi(-q^8)\psi(q^8)}\left(\frac{2}{\psi(-q^2)(\phi(q^2)+\phi(q^4))}\right)^{1/2},$$

$$(vii)\quad \left(\frac{1}{M(q)}-M(q)\right)\left(\frac{1}{N(q)}-N(q)\right)=\frac{\phi^2(q^4)}{q^2\psi^2(q^8)}=\left(\frac{1}{H(q^2)}-H(q^2)\right)^2,$$

$$(viii)\quad \left(\frac{1}{M(q)}-M(q)\right)+\left(\frac{1}{N(q)}-N(q)\right)=\frac{\phi(q^4)\phi(-q^2)\psi(q)}{q^{3/2}\psi(q^8)\psi(q^4)\psi(-q)},$$

$$(ix)\quad \left(\frac{1}{M(q)}-M(q)\right)-\left(\frac{1}{N(q)}-N(q)\right)=\frac{\phi(q^4)\phi(q^2)}{q^{3/2}\psi(q^8)\psi(q^4)},$$

$$(x)\quad \frac{\big(N^{-1}(q)-N(q)\big)}{(M^{-1}(q)-M(q))}=\frac{\phi(q)-\phi(q^2)}{\phi(q)+\phi(q^2)}=H^2(q),$$

$$(xi)\quad \frac{\big(N^{-1}(q)+N(q)\big)}{(M^{-1}(q)+M(q))}=\left(\frac{\phi(q^2)-\phi(q^4)}{\phi(q^2)+\phi(q^4)}\right)^{1/2}=H(q^2),$$

$$(xii)\quad \left(\frac{1}{M(q)}+M(q)\right)\left(\frac{1}{N(q)}+N(q)\right)=\frac{\phi^2(-q^4)\psi(q)}{q^2\psi^2(q^8)\psi(-q)}.$$

Proof. From (11), we obtain

$$\frac{1}{\sqrt{M(q)}}-\sqrt{M(q)}=\frac{\mathfrak{f}(-q^7,-q^9)-q^{3/2}\mathfrak{f}(-q,-q^{15})}{\sqrt{q^{3/2}\mathfrak{f}(-q,-q^{15})\mathfrak{f}(-q^7,-q^9)}}. \tag{14}$$

From [5, p. 46, Entry 30 (ii) and (iii)], we note that

$$\mathfrak{f}(a,b)=\mathfrak{f}(a^3b,ab^3)+a\mathfrak{f}(b/a,a^5b^3). \tag{15}$$

Setting $a=-q^{3/2}$ and $b=q^{5/2}$ in (15), we obtain

$$\mathfrak{f}(-q^{3/2},q^{5/2})=\mathfrak{f}(-q^7,-q^9)-q^{3/2}\mathfrak{f}(-q,-q^{15}). \tag{16}$$

Employing (16) in (14), we find that

$$\frac{1}{\sqrt{M(q)}}-\sqrt{M(q)}=\frac{\mathfrak{f}(-q^{3/2},q^{5/2})}{\sqrt{q^{3/2}\mathfrak{f}(-q,-q^{15})\mathfrak{f}(-q^7,-q^9)}}. \tag{17}$$

Similarly, from (11), we deduce that

$$\frac{1}{\sqrt{M(q)}}+\sqrt{M(q)}=\frac{\mathfrak{f}(-q^7,-q^9)+q^{3/2}\mathfrak{f}(-q,-q^{15})}{\sqrt{q^{3/2}\mathfrak{f}(-q,-q^{15})\mathfrak{f}(-q^7,-q^9)}}. \tag{18}$$

Employing (15) with $a=q^{3/2}$ and $b=-q^{5/2}$ in (18), we obtain

$$\frac{1}{\sqrt{M(q)}}+\sqrt{M(q)}=\frac{\mathfrak{f}(q^{3/2},-q^{5/2})}{\sqrt{q^{3/2}\mathfrak{f}(-q,-q^{15})\mathfrak{f}(-q^7,-q^9)}}. \tag{19}$$

Combining (17) and (19), we arrive at

$$\frac{1}{M(q)}-M(q)=\frac{\mathfrak{f}(-q^{3/2},q^{5/2})\mathfrak{f}(q^{3/2},-q^{5/2})}{q^{3/2}\mathfrak{f}(-q,-q^{15})\mathfrak{f}(-q^7,-q^9)}. \tag{20}$$

Again, from [5, p. 46, Entry 30 (i),(iv)], we note that

$$\mathfrak{f}(a,ab^2)\mathfrak{f}(b,a^2b)=\mathfrak{f}(a,b)\psi(ab) \tag{21}$$

and

$$\mathfrak{f}(a,b)\mathfrak{f}(-a,-b)=\mathfrak{f}(-a^2,-b^2)\phi(-ab). \tag{22}$$

Setting $\{a=-q,\, b=-q^7\}$ and $\{a=-q,\, b=-q^3\}$ in (21) and using (5), we obtain

$$\mathfrak{f}(-q,-q^{15})\mathfrak{f}(-q^7,-q^9)=\mathfrak{f}(-q,-q^7)\psi(q^8) \tag{23}$$

and

$$\mathfrak{f}(-q,-q^7)\mathfrak{f}(-q^3,-q^5)=\mathfrak{f}(-q,-q^3)\psi(q^4)=\psi(-q)\psi(q^4), \tag{24}$$

respectively. Also, setting $a=-q^{3/2}$ and $b=q^{5/2}$ in (22), we obtain

$$\mathfrak{f}(-q^{3/2},q^{5/2})\mathfrak{f}(q^{3/2},-q^{5/2})=\mathfrak{f}\left(-q^3,-q^5\right)\phi(q^4). \tag{25}$$

Employing (23) and (25) in (20), we obtain

$$\frac{1}{M(q)}-M(q)=\frac{\mathfrak{f}(-q^3,-q^5)\phi(q^4)}{q^{3/2}\mathfrak{f}(-q,-q^7)\psi(q^8)}. \tag{26}$$

Again, employing (24) in (26), we arrive at

$$\frac{1}{M(q)} - M(q) = \frac{\mathfrak{f}^2(-q^3, -q^5)\phi(q^4)}{q^{3/2}\psi(q^8)\psi(q^4)\psi(-q)}. \tag{27}$$

From [5, p. 51] (with q by $-q$), we note that

$$\phi(q) + \phi(q^2) = \frac{2\mathfrak{f}^2(-q^3, -q^5)}{\psi(-q)} \tag{28}$$

and

$$\phi(q) - \phi(q^2) = \frac{2q\mathfrak{f}^2(-q, -q^7)}{\psi(-q)}. \tag{29}$$

Combining (28) and (27), we arrive at (i). Employing (8) in (26), we arrive at (ii). Proceeding as in the proofs of (i) and (ii), and using (29), we arrive at (iii) and (iv), respectively.

Squaring (19), we obtain

$$\frac{1}{M(q)} + M(q) = \frac{\mathfrak{f}^2(q^{3/2}, -q^{5/2})}{q^{3/2}\mathfrak{f}(-q, -q^{15})\mathfrak{f}(-q^7, -q^9)} - 2. \tag{30}$$

From [5, p. 46, Entry 30 (v),(vi)], we note that

$$\mathfrak{f}^2(a, b) = \mathfrak{f}(a^2, b^2)\phi(ab) + 2a\mathfrak{f}(b/a, a^3b)\psi(a^2b^2). \tag{31}$$

Setting $a = q^{3/2}$ and $b = -q^{5/2}$ in (31), we obtain

$$\mathfrak{f}^2(q^{3/2}, -q^{5/2}) = \mathfrak{f}(q^3, q^5)\phi(-q^4) + 2q^{3/2}\mathfrak{f}(-q, -q^7)\psi(q^8). \tag{32}$$

Employing (23) and (32) in (30) and simplifying, we arrive at the first equality of (v). Simplifying the first equality can be expressed as

$$\frac{1}{M(q)} + M(q) = \frac{\mathfrak{f}(q, q^3)\psi(q^4)\phi(-q^4)}{q^{3/2}\mathfrak{f}(-q^2, -q^{14})\psi(q^8)\phi(-q^8)}. \tag{33}$$

Replacing q by q^2 in (29), we obtain

$$\mathfrak{f}(-q^2, -q^{14}) = \left(\frac{\psi(-q^2)\left(\phi(q^2) - \phi(q^4)\right)}{2q^2}\right)^{1/2}. \tag{34}$$

Employing (5) and (34) in (33), we arrive at the second equality of (v). Proof of (vi) is identical to the proof of (v), so we omit it. Combining (v) and (vi), we obtain (xi) and (xii). Proofs of (vii), (viii), (ix), and (x) follow from (i) and (ii). □

Theorem 4. *Let n be a positive integer. Then*

(i) $M^n(q)M^n(-q) = \begin{cases} M^n(q^2), & if \quad n \equiv 0 \pmod 4 \\ -M^n(q^2), & if \quad n \equiv 2 \pmod 4, \end{cases}$

(ii) $N^n(q)N^n(-q) = \begin{cases} N^n(q^2), & if \quad n \equiv 0 \pmod 4 \\ -N^n(q^2), & if \quad n \equiv 2 \pmod 4. \end{cases}$

Proof. From (11), we obtain

$$M^n(q)M^n(-q) = (-1)^{3n/2}q^{3n}\frac{\mathfrak{f}^n(-q,-q^{15})}{\mathfrak{f}^n(-q^7,-q^9)} \times \frac{\mathfrak{f}^n(q,q^{15})}{\mathfrak{f}^n(q^7,q^9)}. \tag{35}$$

Setting $\{a = q, b = q^{15}\}$ and $\{a = q^7, b = q^9\}$ in (22), we find that

$$\mathfrak{f}(q,q^{15})\mathfrak{f}(-q,-q^{15}) = \mathfrak{f}(-q^2,-q^{30})\phi(-16) \tag{36}$$

and

$$\mathfrak{f}(q^7,q^9)\mathfrak{f}(-q^7,-q^9) = \mathfrak{f}(-q^{14},-q^{18})\phi(-16), \tag{37}$$

respectively.

Employing (36) and (37) in (35), we obtain

$$\begin{aligned} M^n(q)M^n(-q) &= (-1)^{3n/2}q^{3n}\frac{\mathfrak{f}^n(-q^2,-q^{30})}{\mathfrak{f}^n(-q^{14},-q^{18})} \\ &= (-1)^{3n/2}M^n(q^2). \end{aligned} \tag{38}$$

Now the desired result follows from (38) and noting the fact that $3n/2$ is even if $n \equiv 0 \pmod 4$ and odd if $n \equiv 2 \pmod 4$. The proof of (ii) is identical to the proof of (i), so we omit it. □

3. Explicit Evaluation of $M(\pm q)$ and $N(\pm q)$

In this section, we give general theorems for the explicit evaluations of $M(q)$, $N(q)$, $M^2(-q)$ and $N^2(-q)$ by the method of parametrization. We will use the parameter $s_{4,n}$ defined by

$$s_{4,n} = \frac{\mathfrak{f}(q)}{\sqrt{2}q^{1/8}\mathfrak{f}(-q^4)}, \tag{39}$$

where n is a positive real number. The parameter $s_{4,n}$ is the particular case $k = 4$ of the parameter $s_{k,n}$ defined by Berndt [6, p. 9, (4.7)]. Baruah and Saikia [4] proved the following formula for the explicit evaluation of $H(q)$ [4, p. 275, Theorem 3.1]:

$$H(e^{-\pi\sqrt{n}/4}) = -s_{4,n}^2 + \sqrt{s_{4,n}^4 + 1}. \tag{40}$$

Baruah and Saikia [4] calculated many values of the parameter $s_{4,n}$ to evaluate explicit values of $H(q)$ by appealing to (40).

Theorem 5. *If $U(q) = H(q^2)H(q)$ and $V(q) = H(q^2)/H(q)$, then*

(i) $\dfrac{1}{M(q)} - M(q) = \dfrac{1}{U(q)} - V(q),$

(ii) $\dfrac{1}{N(q)} - N(q) = \dfrac{1}{V(q)} - U(q).$

Proof. Employing (9) in Theorem 3(iii) and (iv), we arrive at (i) and (ii), respectively. □

From the above theorem, it is clear that if we know the values of $H(q)$ and $H(q^2)$, we can easily evaluate $U(q)$ and $V(q)$, and hence the explicit values of $M(q)$ and $N(q)$ can be evaluated by solving the corresponding quadratic equations. For that, we have Theorem 6, which follows from (40) and the definitions of $U(q)$ and $V(q)$.

Theorem 6. *We have*

(i) $U(e^{-\pi\sqrt{n}/4}) = \left(-s_{4,4n}^2 + \sqrt{s_{4,4n}^4 + 1}\right)\left(-s_{4,n}^2 + \sqrt{s_{4,n}^4 + 1}\right),$

(ii) $V(e^{-\pi\sqrt{n}/4}) = \left(-s_{4,4n}^2 + \sqrt{s_{4,4n}^4 + 1}\right) / \left(-s_{4,n}^2 + \sqrt{s_{4,n}^4 + 1}\right).$

Remark 1. From Theorem 6, it is easily seen that to evaluate the explicit values of $U(e^{-\pi\sqrt{n}/4})$ and $V(e^{-\pi\sqrt{n}/4})$, it is sufficient to know the values of $s_{4,n}$ and $s_{4,4n}$. Baruah and Saikia [4] evaluated explicit values of the parameters $s_{4,n}$ and $s_{4,4n}$ for $n = 1, 2, 3, 4, 5, 7, 8, 9, 10, 13, 15, 16, 18, 25,$ and 36. For example, employing the values $s_{4,1} = 2^{-5/16}(1+\sqrt{2})^{1/4}$ and $s_{4,4} = 2^{1/8}$ in Theorem 6(i) and (ii), we evaluate

$$U(e^{-\pi/4}) = \left(-2^{1/4} + \sqrt{1+\sqrt{2}}\right)\left(\frac{\sqrt{1+\sqrt{2}+2^{5/4}} - \sqrt{1+\sqrt{2}}}{2^{5/8}}\right)$$

and

$$V(e^{-\pi/4}) = \frac{2^{5/8}\left(-2^{1/4} + \sqrt{1+\sqrt{2}}\right)}{\sqrt{1+\sqrt{2}+2^{5/4}} - \sqrt{1+\sqrt{2}}},$$

respectively.

Next, employing above values of $U(e^{-\pi/4})$ and $V(e^{-\pi/4})$ in Theorem 5(i) and (ii), and solving the resulting equations, we evaluate

$$M(e^{-\pi/4}) = \frac{-2^{7/8} + \sqrt{2(1+2^{1/4})(1+\sqrt{2}-\sqrt{1+\sqrt{2}})}}{1+2^{1/4}-\sqrt{1+\sqrt{2}}} \tag{41}$$

and

$$N(e^{-\pi/4}) = \frac{-2^{9/8} + 2^{7/8}\sqrt{1+\sqrt{2}}}{-1+2^{1/4}-\sqrt{1+\sqrt{2}}}$$

$$-\frac{\sqrt{2(1+3\cdot 2^{1/4}+\sqrt{2}+2^{3/4}-(3+2^{1/4})\sqrt{1+\sqrt{2}})}}{-1+2^{1/4}-\sqrt{1+\sqrt{2}}}, \tag{42}$$

respectively.

Similarly, with the help of the Theorem 5 and Theorem 6, and the values $s_{4,4} = 2^{1/8}$ and $s_{4,16} = (1+\sqrt{2})^{1/2}$, we evaluate

$$M(e^{-\pi/2}) = \frac{1+\sqrt{2}-\sqrt{4+4\cdot\sqrt{2}-2^{5/4}\sqrt{1+\sqrt{2}}}}{2^{1/4}-\sqrt{1+\sqrt{2}}} \tag{43}$$

and

$$N(e^{-\pi/2}) = \frac{-1-\sqrt{2}+\sqrt{4+4\cdot\sqrt{2}+2^{5/4}\sqrt{1+\sqrt{2}}}}{2^{1/4}+\sqrt{1+\sqrt{2}}}. \tag{44}$$

Theorem 7. *We have*

(*i*) $M^2(-q) = -\dfrac{M^2(q^2)}{M^2(q)}$

(*ii*) $N^2(-q) = -\dfrac{N^2(q^2)}{N^2(q)}$

Proof. Setting $n = 2$ in Theorem 4 (i) and (ii), we arrive at (i) and (ii), respectively. □

Remark 2. From Theorem 7, it is obvious that if we know the explicit values of $M(q^2)$ and $M(q)$ (or $N(q^2)$ and $N(q)$), then explicit values of $M^2(-q)$ (or $N^2(-q)$) can be evaluated. For example, employing the values of $M(e^{-\pi/2})$ and $M(e^{-\pi/4})$ from (41) and (43) in Theorem 7(i), we evaluate

$$M^2(-e^{-\pi/4}) = -(\lambda_1/\lambda_2)^2 \cdot \left(\frac{1+2^{1/4}-\sqrt{1+\sqrt{2}}}{2^{1/4}-\sqrt{1+\sqrt{2}}}\right)^2, \tag{45}$$

where

$$\lambda_1 = 1+\sqrt{2}-\sqrt{4+4\cdot\sqrt{2}-2^{5/4}\sqrt{1+\sqrt{2}}}$$

and

$$\lambda_2 = -2^{7/8}+\sqrt{2(1+2^{1/4})\left(1+\sqrt{2}-\sqrt{1+\sqrt{2}}\right)}.$$

Similarly, employing the values of $N(e^{-\pi/2})$ and $N(e^{-\pi/4})$ from (42) and (44) in Theorem 7(ii), we obtain

$$N^2(-e^{-\pi/4}) = -(\lambda_3/\lambda_4)^2 \cdot \left(\frac{-1+2^{1/4}-\sqrt{1+\sqrt{2}}}{2^{1/4}+\sqrt{1+\sqrt{2}}}\right)^2, \tag{46}$$

where

$$\lambda_3 = 1+\sqrt{2}-\sqrt{4+4\cdot\sqrt{2}+2^{5/4}\sqrt{1+\sqrt{2}}}$$

and

$$\lambda_4 = 2^{9/8}+2^{7/8}\sqrt{1+\sqrt{2}}+\sqrt{2\left(1+3\cdot 2^{1/4}+\sqrt{2}+2^{3/4}-(3+2^{1/4})\sqrt{1+\sqrt{2}}\right)}.$$

4. Color Partition Identities

In this section, we see that color partition identities can be obtained from the theta-function identities of $M(q)$ and $N(q)$ established in Theorem 3. We demonstrate this by deriving a color partition identity from Theorem 3(iv). Similarly, color partition identities can be obtained from remaining theta-function identities of Theorem 3.

First, we define the partition and the color partition of a positive integer. A *partition* of a positive integer n is a non-increasing sequence of positive integers, called parts, whose sum equals n. For example, $n = 3$ has three partitions, namely, $3, \quad 2+1, \quad 1+1+1$. If $p(n)$ denotes the number of partitions of n, then $p(3) = 3$. The generating function for $p(n)$ due to Euler is given by

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q;q)_\infty}. \tag{47}$$

An *r-color partition* of a positive integer n is a partition in which each part appears in r distinct colors. For any positive integer n and r, let $C_r(n)$ denote the number of partitions of n with each part having r distinct colors. For example, if each part of a partition of 3 has 2 colors, say white (indicated by the suffix w) and black (indicated by the suffix b), then the number of 2-color partitions of 3 is 10, namely, $3_w, \quad 3_b, \quad 2_w+1_w, \quad 2_w+1_b, \quad 2_b+1_w, \quad 2_b+1_b, \quad 1_w+1_w+1_w, \quad 1_w+1_w+1_b, \quad 1_w+1_b+1_b, \quad 1_b+1_b+1_b$. The generating function of $C_r(n)$ is given by

$$\sum_{n=0}^{\infty} C_r(n)q^n = \frac{1}{(q;q)_\infty^r}. \tag{48}$$

For positive integers s, m and r,

$$\frac{1}{(q^s;q^m)_\infty^r} \tag{49}$$

is the generating function of the number of partitions of n with parts congruent to s modulo m and each part having r colors. For example, if s_1 and s_2 are positive integers, then

$$\frac{1}{(q^{s_1};q^m)_\infty^r (q^{s_2};q^m)_\infty^r} = \frac{1}{(q^{s_1},q^{s_2};q^m)_\infty^r} \tag{50}$$

is the generating function of the number of partitions of positive integer with parts congruent to s_1 or s_2 modulo m and each part having r colors. Here we use the notation

$$(q^{r\pm};q^m) := (q^r, q^{m-r}; q^m)_\infty, \tag{51}$$

where r and m are positive integers and $r < m$.

Theorem 8. *Let $C_1(n)$ denote the number of partitions of n into parts congruent to ±1,±3, ±7 or ±8 $\pmod{16}$ such that the parts congruent to ±3 and ±8 $\pmod{16}$ have 2 colors. Let $C_2(n)$ denote the number of partitions of n into parts congruent to ±1, ±5, ±7 or ±8 $\pmod{16}$ such that parts congruent to ±5 and ±8 $\pmod{16}$ have 2 colors. Let $C_3(n)$ denote the number of partitions of n into parts congruent to ±3, ±4 or ±5 $\pmod{16}$ with 2 colors.*
Then for any integer $n \geq 1$,

$$C_1(n) - C_2(n-1) - C_3(n) = 0. \tag{52}$$

Proof. Employing (4), (5), (8) and (12) in Theorem 3(iv), we obtain

$$\frac{(q^{5\pm};q^{16})_\infty}{(q^{3\pm};q^{16})_\infty} - q\frac{(q^{3\pm};q^{16})_\infty}{(q^{5\pm};q^{16})_\infty} - \frac{(q^{1\pm,7\pm};q^{16})_\infty(q^{8\pm};q^{16})^2_\infty}{(q^{3\pm,5\pm};q^{16})_\infty(q^{4\pm};q^{16})^2_\infty} = 0. \tag{53}$$

Dividing (53) by $(q^{1\pm,3\pm,5\pm,7\pm};q^{16})_\infty(q^{8\pm},q^{16})^2_\infty$, we obtain

$$\frac{1}{(q^{1\pm,7\pm};q^{16})_\infty(q^{3\pm,8\pm};q^{16})^2_\infty} - \frac{q}{(q^{1\pm,7\pm};q^{16})_\infty(q^{5\pm,8\pm};q^{16})^2_\infty} = \frac{1}{(q^{3\pm,4\pm,5\pm};q^{16})^2_\infty}. \tag{54}$$

The above quotients of (54) represent the generating functions for $C_1(n)$, $C_2(n)$ and $C_3(n)$, respectively. Hence, (54) is equivalent to

$$\sum_{n=0}^{\infty} C_1(n)q^n - q\sum_{n=0}^{\infty} C_2(n)q^n - \sum_{n=0}^{\infty} C_3(n)q^n = 0, \tag{55}$$

where we set $C_1(0) = C_2(0) = C_3(0) = 1$. Equating coefficients of q^n on both sides of (55), we obtain the desired result. □

Example: The following table illustrates the case $n = 3$ in Theorem 8:

$C_1(3) = 3$	$C_2(2) = 1$	$C_3(3) = 2$
3_r	$1+1$	3_r
3_g		3_g
$1+1+1$		

5. Matching Coefficients

In this section, we offer matching coefficient results arising from the theta-function identities of the continued fractions $M(q)$ and $N(q)$ with their reciprocals. Recently, Baruah and Das [3] established several matching coefficient results for the series expansion of certain q-products and their reciprocals. We first give the definition of the matching coefficients from [3].

For any two power series $\sum_{n=0}^{\infty} C_n q^n$ and $\sum_{n=0}^{\infty} D_n q^n$, if for some positive integers s, t and k, and non-negative integers u and v, $C(sn+u) = \pm kD(tn+v)$, for all $n \geq 0$, then the two power series are said to have *matching coefficients*.

Theorem 9. *If*

$$q^{3/2}M(q) = q^3 \frac{(q, q^{15}; q^{16})_\infty}{(q^7, q^9; q^{16})_\infty} = \sum_{n=0}^{\infty} a_n q^n$$

and

$$q^{3/2}\frac{1}{M(q)} = \frac{(q^7, q^9; q^{16})_\infty}{(q, q^{15}; q^{16})_\infty} = \sum_{n=0}^{\infty} a'_n q^n,$$

then

$$a_{4n+3} = a'_{4n+3}.$$

Proof. Employing (11) and Lemma 1 in Theorem 3(i), we obtain

$$\frac{(q^7, q^9; q^{16})_\infty}{q^{3/2}(q, q^{15}; q^{16})_\infty} - q^{3/2}\frac{(q, q^{15}; q^{16})_\infty}{(q^7, q^9; q^{16})_\infty} = \frac{f_2^5 f_8^4}{2q^{3/2} f_1^2 f_4^3 f_{16}^4} + \frac{f_4^4 f_8^2}{2q^{3/2} f_2^2 f_{16}^4}. \tag{56}$$

Multiplying both sides of (56) by $2q^{3/2}$ and then employing Lemma 2, we obtain

$$2\frac{(q^7, q^9; q^{16})_\infty}{(q, q^{15}; q^{16})_\infty} - 2q^3\frac{(q, q^{15}; q^{16})_\infty}{(q^7, q^9; q^{16})_\infty} = \frac{f_2^5 f_8^4}{f_4^3 f_{16}^4}\left(\frac{f_8^5}{f_2^5 f_{16}^2} + 2q\frac{f_4^2 f_{16}^2}{f_2^5 f_8}\right) + \frac{f_4^4 f_8^2}{f_2^2 f_{16}^4}. \tag{57}$$

Simplifying (57), we obtain

$$2\sum_{n=0}^{\infty} a'_n q^n - 2\sum_{n=0}^{\infty} a_n q^n = \frac{f_8^9}{f_4^3 f_{16}^6} + 2q\frac{f_8^3}{f_4 f_{16}^2} + \frac{f_4^4 f_8^2}{f_2^2 f_{16}^4}. \tag{58}$$

Extracting the terms involving q^{2n+1}, dividing by q and then replacing q^2 by q, we obtain

$$\sum_{n=0}^{\infty} a'_{2n+1} q^n - \sum_{n=0}^{\infty} a_{2n+1} q^n = \frac{f_4^3}{f_2 f_8^2}. \tag{59}$$

The right hand side of (59) contains no term involving q^{2n+1}, so extracting terms involving q^{2n+1}, dividing by q and replacing q^2 by q, we obtain

$$\sum_{n=0}^{\infty} a'_{4n+3} q^n - \sum_{n=0}^{\infty} a_{4n+3} q^n = 0. \tag{60}$$

Equating the coefficients of q^n, we obtain the desired result. □

The proof of Theorem 10 is similar to the proof of Theorem 9 and follows from (12), Lemma 1, Lemma 2 and Theorem 3(iii).

Theorem 10. *If*

$$q^{3/2}N(q) = q^2\frac{(q^3, q^{13}; q^{16})_\infty}{(q^5, q^{11}; q^{16})_\infty} = \sum_{n=0}^{\infty} b_n q^n$$

and

$$q^{3/2}\frac{1}{N(q)} = q\frac{(q^5, q^{11}; q^{16})_\infty}{(q^3, q^{13}; q^{16})_\infty} = \sum_{n=0}^{\infty} b'_n q^n,$$

then

$$b_{4n+3} = b'_{4n+3}.$$

Acknowledgements. The authors are grateful to the anonymous referee, who read our manuscript with great care and offered useful suggestions. The first author acknowledges the financial support received from the Department of Science and Technology (DST), Government of India through INSPIRE Fellowship [DST/INSPIRE Fellowship/2021 /IF210210].

References

[1] C. Adiga, B. C. Berndt, S. Bhargava and G. N. Watson, Chapter 16 of Ramanujan's second notebook: Theta functions and q-series, *Mem. Am. Math. Soc.* **315** (1985), 1-91.

[2] G. E. Andrews, On q-difference equations for certain well-poised basic hypergeometric series, *Quart. J. Math. (Oxford)* **19** (1968), 433-447.

[3] N. D. Baruah and H. Das, Matching coefficients in the series expansions of certain q-products and their reciprocals, *Ramanujan J.* **59** (2022), 511-548.

[4] N. D. Baruah and N. Saikia, Explicit evaluations of Ramanujan-Göllnitz-Gordon continued fraction, *Monatsh. Math.* **154**(4) (2008), 271-288.

[5] B. C. Berndt, *Ramanujan's Notebooks, Part III*, Springer-Verlag, New York, 1991.

[6] B. C. Berndt, Flowers which we cannot yet see growing in Ramanujan's garden of hypergeometric series, elliptic functions, and q's, in Special Functions 2000: Current Perspective and Future Directions (J. Bustoz, M. E. H. Ismail, and S. K. Suslov, eds.), Kluwer, Dordrecht, (2001), 61-85.

[7] H. H. Chan and S.-S. Huang, On the Ramanujan-Göllnitz-Gordon continued fraction, *Ramanujan J.* **1** (1997), 75-90.

[8] W. Chu, Common source of numerous theta function identities, *Glasgow Math. J.* **49** (2007), 61-79.

[9] J. A. Ewell, Arithmetical consequences of a sextuple product identity, *Rocky Mountain J. Math.* **25(4)** (1995), 1287-1293.

[10] J. A. Ewell, A note on a Jacobian identity, *Proc. Amer. Math. Soc.* **126(2)** (1998), 421-423.

[11] H. Göllnitz, Partitionen mit Differenzenbedingungen, *J. Reine Angew. Math.* **225** (1967), 154-190.

[12] B. Gordon, Some continued fractions of Rogers-Ramanujan's type, *Duke Math.* **16** (1985), 695-724.

[13] M. D. Hirschhorn, *The Power of q*, Springer International Publishing, Switzerland, 2017.

[14] K. G. Ramanathan, Ramanujan's continued fraction, *Indian J. Pure Appl. Math.* **16** (1985), 695-724.

[15] S. Ramanujan, *Notebooks* (2 volumes), Tata Institute of Fundamental Research, Bombay, 1957.

[16] M. S. Surekha, On the modular relations and dissections for a continued fraction of order sixteen, *Palestine J. Math.* **6**(1) (2017), 119-132.

[17] A. Vanitha, On the expansion of Ramanujan's continued fraction of order sixteen, *Tamsui Oxf. J. Inf. Math.* **31**(1) (2017), 81-99.

[18] K. R. Vasuki and B. R. Srivatsa Kumar, Certain identities for Ramanujan-Göllnitz Gordon continued fraction, *J. Comput. Appl. Math.* **187** (2006), 87-95.