A UNIFIED GENERALIZATION OF TOUCHARD AND FUBINI POLYNOMIAL EXTENSIONS

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Abstract

In this paper, a unified generalization for several extensions of Fubini and Touchard polynomials is considered. A combinatorial interpretation of the generalization in terms of generalized hierarchical orderings is examined. Also, asymptotic results are obtained, and a linear operator approach is used to obtain closed form expressions.

1. Introduction

Inserting bars between blocks of an ordered set partition induces a *barred preferential arrangement* (see [3, 26]). The magnitude of the spaces between elements indicates which elements belong to the same block. For instance, for the set $[6] = \{1, 2, \ldots, 6\}$, the following are possible barred preferential arrangements with 2 and 3 bars respectively:

(I) $26|1 \quad 5|34$,

(II) $|14 \quad 36| \quad |23 \quad 5$.

In the barred preferential arrangement in (I), the two bars induce three sections. In the first section (section to the left of the first bar), there is only one block $\{2, 6\}$. In the second section (between the two bars), there are two singleton blocks $\{1\}$ and $\{5\}$. In the third section, we have only one block $\{3, 4\}$. In the barred preferential arrangement in (II), the three bars induce four sections. The first section is empty. In the second section, we have two blocks: the first block is $\{1, 4\}$, and the second

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block is $\{3, 6\}$. The third section is empty. In the fourth section, we have two blocks. The first block is $\{2, 3\}$, and the second block is the singleton $\{5\}$. Note that in forming the barred preferential arrangements in the current study, the bars are taken to be identical.

The Hsu-Shiue generalized Stirling numbers $S(n, k, q, \beta, \gamma)$ (see [16]) can be stated in the following way:

$$(t|(1-q))_n = \sum_{k=0}^{n} S(n, k, q, \beta, \gamma)(t-\gamma|\beta)_n,$$

where $(t|(1-q))_n$ is the generalized factorial polynomial

$$(t|(1-q))_n = \prod_{k=0}^{n-1}(t-k(1-q)),$$

such that $n \geq 1$ and q, β, γ are real or complex numbers.

For real or complex q, β, λ, a generalization of Fubini polynomials (see [17, 18]) is

$$\sum_{n=0}^{\infty} \omega_n^{(\lambda)}(x; q, \beta, \gamma)\frac{t^n}{n!} = \frac{[1+(1-q)t]^{\frac{\gamma}{(1-q)}}}{(1-x((1+(1-q)t)^{\beta/(1-q)}-1))^{\lambda}}. \tag{1}$$

For $\beta, \lambda \in \mathbb{N}$ and $q \in \mathbb{Z}^- = \{-1, -2, -3, \ldots\}$, combinatorial properties of the polynomials in Equation (1) have been studied in [24], and related polynomials in [8, 25].

Lemma 1. *[10] Let β, γ be nonnegative integers. Let $q \in \mathbb{Z}^-$ be such that $(1-q)|\beta$ and $(1-q)|\gamma$. Suppose the following conditions are true:*

- *there are $k+1$ distinct cells,*
- *each of the first k cells (from left to right) contains β labelled compartments (the cells are called ordinary cells),*
- *the $(k+1)th$ cell contains γ labelled compartments (the cell is called a special cell),*
- *in each cell, compartments have cyclic ordered numbering,*
- *the capacity of each compartment is limited to one element,*
- *in each consecutive available $(1-q)$ compartments only the first compartment would be occupied by an element, the $(1-q)$ compartments are called a $(1-q)$-block.*

Then the number of ways of distributing elements of $[n]$ into the compartments satisfying the above conditions, one element at a time, such that the first k cells are nonempty, is $\beta^k k! S(n, k, q, \beta, \gamma)$.

In [24], it is noted that for all $k = 0, 1, \ldots, n$, the number of ways of distributing $[n]$ into cells satisfying the conditions of Lemma 1 above, such that $\lambda - 1$ identical bars are inserted before, in between, and immediately after the first k cells, and each of the ordinary cells is colored, each with one of x available colors, is

$$\omega_n^{(\lambda)}(x; q, \beta, \gamma) = \sum_{k=0}^{n} \binom{k+\lambda-1}{k} k! S(n, k, q, \beta, \gamma) \beta^k x^k. \tag{2}$$

The blocks of an *ordered set partition* can be viewed as a *preferential arrangement* of the elements of the set. For instance, the first block of an ordered set partition from left to right can be considered the least preferred elements. Hence, throughout the paper the use of the terms ordered set partition and preferential arrangement is meant to mean the same thing. Blocks of a preferential arrangement can also be viewed from left to right as elements being ranked based on some relative status. For instance, the first block of an ordered set partition can be considered as the lowest level of a hierarchy. Hence, by a *hierarchy* we also mean an ordered set partition. In different parts of this paper we use the more appropriate word depending on the context. Partitioning $[n]$ into nonempty disjoint subsets is done, such that on each of the subsets a preferential arrangement/hierarchy is formed, following which a fixed collection of all those preferential arrangements/hierarchies forms a single *hierarchical ordering*. The integer sequence for the number of hierarchical orderings is A075729 in [28]. The generating function for the number of hierarchical orderings on $[n]$ is (see [29])

$$\sum_{n=0}^{\infty} H_n \frac{t^n}{n!} = \exp\left(\frac{1}{2-e^t} - 1\right). \tag{3}$$

As an example, let us consider the possible hierarchical orderings of the set [2]. There are four possible hierarchical orderings:

- when we have only one hierarchy, there are three possibilities; 1 2, 2 1, 12;
- when we have two hierarchies, one hierarchy will be composed of the element $\{1\}$ and the other of the element $\{2\}$.

Hierarchical orderings are studied in Motzkin's classical paper [23]. Touchard polynomials, also known as exponential or Bell polynomials, are defined in the following way (see [5, 13, 30, 31]):

$$\sum_{n=0}^{\infty} T_n(u) \frac{t^n}{n!} = \exp(u(e^t - 1)). \tag{4}$$

As a generalization of Equations (3) and (4), we define (using Tsallis' generalized exponential function (see [32]))

$$\sum_{n=0}^{\infty} H_n^{(\lambda,u,p,\delta)}(x;q,\beta,\gamma)\frac{t^n}{n!} = \left[1+(1-p)u\left[\frac{(1+(1-q)t)^{\frac{\gamma}{1-q}}}{(1-x((1+(1-q)t)^{\frac{\beta}{1-q}}-1))^{\lambda}} - 1\right]\right]^{\frac{\delta}{1-p}}, \quad (5)$$

for real or complex numbers $\lambda, \delta, \beta, \gamma, p, q, x, u$.

The generating function in Equation (5) is the main generating function in this paper. The polynomials in Equation (5) are a unified generalization of both Fubini and Touchard polynomial extensions. Both Fubini and Touchard polynomials have a rich history. Over the years they have cropped up in many different branches of mathematics and science, and are continuously being rediscovered in many branches of mathematics. In recent years, many extensions of both kinds of polynomials have been investigated (see, for instance, [3, 4, 6, 9, 11, 13, 17, 19, 20, 21, 25, 26]). The following are special cases of the polynomials in Equation (5).

- The higher order degenerate Euler polynomials discussed in [17] are obtained when $(\lambda, \gamma, \delta, p, q, u, x, \beta) = (\lambda, \gamma, 1, 0, q, 1, -\frac{1}{2}, 1)$.
- The two variable higher order Fubini polynomials discussed in [20, 27] are obtained when $(\lambda, \gamma, \delta, p, q, u, x, \beta) = (\lambda, \gamma, 1, 0, 1, 1, x, 1)$.
- The (p, q)-deformed Touchard polynomials discussed in [13] are obtained when $(\lambda, \gamma, \delta, p, q, u, x, \beta) = (0, 1, 1, p, q, u, x, \beta)$.
- The partially degenerate Bell polynomials discussed in [21, 22] are obtained when $(\lambda, \gamma, \delta, p, q, u, x, \beta) = (0, 1, 1, 1, q, u, x, \beta)$.
- The higher order generalized geometric polynomials discussed in [17] are obtained when $(\lambda, \gamma, \delta, p, q, u, x, \beta) = (\lambda, \gamma, 1, 0, q, 1, x, \beta)$.

In the following section, we propose a combinatorial interpretation of the polynomials in Equation (5). Thereafter, we discuss some asymptotic results using a method developed in [14, 15]. Finally, we offer a linear operator interpretation by using linear operators having Poisson and negative binomial weights.

2. Main Generating Function and Combinatorial Interpretations

In this section, we study some special cases of the main generating function given in Equation (5). We also discuss its possible combinatorial interpretation.

2.1. The Case $(p,\delta)=(1,1)$, $\beta,\gamma\in\mathbb{N}_0=\{0,1,2,\ldots\}$, and $q\in\mathbb{Z}^-$

The following formula is a consequence of Equation (5):

$$\sum_{n=0}^{\infty} H_n^{(\lambda,u,1,1)}(x;q,\beta,\gamma)\frac{t^n}{n!}=\exp\left(\frac{u(1+(1-q)t)^{\frac{\gamma}{1-q}}}{(1-x((1+(1-q)t)^{\frac{\beta}{1-q}}-1))^{\lambda}}-u\right). \tag{6}$$

Clearly, by Equations (1) and (3), $H_n^{(\lambda,u,1,1)}(x;q,\beta,\gamma)$ is the number of hierarchies of $[n]$ into u hierarchical orderings, where in each hierarchical ordering the hierarchies are the barred preferential arrangements described in Equation (2). We will refer to these hierarchies as *barred hierarchies*. In Equation (6), we assume that $(1-q)$ divides both β and γ.

Let

$$H_{n,k}^{(\lambda,1,1,1)}(x;q,\beta,\gamma)=\left[\tfrac{t^n}{n!}\right]\left[\frac{(1+(1-q)t)^{\frac{\gamma}{1-q}}}{(1-x((1+(1-q)t)^{\frac{\beta}{1-q}}-1))^{\lambda}}-1\right]^k\times\frac{1}{k!}. \tag{7}$$

It follows from Equation (7) that $H_{n,k}^{(\lambda,u,1,1)}(x;q,\beta,\gamma)$ is the number of hierarchies of $[n]$ into a single hierarchical ordering having k barred hierarchies.

Theorem 1. *For $\beta,\lambda\in\mathbb{N}$, $(1-q)|\beta$, and $(1-q)|\gamma$, we have*

$$H_{n+1}^{(\lambda,u,1,1)}(x;q,\beta,\gamma)=u\sum_{k=0}^{n}\binom{n}{k}H_k^{(\lambda,u,1,1)}(x;q,\beta,\gamma)\omega_{n+1-k}^{(\lambda)}(x;q,\beta,\gamma). \tag{8}$$

Proof. Consider the position of the $(n+1)$st element. There are $\binom{n}{k}$ ways of selecting k elements from $[n]$ that are not in the same barred hierarchy as $(n+1)$. The k elements can form u hierarchical orderings in $H_k^{(\lambda,u,1,1)}(x;q,\beta,\gamma)$ ways. Together with the $n-k$ other elements, the element $(n+1)$ can form a single barred hierarchy in $\omega_{n+1-k}^{(\lambda)}(x;q,\beta,\gamma)$ ways. The formed hierarchy can be placed in one of the hierarchical orderings in $\binom{u}{1}$ ways. $\square$

Special cases of Theorem 1 include Equation (2) in [29], Theorem 2 in [22], and Theorem 2.4 in [21].

Theorem 2. *For $u,\beta,\lambda\in\mathbb{N}$, $(1-q)|\beta$, and $(1-q)|\gamma$, we have*

$$\begin{aligned}H_{n+1}^{(\lambda,u,1,1)}(x;q,\beta,\gamma)&=u\gamma\sum_{k=0}^{n}\binom{n}{k}\omega_k^{(\lambda)}(x;q,\beta,\gamma-(1-q))H_{n-k}^{(\lambda,u,1,1)}(x;q,\beta,\gamma)\\&\quad+ux\beta\lambda\sum_{k=0}^{n}\binom{n}{k}\omega_k^{(\lambda-1)}(x;q,\beta,\gamma+\beta-(1-q))H_{n-k}^{(\lambda,u,1,1)}(x;q,\beta,\gamma).\end{aligned} \tag{9}$$

Proof. Recalling Lemma 1, we consider the following two cases.

`Case 1`: Within a barred hierarchy, the $(n+1)$st element is in a special cell. Say there are k other elements together with $(n+1)$ in this barred hierarchy. The k other elements can be chosen in $\binom{n}{k}$ ways. There are γ ways of placing $(n+1)$ within the special cell. The k other elements, together with $(n+1)$, can form a barred hierarchy in $\gamma\omega_k^{(\lambda)}(x;q,\beta,\gamma-(1-q))$ ways. Using the remaining $n-k$ elements that are not in the same barred hierarchy as $(n+1)$, u hierarchical orderings can be formed in $H_{n-k}^{(\lambda,u,1,1)}(x;q,\beta,\gamma)$ ways. The barred hierarchy having $(n+1)$ can be placed in one of the u hierarchical orderings in u ways. Thus, the number of possible arrangements in this case is $u\gamma\sum_{k=0}^{n}\binom{n}{k}\omega_k^{(\lambda)}(x;q,\beta,\gamma-(1-q))H_{n-k}^{(\lambda,u,1,1)}(x;q,\beta,\gamma)$.
`Case 2`: Within a barred hierarchy, the $(n+1)$st element is in an ordinary cell. Within a barred hierarchy, there are λ ways of selecting a section to which $(n+1)$ belongs; there are β ways of choosing a compartment within the cell; and there are x ways of coloring the cell. Consider the barred hierarchy having $(n+1)$. View the cell occupied by $(n+1)$ and the special cell of the barred hierarchy as a single unit, and call it B. Say as part of B there are k other elements (chosen in $\binom{n}{k}$ ways). Such barred hierarchies can be formed in $\omega_k^{(\lambda-1)}(x;q,\beta,\gamma+\beta-(1-q))$ ways. The other $n-k$ elements that are not in the same barred hierarchy as $(n+1)$ can form u hierarchical orderings in $H_{n-k}^{(\lambda,u,1,1)}(x;q,\beta,\gamma)$ ways. Thus, in total, there are $ux\beta\lambda\sum_{k=0}^{n}\binom{n}{k}\omega_k^{(\lambda-1)}(x;q,\beta,\gamma+\beta-(1-q))H_{n-k}^{(\lambda,u,1,1)}(x;q,\beta,\gamma)$ possible hierarchical orderings in this case. □

Theorem 2 is a generalization of Theorem 3.2 in [24]. The following identity can be viewed as a special case of the Bell partition polynomial identity in Equation (11.11) in [7] obtained by using an algebraic technique:

$$H_{n+1,k+1}^{(\lambda,1,1,1)}(x;q,\beta,\gamma)=\sum_{i=k}^{n}\binom{n}{i}H_{i,k}^{(\lambda,1,1,1)}(x;q,\beta,\gamma)\omega_{n-i+1}^{(\lambda)}(x;q,\beta,\gamma). \tag{10}$$

One can obtain a combinatorial proof of Equation (10) in the following way. We construct a single hierarchical ordering of $[n+1]$ having $k+1$ barred hierarchies. Say there are $n-i$ other elements in the barred hierarchy occupied by the $(n+1)$st element. There are $\binom{n}{i}$ ways of choosing the $n-i$ elements. A barred hierarchy can be formed in $\omega_{n-i+1}^{(\lambda)}(x;q,\beta,\gamma)$ ways. The other k barred hierarchies can be formed in $H_{i,k}^{(\lambda,1,1,1)}(x;q,\beta,\gamma)$ ways.

Theorem 3. *For $u,\beta,\lambda\in\mathbb{N}$, $(1-q)|\beta$, and $(1-q)|\gamma$, we have*

$$(k+1)H_{n+1,k+1}^{(\lambda,1,1,1)}(x;q,\beta,\gamma)=\sum_{i=0}^{n-k}\binom{n+1}{i+1}H_{n-i,k}^{(\lambda,1,1,1)}(x;q,\beta,\gamma)\omega_{i+1}^{(\lambda)}(x;q,\beta,\gamma). \tag{11}$$

Proof. We construct a single hierarchical ordering of $[n+1]$ with $k+1$ barred hierarchies. There are $\binom{n+1}{i+1}$ ways of choosing $i+1$ elements to form one of the

$k+1$ barred hierarchies. The barred hierarchy can be formed in $\omega^{(\lambda)}_{i+1}(x;q,\beta,\gamma)$ ways. The other k barred hierarchies can be formed in $H^{(\lambda,1,1,1)}_{n-i,k}(x;q,\beta,\gamma)$ ways. In light of Equation (10), constructing a hierarchical ordering having k hierarchies in this way leads to an overcount. The repetition comes from the fact that the element $(n+1)$ can be in any of the $k+1$ hierarchies. As a result, we need to subtract $kH^{(\lambda,1,1,1)}_{n+1,k+1}(x;q,\beta,\gamma)$. Thus, the number of barred hierarchies is $\sum_{i=0}^{n-k}\binom{n+1}{i+1}H^{(\lambda,1,1,1)}_{n-i,k}(x;q,\beta,\gamma)\omega^{(\lambda)}_{i+1}(x;p,\beta,\gamma)-kH^{(\lambda,1,1,1)}_{n+1,k+1}(x;q,\beta,\gamma)$. □

The identity in Equation (11) can be viewed as a special case of the Bell partition polynomial identity given in Equation (11.12) in [7].

2.2. The General Case

In this subsection, we consider the polynomials generated by

$$\sum_{n=0}^{\infty}H^{(\lambda,u,p,\delta)}_{n}(x;q,\beta,\gamma)\frac{t^n}{n!}=\left[1+(1-p)u\left[\frac{(1+(1-q)t)^{\frac{\gamma}{1-q}}}{(1-x((1+(1-q)t)^{\frac{\beta}{1-q}}-1))^{\lambda}}-1\right]\right]^{\frac{\delta}{1-p}}, \tag{12}$$

for $\delta,\beta,\gamma\in\mathbb{N}_0$ and $p,q\in\mathbb{Z}^-$. Comparing Equations (7) and (12), we have

$$H^{(\lambda,u,p,\delta)}_{n}(x;q,\beta,\gamma)=\sum_{k=0}^{n}(\delta|(1-p))_k u^k H^{(\lambda,1,1,1)}_{n,k}(x;q,\beta,\gamma). \tag{13}$$

From Equation (13), one can give the following combinatorial interpretation of $H^{(\lambda,u,p,\delta)}_{n}(x;q,\beta,\gamma)$. The number $H^{(\lambda,u,p,\delta)}_{n}(x;q,\beta,\gamma)$ is the number of hierarchies of $[n]$ into u hierarchical orderings, such that each hierarchical ordering is composed of barred hierarchies as described in Equation (2). In the hierarchical orderings discussed in Subsection 2.1, the barred hierarchies were unordered with respect to each other. We now consider the case where the barred hierarchies are ordered one at a time into δ positions (cyclically ordered), where in each available $(1-p)$ positions only the first position can be occupied by a barred hierarchy. We will refer to the $(1-p)$ positions as $(1-p)$-blocks. Also, each of the barred hierarchies is colored with one of u available colors. For the case $\delta=1$ and $\lambda=0$, Equations (14), (16), and (17) below were obtained algebraically in [13]. Here, we provide combinatorial proofs of their generalizations.

Theorem 4. *For $\delta,\beta,\gamma\in\mathbb{N}_0$, $p,q\in\mathbb{Z}^-$, and $(1-p)|\delta$, we have*

$$\begin{aligned}H^{(\lambda,u,p,\delta)}_{n+1}(x;q,\beta,\gamma)=&u\sum_{m=0}^{n}\left[\binom{n}{m}\delta-(1-p)\binom{n}{m-1}\right]\omega^{(\lambda)}_{n-m+1}(x;q,\beta,\gamma)\\&\times H^{(\lambda,u,p,\delta)}_{m}(x;q,\beta,\gamma).\end{aligned} \tag{14}$$

Proof. The proof is based on the position of the barred hierarchy having the $(n+1)$st element. We will refer to it as the *chosen barred hierarchy.*

`Case 1`: All elements of $[n+1]$ are in the chosen barred hierarchy. The chosen barred hierarchy can be colored in $\binom{u}{1}$ ways and placed in one of δ positions in $u\delta\omega_{n+1}^{\lambda}(x;\alpha,\beta,\gamma)$ ways.

`Case 2`: We use an inclusion/exclusion argument. From $[n]$, we choose $n-m$ elements in $\binom{n}{n-m}$ ways and form a barred hierarchy as the chosen barred hierarchy, and place it in one of δ positions. This can be done in $\delta\omega_{n-m+1}^{(\lambda)}(x;q,\beta,\gamma)$ ways. Using the unselected m elements without any restriction, one can form hierarchical orderings and place the formed barred hierarchies in one of δ positions in $H_m^{(\lambda,u,p,\delta)}(x;q,\beta,\gamma)$ ways.

Constructing hierarchical orderings in the above way leads to overcount. In fact, the arrangements enumerated by $H_m^{(\lambda,u,p,\delta)}(x;q,\beta,\gamma)$ are unrestricted as in the δ positions, the $(1-p)$-block already occupied by the chosen barred hierarchy was not considered as already occupied. To account for this, without loss of generality, say of the δ positions, the $(1-p)$-block occupied by the chosen barred hierarchy is $1,2,\ldots,1-p$. A way in which at least one of the positions $1,2,\ldots,1-p$ is occupied by a non-chosen barred hierarchy is the following. From the set $[n]$, select $\binom{n}{m-1}$ elements, and form a barred hierarchy of $[n-(m-1)]$ in one of the $(1-p)$ positions. The formed barred hierarchy can be colored and placed in one of the $(1-p)$ positions in $u(1-p)\omega_{n-(m-1)}^{(\lambda)}(x;q,\beta,\gamma)$ ways, and the other $m-1$ elements, together with the element $(n+1)$, can form hierarchical orderings in $H_m^{(\lambda,u,p,\delta)}(x;q,\beta,\gamma)$ ways. Thus, the total number of arrangements in this case is
$u\sum_{m=0}^{n}\left[\binom{n}{m}\delta-(1-p)\binom{n}{m-1}\right]\omega_{n-m+1}^{(\lambda)}(x;q,\beta,\gamma)H_m^{(\lambda,u,p,\delta)}(x;q,\beta,\gamma).$ □

By Equation (12), we have

$$\begin{aligned}&\frac{d}{du}H_n^{(\lambda,u,p,\delta)}(x;q,\beta,\gamma)\\&\quad=\delta\sum_{m=0}^{n-1}\binom{n}{m}\omega_{n-m}^{(\lambda)}(x;q,\beta,\gamma)H_m^{(\lambda,u,p,\delta-(1-p))}(x;q,\beta,\gamma).\end{aligned}\tag{15}$$

Hence, we deduce that $\frac{d}{du}H_n^{(\lambda,u,p,\delta)}(x;q,\beta,\gamma)$ gives the number of arrangements of $[n]$ into hierarchical orderings placed in δ positions, such that only one of the barred hierarchies is not colored and the $(1-p)$-block occupied by the non-colored barred hierarchy ordering is pre-specified.

Theorem 5. *For $\delta,\beta,\gamma\in\mathbb{N}_0$, $p,q\in\mathbb{Z}^-$, and $(1-p)|\delta$, we have*

$$\frac{d}{du}H_n^{(\lambda,u,p,\delta)}(x;q,\beta,\gamma)+(1-p)u\sum_{m=0}^{n-1}\binom{n}{m}\frac{d}{du}H_m^{(\lambda,u,p,\delta)}(x;q,\beta,\gamma)\omega_{n-m}^{\lambda}(x;q,\beta,\gamma)$$
$$=\delta\sum_{m=0}^{n-1}\binom{n}{m}\omega_{n-m}^{\lambda}(x;q,\beta,\gamma)H_m^{(\lambda,u,p,\delta)}(x;q,\beta,\gamma). \tag{16}$$

The above theorem can be proved in a similar way to Theorem 4. The number of hierarchical orderings of $[n]$ where the formed barred hierarchies are placed in $\delta+(1-p)$ positions, such that one barred hierarchy is not colored and the $(1-p)$-block of the noncolored barred hierarchy is pre-specified, is given by the following identity:

$$\begin{aligned}(\delta+(1-p))&\sum_{m=0}^{n-1}\binom{n}{m}H_{m+1}^{(\lambda,u,p,\delta)}(x;q,\beta,\gamma)\omega_{n-m}^{(\lambda)}(x;q,\beta,\gamma)\\ &=(\delta+(1-p))u\sum_{m=0}^{n-1}\binom{n}{m}\omega_{n-m}^{(\lambda)}(x;q,\beta,\gamma)\frac{d}{du}H_{m+1}^{(\lambda,u,p,\delta)}(x;q,\beta,\gamma).\end{aligned} \tag{17}$$

3. Asymptotic Results

In this section, we offer some asymptotic results for the numbers $H_n^{(\lambda,u,p,\delta)}(x;q,\beta,\gamma)$. The results are based on a method developed in [14, 15]. The asymptotic method has also been used in [8, 16, 24]. We let $\overline{k}$ denote the vector $(k_1,k_2,\ldots,k_n)$ which represents the partition $1^{k_1}2^{k_2}\cdots n^{k_n}$, where $k_1+2k_2+\cdots+nk_n=n$, and $k=k_1+k_2+\cdots+k_n$ is the number of parts of the partition $1^{k_1}2^{k_2}\cdots n^{k_n}$. We let $d\in\mathbb{C}$ and $n\in\mathbb{N}$. We let $(d)_n$ denote the product $d(d-1)(d-2)\cdots(d-n+1)$, where $(d)_0=1$. We also let $\sigma(n)$ denote the set of partitions of $n\in\mathbb{N}$ and $\sigma(n,k)$ denote the set of partitions of $n\in\mathbb{N}$ having k parts. We let $\Psi(t)$ denote the formal power series $\sum_{n=0}^{\infty}b_nt^n$ over $\mathbb{C}$ with $\Psi(0)=b_0=1$. We suppose that, for any $\alpha\in\mathbb{C}$ with $\alpha\neq0$, we have a formal power series

$$\Pi(t)=(\Psi(t))^{\alpha}=\sum_{n=0}^{\infty}\begin{Bmatrix}\alpha\\ n\end{Bmatrix}t^n, \tag{18}$$

where $\left\{{\alpha \atop n}\right\}=[t^n]\Pi(t)$ and $\left\{{\alpha \atop 0}\right\}=1$. Then for $\delta\in\mathbb{C}$ with $\delta\neq0$ we have

$$\frac{1}{(\delta)_n}\left[t^n\right](\Pi(t))^{\delta}=\frac{1}{(\delta)_n}\begin{Bmatrix}\alpha\delta\\ n\end{Bmatrix}=\sum_{j=0}^{s}\frac{W(n,j)}{(\delta-n+j)_j}\quad+o\left(\frac{W(n,s)}{(\delta-n+s)_s}\right), \tag{19}$$

where $W(n,i)=\sum\limits_{\sigma(n,n-i)}\frac{b_1^{k_1}b_2^{k_2}\cdots b_n^{k_n}}{k_1!k_2!\cdots k_n!}$, $0\le i<n$, and $n=o(\sqrt{|\delta|})$ as $|\delta|\to\infty$. The following theorem is a consequence of the discussion above.

Theorem 6. *For all integers $n\ge 0$, we have*

$$\frac{\left\{{\alpha\delta \atop n}\right\}}{(\delta)_n}=\sum_{j=0}^{s}\frac{W(n,j)}{(\delta-n+j)_j}+o\left(\tfrac{W(n,s)}{(\delta-n+s)_s}\right). \tag{20}$$

Corollary 1. *For all integers $n\ge 0$, we have*

$$\frac{H_n^{(\lambda,u,p,\delta)}(x;q,\beta,\gamma)}{(\delta)_n n!}=\sum_{j=0}^{s}\frac{W(n,j)}{(\delta-n+j)_j}+o\left(\tfrac{W(n,s)}{(\delta-n+s)_s}\right), \tag{21}$$

where

$$W(n,j)=\sum_{\sigma(n,n-j)}\prod_{i=1}^{n}\frac{1}{k_i!}\left[\tfrac{H_i^{(\lambda,u,p,1)}(x;q,\beta,\gamma)}{i!}\right]^{k_i}, \tag{22}$$

and $n=o(\sqrt{|\delta|})$ as $|\delta|\to\infty$.

The numbers $H_n^{(\lambda,u,p,\delta)}(x;q,\beta,\gamma)$ satisfy the following identity (see [14]):

$$\frac{H_n^{(\lambda,u,p,\delta)}(x;q,\beta,\gamma)}{n!}=\sum_{\sigma(n)}\binom{\delta}{k_1,k_2,\cdots,k_n}\prod_{i=1}^{n}\left[\tfrac{H_i^{(\lambda,u,p,1)}(x;q,\beta,\gamma)}{i!}\right]^{k_i}. \tag{23}$$

We now compute a few values of the numbers $W(n,s)$ exhibited in Theorem 6 above. We have:

$$\begin{aligned}
W(n,0)=&\frac{1}{n!}\left\{\tfrac{H_1^{(\lambda,u,p,1)}(x;q,\beta,\gamma)}{1!}\right\}^n,\\
W(n,1)=&\frac{1}{(n-2)!}\left\{\tfrac{H_1^{(\lambda,u,p,1)}(x;q,\beta,\gamma)}{1!}\right\}^{n-2}\left\{\tfrac{H_2^{(\lambda,u,p,1)}(x;q,\beta,\gamma)}{2!}\right\},\\
W(n,2)=&\frac{1}{(n-3)!}\left\{\tfrac{H_1^{(\lambda,u,p,1)}(x;q,\beta,\gamma)}{1!}\right\}^{n-3}\left\{\tfrac{H_3^{(\lambda,u,p,1)}(x;q,\beta,\gamma)}{3!}\right\}\\
&+\frac{1}{2!(n-4)!}\left\{\tfrac{H_1^{(\lambda,u,p,1)}(x;q,\beta,\gamma)}{1!}\right\}^{n-4}\left\{\tfrac{H_2^{(\lambda,u,p,1)}(x;q,\beta,\gamma)}{2!}\right\}^2,\\
W(n,3)=&\frac{1}{(n-4)!}\left\{\tfrac{H_1^{(\lambda,u,p,1)}(x;q,\beta,\gamma)}{1!}\right\}^{n-4}\left\{\tfrac{H_4^{(\lambda,u,p,1)}(x;q,\beta,\gamma)}{4!}\right\}\\
&+\frac{1}{(n-5)!}\left\{\tfrac{H_1^{(\lambda,u,p,1)}(x;q,\beta,\gamma)}{1!}\right\}^{n-5}\left\{\tfrac{H_2^{(\lambda,u,p,1)}(x;q,\beta,\gamma)}{2!}\right\}\left\{\tfrac{H_3^{(\lambda,u,p,1)}(x;q,\beta,\gamma)}{3!}\right\}\\
&+\frac{1}{3!(n-6)!}\left\{\tfrac{H_1^{(\lambda,u,p,1)}(x;q,\beta,\gamma)}{1!}\right\}^{n-6}\left\{\tfrac{H_2^{(\lambda,u,p,1)}(x;q,\beta,\gamma)}{2!}\right\}^3.
\end{aligned}$$

Finally, Theorem 6 implies the following:

$$\frac{H_n^{(\lambda,u,p,\delta)}(x;q,\beta,\gamma)}{n!} \sim (\delta)_n W(n,0) + (\delta)_{n-1} W(n,1) + (\delta)_{n-2} W(n,2) \\ + (\delta)_{n-3} W(n,3) + (\delta)_{n-4} W(n,4) + (\delta)_{n-5} W(n,5).$$

4. Linear Operator Interpretation

In this section, we give explicit expressions for the coefficients appearing in Equations (1), (5), and (6). The main feature is that such expressions are computable, since they depend only on the classical Stirling numbers of the first and second kind. To achieve this, we use the linear operator approach below.

Let $\mathbb{N}_0$ be the set of nonnegative integers. We consider functions $\phi : \mathbb{N}_0 \to \mathbb{C}$ having at most polynomial growth, that is,

$$|\phi(m)| \leq C m^s, \quad m \in \mathbb{N}_0,$$

for some positive constants C and s. We define the linear operators

$$L\phi(u) = \sum_{m=0}^{\infty} \phi(m) \frac{u^m}{m!} e^{-u}, \quad u \in \mathbb{C}, \tag{24}$$

and

$$N_r\phi(v) = \sum_{m=0}^{\infty} \phi(m) \binom{r}{m} v^m (1+v)^{-r}, \quad r, v \in \mathbb{C}, \quad |v| < 1. \tag{25}$$

The weights in Equations (24) and (25) are the Poisson and the negative binomial weights, provided that $u \geq 0$, and $r \leq 0$, $-1 < v \leq 0$, respectively.

The series in Equations (24) and (25) can be computed by means of forward differences. We let $k, l \in \mathbb{N}_0$. Recall that the usual kth forward difference of ϕ is recursively defined as

$$\begin{aligned} \Delta^0\phi(l) &= \phi(l), \\ \Delta^1\phi(l) &= \phi(l+1) - \phi(l), \\ \Delta^{k+1}\phi(l) &= \Delta^1(\Delta^k\phi)(l), \end{aligned} \tag{26}$$

or, equivalently, as

$$\Delta^k\phi(l) = \sum_{i=0}^{k} \binom{k}{i} (-1)^{k-i} \phi(l+i), \quad k, l \in \mathbb{N}_0. \tag{27}$$

As noted by Flajolet and Vepstas [12], the dual formula of Equation (27) is

$$\phi(m) = \sum_{k=0}^{m} \binom{m}{k} \Delta^k\phi(0). \tag{28}$$

Lemma 2. *For $u, r, v \in \mathbb{C}$, we have*

$$L\phi(u) = \sum_{k=0}^{\infty} \frac{\Delta^k \phi(0)}{k!} u^k, \tag{29}$$

and

$$N_r\phi(v) = \sum_{k=0}^{\infty} \Delta^k \phi(0) \binom{r}{k} \left(\frac{v}{1+v} \right)^k, \quad |v| < \min(1, |1+v|). \tag{30}$$

Proof. The proof of Equation (29) being similar, we will only prove Equation (30). Since

$$\binom{r}{m}\binom{m}{k} = \binom{r}{k}\binom{r-k}{m-k},$$

we have from Equations (25) and (28)

$$\begin{aligned} N_r\phi(v) =& (1+v)^{-r} \sum_{m=0}^{\infty} \binom{r}{m} v^m \sum_{k=0}^{m} \binom{m}{k} \Delta^k \phi(0) \\ =& (1+v)^{-r} \sum_{k=0}^{\infty} \binom{r}{k} \Delta^k \phi(0) \sum_{m=k}^{\infty} \binom{r-k}{m-k} v^m \\ =& (1+v)^{-r} \sum_{k=0}^{\infty} \binom{r}{k} \Delta^k \phi(0) v^k (1+v)^{r-k}. \end{aligned}$$

□

Equations (29) and (30) for the Poisson and negative binomial weights were previously proved in [2] and [1], respectively.

Remark 1. The sums in Equations (29) and (30) are actually finite sums when ϕ is a polynomial of degree n, since $\Delta^k \phi = 0$, for all $k \geq n+1$. In such a case, the assumption $|v| < |1+v|$ is unnecessary.

Recall that for $d \in \mathbb{C}$ and $n \in \mathbb{N}$, $(d)_n = d(d-1)(d-2)\cdots(d-n+1)$, is the falling factorial with $(d)_0 = 1$. Denote

$$\phi_{n,c}(l) = (l+c)_n, \quad \phi_n(l) = (l)_n, \quad n, l \in \mathbb{N}_0, \quad c \in \mathbb{C}. \tag{31}$$

Lemma 3. *For any $c \in \mathbb{C}$ and $n \in \mathbb{N}_0$, we have*

$$L\phi_{n,c}(u) = \sum_{k=0}^{n} \binom{n}{k} \binom{c}{k} k! u^{n-k}, \quad u \in \mathbb{C}, \tag{32}$$

and

$$N_r\phi_{n,c}(v) = \sum_{k=0}^{n} \binom{n}{k} \binom{r}{k} k! (c)_{n-k} \left(\frac{v}{1+v} \right)^k, \quad r, v \in \mathbb{C}, \quad |v| < 1. \tag{33}$$

Proof. Using induction and Equation (26), it can be checked that

$$\Delta^k \phi_{n,c}(l) = (n)_k (l+c)_{n-k}, \quad k = 0, 1, \ldots, n,$$

whereas $\Delta^k \phi_{n,c} = 0$, for all $k \geq n+1$. Hence, Equations (32) and (33) follow from Equations (29) and (30), respectively. □

Recall that the classical *Stirling numbers of the first* and *second kind*, respectively denoted by $s(n,m)$ and $S(n,m)$, are defined by

$$(z)_n = \sum_{m=0}^{n} s(n,m) z^m, \text{ and } \quad z^n = \sum_{m=0}^{n} S(n,m)(z)_m, \quad z \in \mathbb{C}. \tag{34}$$

We consider the numbers

$$t(n,i;a) = \sum_{l=i}^{n} s(n,l) S(l,i) a^l, \quad a \in \mathbb{C}, \quad n, i \in \mathbb{N}_0, \quad i \leq n. \tag{35}$$

The interest of these numbers comes from the following lemma.

Lemma 4. *Let $a, z \in \mathbb{C}$ and $n \in \mathbb{N}_0$. Then,*

$$(az)_n = \sum_{i=0}^{n} t(n,i;a)(z)_i.$$

Proof. We have from Equation (34)

$$(az)_n = \sum_{l=0}^{n} s(n,l) a^l z^l = \sum_{l=0}^{n} s(n,l) a^l \sum_{i=0}^{l} S(l,i)(z)_i = \sum_{i=0}^{n} (z)_i \sum_{l=i}^{n} s(n,l) S(l,i) a^l.$$

□

The numbers defined in Equation (35) are actually a particular case of the generalized Stirling numbers introduced by Hsu and Shiue [16] mentioned in Section 1. The numbers $t(n,i;a)$ play an important role in the following two auxiliary results. To this end, denote $i \wedge j = \min(i,j)$ and $i \vee j = \max(i,j)$. We define

$$C(n,m;a,b,u) := \sum_{k=0}^{\infty} (ak)_n (bk)_m \frac{u^k}{k!} e^{-u}.$$

Lemma 5. *Let $a, b, u \in \mathbb{C}$ and $n, m \in \mathbb{N}_0$. Then,*

$$C(n,m;a,b,u) = \sum_{i=0}^{n} \sum_{j=0}^{m} t(n,i;a) t(m,j;b) \sum_{k=0}^{i \wedge j} \binom{i}{k} \binom{j}{k} k! u^{i+j-k}.$$

Proof. By Lemma 4, we have

$$\sum_{k=0}^{\infty}(ak)_n(bk)_m\frac{u^k e^{-u}}{k!}=\sum_{i=0}^{n}\sum_{j=0}^{m}t(n,i;a)t(m,j;b)\sum_{k=0}^{\infty}(k)_i(k)_j\frac{u^k e^{-u}}{k!}. \tag{36}$$

Suppose that $i\le j$. By Equations (31) and (32), the last sum in Equation (36) equals

$$\begin{aligned}\sum_{k=j}^{\infty}(k)_i\frac{u^k e^{-u}}{(k-j)!}&=u^j\sum_{l=0}^{\infty}(l+j)_i\frac{u^l e^{-u}}{l!}\\&=u^j L\phi_{i,j}(u)=u^j\sum_{k=0}^{i}\binom{i}{k}\binom{j}{k}k!u^{i-k}.\end{aligned}$$

This and Equation (36) show the result. We define

$$D(n,m;a,b,r,v):=\sum_{k=0}^{\infty}(ak)_n(bk)_m\binom{r}{m}v^m(1+v)^{-r}.$$

□

Lemma 6. *Let $a,b,r,v\in\mathbb{C}$ with $|v|<1$, and $n,m\in\mathbb{N}_0$. Then,*

$$\begin{aligned}&D(n,m;a,b,r,v)\\&\quad=\sum_{i=0}^{n}\sum_{j=0}^{m}t(n,i;a)t(m,j;b)\sum_{k=0}^{i\wedge j}\binom{i}{k}\binom{j}{k}k!(r)_{i+j-k}\left(\frac{v}{1+v}\right)^{i+j-k}.\end{aligned}$$

Proof. From Lemma 4, we see that

$$D(n,m;a,b,r,v)=\sum_{i=0}^{n}\sum_{j=0}^{m}t(n,i;a)t(m,j;b)\sum_{k=0}^{\infty}(k)_i(k)_j\binom{r}{k}v^k(1+v)^{-r}. \tag{37}$$

Assume that $i\le j$. By Equations (31) and (33), the last sum in Equation (37) equals

$$\sum_{k=j}^{\infty}(k)_i\frac{1}{(k-j)!}(r)_k v^k(1+v)^{-r}$$
$$=\left(\frac{v}{1+v}\right)^j(r)_j\sum_{l=0}^{\infty}(l+j)_i\binom{r-j}{l}v^l(1+v)^{-(r-j)}$$
$$=\left(\frac{v}{1+v}\right)^j(r)_j N_{r-j}\phi_{i,j}(v)$$
$$=\left(\frac{v}{1+v}\right)^j(r)_j\sum_{k=0}^{i}\binom{i}{k}\binom{r-j}{k}k!(j)_{i-k}\left(\frac{v}{1+v}\right)^k$$
$$=\sum_{k=0}^{i}\binom{i}{k}(j)_{i-k}(r)_{j+k}\left(\frac{v}{1+v}\right)^{j+k}$$
$$=\sum_{l=0}^{i}\binom{i}{l}\binom{j}{l}l!(r)_{i+j-l}\left(\frac{v}{1+v}\right)^{i+j-l}.$$

This in conjunction with Equation (37) completes the proof. □

From now on, we assume that λ, γ, δ, p, q, u, x, and β are complex numbers such that

$$\delta\neq 0,\quad \beta\neq 0,\quad p\neq 1,\quad q\neq 1,\quad \left|\frac{x}{x+1}\right|<1,\quad \left|\frac{(1-p)u}{1-(1-p)u}\right|<1. \tag{38}$$

We denote by $G^{(\lambda)}(t,x;q,\beta,\gamma)$ the generating function in Equation (1) i.e.

$$G^{(\lambda)}(t,x;q,\beta,\gamma)=\frac{(1+(1-q)t)^{\gamma/(1-q)}}{\left(1-x\left((1+(1-q)t)^{\beta/(1-q)}-1\right)\right)^{\lambda}}. \tag{39}$$

The last auxiliary result is the following.

Lemma 7. *For any $m\in\mathbb{N}_0$, we have*

$$\left[G^{(\lambda)}(t,x;q,\beta,\gamma)\right]^m=\sum_{n=0}^{\infty}(1-q)^n\frac{t^n}{n!}\sum_{i=0}^{n}t\left(n,i;\frac{\beta}{1-q}\right)\sum_{k=0}^{i}\binom{i}{k}(-x)^k(-\lambda m)_k \times\left(\frac{\gamma m}{\beta}\right)_{i-k}.$$

Proof. Choose $t\in\mathbb{C}$ small enough so that

$$\left|\frac{x}{x+1}(1+(1-q)t)^{\beta/(1-q)}\right|<1, \tag{40}$$

which is possible by the assumptions in Equation (38). We define

$$s = \lambda m, \quad a = \beta/(1-q), \quad b = \gamma m/(1-q), \quad v = -\frac{x}{x+1}. \tag{41}$$

On the one hand, we have from Equation (40)

$$\begin{aligned} &\sum_{k=0}^{\infty} (1+(1-q)t)^{ak+b} \binom{-s}{k} v^k (1+v)^s \\ &\quad = (1+(1-q)t)^b (1+v)^s \sum_{k=0}^{\infty} \binom{-s}{k} \left(v\,(1+(1-q)t)^a\right)^k \\ &\quad = (1+v)^s \frac{(1+(1-q)t)^b}{(1+v\,(1+(1-q)t)^a)^s} = \left[G^{(\lambda)}(t,x;q,\beta,\gamma)\right]^m, \end{aligned} \tag{42}$$

where the last equality follows from Equations (39) and (41).

On the other hand, we have from the binomial expansion

$$\begin{aligned} &\sum_{k=0}^{\infty} (1+(1-q)t)^{ak+b} \binom{-s}{k} v^k (1+v)^s \\ &\quad = \sum_{k=0}^{\infty} \binom{-s}{k} v^k (1+v)^s \sum_{n=0}^{\infty} (ak+b)_n (1-q)^n \frac{t^n}{n!} \\ &\quad = \sum_{n=0}^{\infty} (1-q)^n \frac{t^n}{n!} \sum_{k=0}^{\infty} (ak+b)_n \binom{-s}{k} v^k (1+v)^s. \end{aligned} \tag{43}$$

By Lemma 4, we can write

$$(ak+b)_n = \left(a\left(k+\frac{b}{a}\right)\right)_n = \sum_{i=0}^{n} t(n,i;a) \left(k+\frac{b}{a}\right)_i. \tag{44}$$

Finally, we have from Equations (31) and (33)

$$\begin{aligned} &\sum_{k=0}^{\infty} \left(k+\frac{b}{a}\right)_i \binom{-s}{k} v^k (1+v)^s \\ &\quad = N_{-s}\phi_{i,b/a}(v) \\ &\quad = \sum_{k=0}^{i} \binom{i}{k} \binom{-s}{k} k! \left(\frac{b}{a}\right)_{i-k} \left(\frac{v}{1+v}\right)^k \\ &\quad = \sum_{k=0}^{i} \binom{i}{k} (-x)^k (-\lambda m)_k \left(\frac{\gamma m}{\beta}\right)_{i-k}, \end{aligned} \tag{45}$$

where the last equality follows from Equation (41). Gathering Equations (42)–(45), the result follows. □

We now state the main results of this section.

Theorem 7. *In the setting of Equation* (1), *we have*

$$\omega_n^{(\lambda)}(x;q,\beta,\gamma) = (1-q)^n \sum_{i=0}^{n} t\left(n,i;\frac{\beta}{1-q}\right) \sum_{k=0}^{i} \binom{i}{k} (-x)^k (-\lambda)_k \left(\frac{\gamma}{\beta}\right)_{i-k}.$$

Proof. The proof is an immediate consequence of Equations (1) and (35), and Lemma 7 for $m=1$. □

Theorem 8. *In the setting of Equation* (6), *we have*

$$\begin{aligned} H_n^{(\lambda,u,1,1)}(x;q,\beta,\gamma) =&(1-q)^n \sum_{i=0}^{n} t\left(n,i;\frac{\beta}{1-q}\right) \sum_{k=0}^{i} \binom{i}{k} (-x)^k \\ &\times C\left(k,i-k;-\lambda,\frac{\gamma}{\beta}.u\right) \end{aligned}$$

where the quantity $C(\cdot)$ *is computed in Lemma 5.*

Proof. Starting from the generating function in Equation (6) and recalling Equation (39) and Lemma 7, we have

$$\begin{aligned} &e^{\left(uG^{(\lambda)}(t,x;q,\beta,\gamma)-u\right)} = \sum_{m=0}^{\infty} \frac{u^m e^{-u}}{m!} \left[G^{(\lambda)}(t,x;q,\beta,\gamma)\right]^m \\ &= \sum_{n=0}^{\infty}(1-q)^n \frac{t^n}{n!} \sum_{i=0}^{n} t\left(n,i;\frac{\beta}{1-q}\right) \sum_{k=0}^{i} \binom{i}{k} (-x)^k \sum_{m=0}^{\infty} (-\lambda m)_k \left(\frac{\gamma m}{\beta}\right)_{i-k} \frac{u^m e^{-u}}{m!} \\ &= \sum_{n=0}^{\infty}(1-q)^n \frac{t^n}{n!} \sum_{i=0}^{n} t\left(n,i;\frac{\beta}{1-q}\right) \sum_{k=0}^{i} \binom{i}{k} (-x)^k C\left(k,i-k;-\lambda,\frac{\gamma}{\beta},u\right), \end{aligned}$$

where the last equality follows from Lemma 5. □

Theorem 9. *In the setting of Equation* (5), *we have*

$$\begin{aligned} H_n^{(\lambda,u,p,\delta)}(x;q,\beta,\gamma) =&(1-q)^n \sum_{i=0}^{n} t\left(n,i;\frac{\beta}{1-q}\right) \sum_{k=0}^{i} \binom{i}{k} (-x)^k \\ &\times D\left(k,i-k;-\lambda,\frac{\gamma}{\beta},\frac{\delta}{1-p},\frac{(1-p)u}{1-(1-p)u}\right), \end{aligned}$$

where the quantity $D(\cdot)$ *is computed in Lemma 6.*

Proof. Choose $t \in \mathbb{C}$ small enough so that

$$\left|\frac{(1-p)u}{1-(1-p)u} G^{(\lambda)}(t,x;q,\beta,\gamma)\right| < 1, \tag{46}$$

which is possible by virtue of Equations (38) and (39). We define v as follows:

$$v = \frac{(1-p)u}{1-(1-p)u}. \tag{47}$$

By Equations (39), (46), and (47), as well as Lemma 7, we can write the generating function in Equation (5) as

$$\begin{aligned}
&\left(1 + (1-p)u\left(G^{(\lambda)}(t,x;q,\beta,\gamma\right) - 1\right)^{\delta/(1-p)} \\
&\quad = (1+v)^{-\delta/(1-p)}\left(1 + vG^{(\lambda)}(t,x;q,\beta,\gamma\right)^{\delta/(1-p)} \\
&\quad = (1+v)^{-\delta/(1-p)}\sum_{m=0}^{\infty}\binom{\delta/(1-p)}{m}v^m\left[G^{(\lambda)}(t,x;q,\beta,\gamma\right]^m \\
&\quad = \sum_{n=0}^{\infty}(1-q)^n\frac{t^n}{n!}\sum_{i=0}^{n} t\left(n,i;\frac{\beta}{1-q}\right)\sum_{k=0}^{i}\binom{i}{k}(-x)^k \\
&\qquad \times \sum_{m=0}^{\infty}(-\lambda m)_k\left(\frac{\gamma m}{\beta}\right)_{i-k}\binom{\delta/(1-p)}{m}v^m(1+v)^{-\delta/(1-p)} \\
&\quad = \sum_{n=0}^{\infty}(1-q)^n\frac{t^n}{n!}\sum_{i=0}^{n} t\left(n,i;\frac{\beta}{1-q}\right)\sum_{k=0}^{i}\binom{i}{k}(-x)^k \\
&\qquad \times D\left(k, i-k; -\lambda, \frac{\gamma}{\beta}, \frac{\delta}{1-p}, \frac{(1-p)u}{1-(1-p)u}\right),
\end{aligned}$$

where the last equality follows from Equation (47) and Lemma 6. □

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