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A PROBABILISTIC PROOF OF A RECURSION FORMULA FOR SUMS OF INTEGER POWERS

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Abstract

In this note, we provide a probabilistic proof for a recursion formula for the sum of powers of the first n positive integers.

1. Introduction

In mathematics, the sum of powers of the first n positive integers has attracted the attention of mathematicians for centuries. Formulae for the first few powers are fairly well-known. Denoting the sum $1^k + 2^k + \cdots + n^k$ by $S_k(n)$, the first three such sum formulae are (for any positive integers n),

$$S_1(n) = \frac{n(n+1)}{2}, \tag{1}$$

$$S_2(n) = \frac{n(n+1)(2n+1)}{6}, \tag{2}$$

$$S_3(n) = \frac{n^2(n+1)^2}{4}. \tag{3}$$

In general, a closed-form formula for the sum $S_k(n)$ (for $k \geq 1$) is of interest in the relevant literature. Existing closed-form formulae are usually polynomials that are expressed in terms of special numbers such as Bernolli numbers, Stirling numbers, and Eulerian numbers (see, e.g., [2], [4], [5], and [8]). There are also some formulae for $S_k(n)$ that do not involve these special numbers, and in fact they are often recursion relations. One of the most popular recurrence formulae is Pascal's Identity, which was found in the middle of the 17th century by Pascal. Using the telescoping idea and the binomial theorem, Pascal found the following identity:

$$(n+1)^{k+1} - 1 = \sum_{r=0}^{k} \binom{k+1}{r} S_r(n), \quad k \geq 1.$$

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From Pascal's Identity it immediately follows that

$$S_k(n) = \frac{(n+1)^{k+1}-1}{k+1} - \frac{1}{k+1}\sum_{r=0}^{k-1}\binom{k+1}{r}S_r(n), \quad k \geq 1. \tag{4}$$

Furthermore, using probabilistic methods, the following related identity was proved in [3]:

$$S_k(n) = \frac{n^{k+1}}{k+1} + \sum_{r=0}^{k-1}\binom{k}{r}\frac{(-1)^{k-r+1}}{k-r+1}S_r(n), \quad k \geq 1. \tag{5}$$

For more related and similar recursive formulae, see, for example, [7].

In this note, we aim to give a probabilistic proof of the following recursion formula for the sum of powers of integers:

$$S_k(n) = \frac{n(n+1)^k}{k+1} - \frac{1}{k+1}\sum_{r=1}^{k-1}\binom{k}{r-1}S_r(n), \quad k \geq 1. \tag{6}$$

As can be seen, Equation (6) is very similar to Pascal's Identity in Equation (4), and at first glance, the apparent differences are insignificant. Computationally, the summation in Equation (4), as well as the summation in Equation (5), contain k terms, while the summation in Equation (6) contains $k-1$ terms.

To the best of our knowledge, Equation (6) in its present form has not been mentioned in the literature, however, it can be derived from [6, Equation (3)]. Our probabilistic proof of Equation (6) will be presented in the next section. To this end, we denote by $\mathbb{P}(X = x)$ and $\mathbb{E}[X]$, respectively, the probability mass function and the expected value of a random variable X.

2. Probabilistic Proof of Equation (6)

A probabilistic proof of Equation (6) is given below:

Proof of Equation (6). Let X be a nonnegative integer-valued random variable. The well-known tail sum formula for the kth moment is defined as (see, e.g., [1, Equation (3)]):

$$\mathbb{E}[X^k] = \sum_{x=0}^{\infty}((x+1)^k - x^k)\mathbb{P}(X > x), \quad k \geq 1.$$

Hence, if X has support in $\{0, 1, 2, \ldots, n\}$, we have

$$\sum_{x=0}^{n} x^k \mathbb{P}(X = x) = \sum_{x=0}^{n}\left(\left((x+1)^k - x^k\right)\sum_{y=x+1}^{n}\mathbb{P}(X = y)\right).$$

Now, let X have the uniform distribution on $\{0, 1, 2 \ldots, n\}$, namely $\mathbb{P}(X = x) = \frac{1}{n+1}$. Since $\sum_{y=x+1}^{n} \mathbb{P}(X = y) = \sum_{y=x+1}^{n} \frac{1}{n+1} = \frac{n-x}{n+1}$, this gives

$$\begin{aligned}\sum_{x=0}^{n} x^k &= \sum_{x=0}^{n}((x+1)^k - x^k)(n-x) \\ &= n\left(\sum_{x=0}^{n}(x+1)^k - x^k\right) - \sum_{x=0}^{n}(x(x+1)^k - x^{k+1}). \qquad (7)\end{aligned}$$

A well-known consequence of the binomial theorem is the identity $\sum_{x=0}^{n}(x+1)^k - x^k = (n+1)^k$. Using this identity and the binomial theorem $(x+1)^k = \sum_{r=0}^{k}\binom{k}{r}x^r$ in Equation (7), we obtain

$$\begin{aligned}\sum_{x=0}^{n} x^k &= n(n+1)^k - \sum_{x=0}^{n}\left(\sum_{r=0}^{k}\binom{k}{r}x^{r+1} - x^{k+1}\right) \\ &= n(n+1)^k - \sum_{x=0}^{n}\left(\sum_{r=0}^{k-2}\binom{k}{r}x^{r+1} + kx^k\right) \\ &= n(n+1)^k - \sum_{r=0}^{k-2}\binom{k}{r}\sum_{x=0}^{n}x^{r+1} - k\sum_{x=0}^{n}x^k.\end{aligned}$$

We may rearrange this to obtain

$$\begin{aligned}(k+1)\sum_{x=0}^{n} x^k &= n(n+1)^k - \sum_{r=0}^{k-2}\binom{k}{r}\sum_{x=0}^{n}x^{r+1} \\ &= n(n+1)^k - \sum_{r=1}^{k-1}\binom{k}{r-1}\sum_{x=0}^{n}x^{r}.\end{aligned}$$

Note that $\sum_{x=0}^{n} x^k = \sum_{x=1}^{n} x^k = S_k(n)$. The proof is complete. □

As an example, take $k = 3$. Applying Equation (6) gives (use also Equation (1) and Equation (2))

$$\begin{aligned}S_3(n) &= \frac{n(n+1)^3}{4} - \frac{1}{4}\left(S_1(n) + 3S_2(n)\right) \\ &= \frac{n(n+1)^3}{4} - \frac{1}{4}\left(\frac{n(n+1)}{2} + 3\left(\frac{n(n+1)(2n+1)}{6}\right)\right) \\ &= \frac{n^2(n+1)^2}{4},\end{aligned}$$

which is Equation (3).

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References

[1] S. Chakraborti, F. Jardim and E. Epprecht, Higher order moments using the survival function: The alternative expectation formula. *Amer. Statist.* **73** (2019), 191–194.

[2] H.W. Gould, Evaluation of sums of convolved powers using Stirling and Eulerian numbers, *Fibonacci Quart.* **16** (1978), 488–497.

[3] X. Hu and Y. Zhong, A probabilistic proof of a recursion formula for sums of powers, *Amer. Math. Monthly.* **127** (2020), 166–168.

[4] H.K. Krishnapriyan, Eulerian polynomials and Faulhaber's result on sums of powers of integers, *College Math. J.* **26** (1995), 118–123.

[5] D. Laissaoui and M. Rahmani, An explicit formula for sums of powers of integers in terms of Stirling numbers, *J. Integer. Seq.* **20** (2017), Article 17.4.8.

[6] M. Lentner, On the sum of powers of the first N integers, *Amer. Statist.* **27** (1973), p. 87.

[7] B. Turner, Sums of powers of integers via the Binomial theorem, *Math. Mag.* **53** (1980), 92–96.

[8] D.W. Wu, Bernoulli numbers and sums of powers, *Internat. J. Math. Ed. Sci. Tech.* **32** (2001), 440–443.