A NOTE ON THE NUMBER OF EGYPTIAN FRACTIONS

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Abstract

Improving an estimate of Croot, Dobbs, Friedlander, Hetzel, and Pappalardi, we show that for all $k \geqslant 2$, the number of integers $1 \leqslant a \leqslant n$, such that the equation $a/n = 1/m_1 + \cdots + 1/m_k$ has a solution in positive integers $m_1, \ldots, m_k$ is bounded above by $n^{1-1/2^{k-2}+o(1)}$ as n goes to infinity.

1. The Result

For a positive integer k, let $A_k(n)$ be the number of integers a, $1 \leqslant a \leqslant n$ for which a/n has a k-term Egyptian fraction representation

$$\frac{a}{n} = \frac{1}{m_1} + \cdots + \frac{1}{m_k}, \tag{1}$$

where the m_i are positive integers with $m_1 \leqslant \cdots \leqslant m_k$. Decompositions of the form (1), often with the m_i's required to be distinct, have been extensively studied in number theory. See Bloom and Elsholtz [1] for a recent survey of the subject and Guy [3, Section D11] for a comprehensive collection of open problems.

Croot, Dobbs, Friedlander, Hetzel, and Pappalardi [2] proved that for any $k \geqslant 2$,

$$A_k(n) \leqslant n^{\alpha_k + o(1)} \tag{2}$$

as $n \to \infty$, where $\alpha_k = 1 - 2/(3^{k-2}+1)$. In particular, (2) shows that $A_2(n) = n^{o(1)}$. Improving upon their strategy, we get the following bounds.

Theorem 1. *For every $k \geqslant 2$, we have $A_k(n) \leqslant n^{\beta_k + o(1)}$, where $\beta_k = 1 - 1/2^{k-2}$.*

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Proof. We apply induction on k. The case $k = 2$ follows from (2), so assume $k \geqslant 3$.

Set $\gamma_i = 2^{i-1}/2^{k-2}$ and $\beta_k = 1 - 1/2^{k-2}$. For $1 \leqslant j \leqslant k-2$, we say that a k-tuple $(m_1, \ldots, m_k)$ with $m_1 \leqslant \cdots \leqslant m_k$ is of *type* j if $m_j \geqslant n^{\gamma_j}$ and $m_i < n^{\gamma_i}$ for $1 \leqslant i < j$. In addition, we say that such a k-tuple is of *type* $k-1$ if $m_i < n^{\gamma_i}$ for all $1 \leqslant i \leqslant k-2$, with no restrictions being placed on m_{k-1} or m_k. It follows from these definitions that any k-tuple $(m_1, \ldots, m_k)$ is of type j for some $1 \leqslant j \leqslant k-1$.

We now show that the number of solutions to (1) of any given type is at most $n^{\beta_k + o(1)}$, as this implies the theorem. Note that type 1 solutions satisfy

$$\frac{a}{n} = \frac{1}{m_1} + \cdots + \frac{1}{m_k} \leqslant \frac{k}{m_1} \leqslant kn^{-\gamma_1}.$$

Thus, $a \leqslant kn^{1-\gamma_1} = kn^{\beta_k}$. That is, there are no more than kn^{β_k} solutions of type 1.

For the type j solutions, $2 \leqslant j \leqslant k-2$, we have

$$0 \leqslant \frac{a}{n} - \frac{1}{m_1} - \cdots - \frac{1}{m_{j-1}} = \frac{1}{m_j} + \cdots + \frac{1}{m_k} \leqslant \frac{k-j+1}{m_j} \leqslant kn^{-\gamma_j}.$$

Therefore,

$$0 \leqslant a - \frac{n}{m_1} - \cdots - \frac{n}{m_{j-1}} \leqslant kn^{1-\gamma_j};$$

thus there are at most $kn^{1-\gamma_j}$ possible values of a given any $(m_1, \ldots, m_{j-1})$. Consequently, the number of solutions of (1) of type j is at most

$$kn^{1-\gamma_j} m_1 \cdots m_{j-1} \leqslant kn^{1+\gamma_1+\cdots+\gamma_{j-1}-\gamma_j} = kn^{\beta_k}.$$

The solutions of type $k-1$ are handled more efficiently via binary Egyptian fractions. Note that

$$0 \leqslant \frac{a}{n} - \frac{1}{m_1} - \cdots - \frac{1}{m_{k-2}} = \frac{1}{m_{k-1}} + \frac{1}{m_k}.$$

Hence, writing $a' = am_1 \cdots m_{k-2} - nm_1 \cdots m_{k-2}(1/m_1 + \cdots + 1/m_{k-2})$, we see that a' is an integer satisfying $0 \leqslant a' \leqslant nm_1 \cdots m_{k-2}$ and

$$\frac{a'}{nm_1 \cdots m_{k-2}} = \frac{1}{m_{k-1}} + \frac{1}{m_k}.$$

Therefore, given $(m_1, \ldots, m_{k-2})$, there are at most $A_2(nm_1 \cdots m_{k-2})$ possible values of a that can satisfy (1). In total, the number of possible values of a is bounded by

$$\sum_{\substack{m_i \leqslant n^{\gamma_i} \\ 1 \leqslant i \leqslant k-2}} A_2(nm_1 \cdots m_{k-2}) \leqslant n^{\gamma_1+\cdots+\gamma_{k-2}+o(1)} = n^{\beta_k+o(1)},$$

where we used again that $A_2(n) = n^{o(1)}$ from (2), so

$$A_2(nm_1 \cdots m_{k-2}) = (nm_1 \cdots m_{k-2})^{o(1)} = n^{o(1)}.$$

□

We get an improved exponent of n in the upper bound of $A_k(n)$ for all $k \geqslant 4$. To illustrate the change, the first exponents in (2) are $\alpha_2 = 0$, $\alpha_3 = 1/2$, $\alpha_4 = 4/5$, $\alpha_5 = 13/14$ and $\alpha_6 = 79/81$, whereas the new exponents are $\beta_2 = 0$, $\beta_3 = 1/2$, $\beta_4 = 3/4$, $\beta_5 = 7/8$ and $\beta_6 = 15/16$.

It is still expected, however, that $A_k(n) = n^{o(1)}$ for all $k \geqslant 2$. Nonetheless, even the weaker statement that $\sum_{n \leqslant x} A_k(n) = x^{1+o(1)}$ remains unproven for $k \geqslant 3$.

Using our argument, if one shows that $A_k(n) = n^{o(1)}$ holds for some fixed k, we get that $A_{k+\ell}(n) \leqslant n^{1-1/2^{\ell}+o(1)}$ for all $\ell \geqslant 1$. In fact, any improvement on the exponent in $A_k(n) \leqslant n^{\beta_k+o(1)}$ can be propagated to an improvement on $A_{k+\ell}(n)$.

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