A MULTIPLE HOOK REMOVING GAME WHOSE STARTING POSITION IS A RECTANGULAR YOUNG DIAGRAM WITH UNIMODAL NUMBERING

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Abstract

An impartial combinatorial game based on Young diagrams is introduced which we refer to as "A multiple hook removing game". Examples are discussed and Sprague-Grundy values are found for games whose starting positions are rectangular Young diagrams with unimodal numbering. Some isomorphisms are established between certain starting positions, and some results are given for shifted diagrams.

1. Introduction and Summary

A Hook Removing Game (HRG) is an impartial game whose game positions are Young diagrams. This game is played by two players, A and B, where each player alternately removes one hook from the Young diagram. It was invented in 1970 by Mikio Sato (see [9] and [10]), who also found a formula for the $\mathcal{G}$-values. In the following, we introduce a "multiple hook removing game", MHRG(m, n), whose starting position is a rectangular Young diagram of size $m \times n$. The diagram is complete with "unimodal numbering", a scheme that originates from Tada's paper [12]. In the following, we prove that MHRG(m, n) is isomorphic to MHRG$(m, n+1)$ if $m \leq n$ and $m + n$ is even. Also, we relate MHRG$(n, n+1)$ to both MHRG(n, n) and HRG(S_n) where the latter is described in terms of shifted Young diagrams.

A brief history of the connection between groups/representation theory and combinatorial games is in order. It is well-known that Young diagrams are used in combinatorial representation theory. For example, there exists a one-to-one correspondence between the (isomorphism classes of) irreducible representation for

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the symmetric group S_n and the Young diagrams having n boxes. Under this correspondence, the dimension of an irreducible representation for S_n is equal to the number of standard Young tableaux of the corresponding shape.

Following Sato's work ([9], [10]), Irie [4] gave an interesting relation between the $\mathcal{G}$-values of HRG and representation theory of S_n. Then, in 1999, Proctor [8] introduced the notion of d-complete posets which are combinatorial generalizations of Young diagrams. In this connection, Kawanaka [5] introduced the notion of a Plain Game which generalizes HRG. In Kawanaka's game, d-complete posets are used instead of Young diagrams, moreover, he gave a closed formula for the $\mathcal{G}$-values of the positions. Finally, following Tada [12], Motegi [6] characterized the game positions that occur in the present paper.

The rules of MHRG are motivated by Tada [12]. In this paper, he described the elements of a minimal parabolic quotient of Weyl groups of types B, C, F, and G in terms of d-complete posets that have "unimodal numbering". Importantly, rectangular Young diagrams with "unimodal numbering" contain diagrams that correspond to the elements of Weyl groups of types B and C; and it happens that MHRG corresponds to the latter types. Table 1 is a list of several combinatorial games along with the corresponding group structure for each.

Game	Diagram	Corresponding Weyl group
Sato-Welter Game	Young diagram	type A
Turning Turtles	shifted Young diagram	type D
Plain Game	d-complete poset	type A,D,E
Multiple Hook Removing Game	**Young diagram with the unimodal numbering**	**type B,C**

Table 1: Correspondence of Game, Diagram, and Weyl group.

2. Preliminaries, The Rules of MHRG(m, n), and Some Examples.

2.1. Impartial Combinatorial Games

Combinatorial games satisfy the requirements stated below. One should consult with Berlekamp, Conway, and Guy [2] for the classical introduction to such games. See Conway [3] and Siegel [11] for more advanced treatments.

- A combinatorial game is played by two players (we will call them "A" and "B").
- Two players alternate in making a move.

- There are no chance elements (no moves are determined by rolling dice, etc.).
- There is no hidden information (both players will have complete knowledge of the game state at all times).
- No position can appear more than once during a game. And, in particular, combinatorial games are *short games* — they always end following a finite number of moves.

In addition, if both players have the same set of options in each position, then the game is an *impartial combinatorial game*. As previously mentioned, MHRG(m, n) is such a game.

Given an impartial game G, a game position is called an $\mathcal{N}$*-position* (resp. $\mathcal{P}$-*position*) if the next (resp. previous) player has a winning strategy, and each game position is either an $\mathcal{N}$-position or a $\mathcal{P}$-position. Additionally, if G is an $\mathcal{N}$-position, then there exists a move from G to a $\mathcal{P}$-position. If G is a $\mathcal{P}$-position, then there exists no move from G to a $\mathcal{P}$-position (see [2], [11]).

Denote by $\mathbb{N}$ the set of all positive integers and $\mathbb{N}_0$ the set of all non-negative integers.

Let G be an impartial game and set

$$\mathcal{C}(G) = \{G' \mid G' \text{ is a game position of } G\} \text{ (of course } G \in \mathcal{C}(G)).$$

If G' is an option of G, then we write $G \to G'$, and we set

$$\mathcal{O}(G) = \{G' \mid G \to G'\} \ (\mathcal{O}(G) \subset \mathcal{C}(G)).$$

A *transition* from G to G' is, by definition, a sequence $G = G_0, G_1, \ldots, G_k = G', k \in \mathbb{N}_0$, of game positions in $\mathcal{C}(G)$ such that

$$G = G_0 \to G_1 \to \cdots \to G_k = G'.$$

Definition 1. Let G and H be impartial games. If there exists a bijection $f : \mathcal{C}(G) \to \mathcal{C}(H)$ such that $f(\mathcal{O}(G')) = \mathcal{O}(f(G'))$ for all $G' \in \mathcal{C}(G)$, then we say that G is *isomorphic* to H, and we call f an *isomorphism* from G to H. In other words, G is isomorphic to H if G and H have identical game trees [1].

Definition 2. For any proper subset T of $\mathbb{N}_0$, we define the *minimal excluded number* $\text{mex}(T)$ as follows:

$$\text{mex}(T) = \min(\mathbb{N}_0 \setminus T).$$

We recall the $\mathcal{G}$-value (or Sprague-Grundy value) of a position in an impartial game.

Definition 3. Let G be a game position. We define $\mathcal{G}(G) \in \mathbb{N}_0$, called the *$\mathcal{G}$-value* (or *Sprague-Grundy value*) of G, by

$$\mathcal{G}(G) := \text{mex}\{\mathcal{G}(G') \mid G \to G'\}.$$

The following proposition is well-known.

Proposition 1 ([11, Chapter IV])**.** *For a game position G, $\mathcal{G}(G) = 0$ if and only if G is a $\mathcal{P}$-position.*

The following proposition can be easily shown.

Proposition 2. *Let G and H be impartial games. If there exists a bijection $f : \mathcal{C}(G) \to \mathcal{C}(H)$, then $\mathcal{G}(G') = \mathcal{G}(f(G'))$ for all $G' \in \mathcal{C}(G)$.*

2.2. Young Diagrams and Numbering

A Young diagram is a finite collection of boxes arranged in left-adjusted rows where the row lengths are in non-increasing order. Let $m \in \mathbb{N}$, and let $\lambda_1, \dots, \lambda_m \in \mathbb{N}_0$ be such that $\lambda_1 \geq \cdots \geq \lambda_m \geq 0$. Then, the set

$$Y = (\lambda_1, \dots, \lambda_m) := \{(i, j) \in \mathbb{N}^2 \mid 1 \leq i \leq m, 1 \leq j \leq \lambda_i\}$$

is called the *Young diagram* corresponding to $(\lambda_1, \dots, \lambda_m)$.

An element of a Young diagram is called a *box* and each box is located by a pair (i, j). For example, the Young diagram $(6, 6, 5, 3, 3)$ is given as follows:

$Y =$

$(1,1)$	$(1,2)$	$(1,3)$	$(1,4)$	$(1,5)$	$(1,6)$
$(2,1)$	$(2,2)$	$(2,3)$	$(2,4)$	$(2,5)$	$(2,6)$
$(3,1)$	$(3,2)$	$(3,3)$	$(3,4)$	$(3,5)$	
$(4,1)$	$(4,2)$	$(4,3)$			
$(5,1)$	$(5,2)$	$(5,3)$			

Figure 1: Young diagram $(6, 6, 5, 3, 3)$

For $i \in \mathbb{N}$, the subset $\{(i, j) \mid j \in \mathbb{N}\} \cap Y$ of Y is called the *i-th row* of Y. Similarly, for $j \in \mathbb{N}$, the subset $\{(i, j) \mid i \in \mathbb{N}\} \cap Y$ of Y is called the *j-th column* of Y. For a Young diagram Y, let $\mathcal{F}(Y)$ denote the set of all Young diagrams contained in Y. Also, let $\#(Y)$ denote the number of boxes contained in Y. It is obvious that if $Y' \subseteq Y$, then $\#(Y') \leq \#(Y)$.

For a Young diagram Y, a map $\alpha : Y \to \mathbb{N}$ is called a *numbering* of Y. For a box $(i,j) \in Y$, if $\alpha(i,j) = x$, then we say that the box (i,j) has the number x. Let Y be a Young diagram with a numbering α. For a subset X of Y, we set $\mathcal{A}_\alpha(X) = [\alpha(i,j) \mid (i,j) \in X]$, where $[x_1, \ldots, x_N]$ denotes the multiset consisting of $x_1, \ldots, x_N$.

For fixed $m, n \in \mathbb{N}$, we denote by

$$Y_{m,n} := \{(i,j) \in \mathbb{N}^2 \mid 1 \le i \le m, 1 \le j \le n\}$$

the *rectangular Young diagram.* For $Y \in \mathcal{F}(Y_{m,n})$, we define a special numbering $\alpha_{m,n} : Y \to \mathbb{N}$, called the *unimodal numbering* of Y, as follows: For $(i,j) \in Y$, we set $\alpha_{m,n}(i,j) := \min\{j-i+m, i-j+n\} \in \mathbb{N}$. In what follows, boxes in $Y \in \mathcal{F}(Y_{m,n})$ are always numbered by the unimodal numbering $\alpha_{m,n}$.

3	3	2	1
2	3	3	2
1	2	3	3

3	4	3	2	1
2	3	4	3	
1	2	3		

Figure 2: unimodal numberings

Remark 1. Let $Y \in \mathcal{F}(Y_{m,n})$. By the definition of unimodal numbering $\alpha_{m,n}$, we can easily check the following.

(1) If Y contains the box $(m,1)$, then it has the number 1. If Y contains the box $(1,n)$, then it has the number 1.

(2) The boxes (i,j) and $(i+1, j+1)$ have the same number (if they exist in Y).

(3) The maximum value of $\alpha_{m,n} : Y_{m,n} \to \mathbb{N}$ is equal to $\hat{\alpha}_{m,n} := \lfloor (n+m)/2 \rfloor$, where $\lfloor x \rfloor := \max\{y \in \mathbb{Z} \mid y \le x\}$ for $x \in \mathbb{R}$.

2.3. Rules of the Multiple Hook Removing Game

In this subsection, we explain the rules of MHRG (in a general setting).

Definition 4. For a box (i,j) of a Young diagram Y,

$$h(i,j) = h_Y(i,j) := \{(i,j)\} \sqcup \{(i',j) \in Y \mid i' > i\} \sqcup \{(i,j') \in Y \mid j' > j\}$$

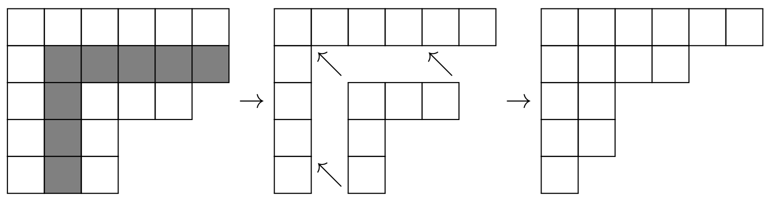
is called the *hook* (in Y) corresponding to the box (i,j).

Definition 5. For a box (i,j) of a Young diagram Y, we remove the hook $h_Y(i,j)$ corresponding to the box (i,j) as follows:

1. Remove each box in the hook $h_Y(i,j)$.
2. Move each box (i',j') satisfying $i' > i$ and $j' > j$ to $(i'-1, j'-1)$.

We denote by $Y \backslash h_Y(i,j)$ the Young diagram obtained by removing the hook $h_Y(i,j)$ corresponding to the box (i,j) from Y.

Example 1. If we remove the hook corresponding to the box $(2,2)$ from the Young diagram $Y = (6,6,5,3,3)$, then we get $Y' = Y \setminus h_Y(2,2) = (6,4,2,2,1)$.



Definition 6. A MHRG is an impartial game. The rules of MHRG are as follows:

(M1) The starting position is a Young diagram Y^s with a numbering $\alpha : Y^s \to \mathbb{N}$. All game positions are Young diagrams Y contained in Y with a numbering $\alpha|_Y$.

(M2) Given a Young diagram Y with the numbering $\alpha|_Y$, each player chooses a box in Y and removes the hook h corresponding to the box on his/her turn. Let $\mathcal{A}_\alpha(h)$ be the multiset of the numbers (in boxes) in the hook h, and let Y' be the Young diagram obtained by removing h from Y, with the numbering $\alpha|_{Y'}$.

(M2a) If there does not exist any box in Y' whose corresponding hook h' satisfies $\mathcal{A}_\alpha(h') = \mathcal{A}_\alpha(h)$ as multisets, then the player's turn is over, and the next player is given Y'.

(M2b) If there exists a box in Y' whose corresponding hook h' satisfies $\mathcal{A}_\alpha(h') = \mathcal{A}_\alpha(h)$, then the player must choose one such boxes, and remove the hook h' corresponding to the box. Let Y'' be the Young diagram obtained by removing h' from Y', with the numbering $\alpha|_{Y''}$.

(M2c) Do the same operation as (M2a) and (M2b), with Y' replaced by Y''. As long as such a box exists, repeat this operation.

(M3) The winner is the player who removes the last remaining hook in the diagram.

In this paper, we mainly treat MHRG(m,n) for $m,n \in \mathbb{N}$ which is MHRG whose starting position Y^s is the rectangular Young diagram $Y_{m,n}$ of size $m \times n$ with the unimodal numbering $\alpha_{m,n}$.

Example 2. At the beginning of MHRG(3, 5) played by A and B, the following Young diagram $Y = Y_{3,5}$ with the numbering $\alpha_{3,5}$ is given to the player, say A, having the first move, as the starting position.

$$Y = \begin{array}{|c|c|c|c|c|}\hline 3&4&3&2&1\\\hline 2&3&4&3&2\\\hline 1&2&3&4&3\\\hline\end{array}$$

If the player A removes the hook h corresponding to the box $(2, 4)$ from Y, then A obtains Y' (with $\alpha_{3,5}|_{Y'}$) below:

$$Y = \begin{array}{|c|c|c|c|c|}\hline 3&4&3&2&1\\\hline 2&3&4&\colorbox{gray}{3}&\colorbox{gray}{2}\\\hline 1&2&3&\colorbox{gray}{4}&3\\\hline\end{array} \quad\rightarrow\quad Y' = \begin{array}{|c|c|c|c|c|}\hline 3&4&3&2&1\\\hline 2&3&4&3\\\cline{1-4} 1&2&3\\\cline{1-3}\end{array}$$

Note that $\mathcal{A}_{\alpha_{3,5}}(h) = [2, 3, 4]$. Since there does not exist a box in Y' whose corresponding hook h' satisfies $\mathcal{A}_{\alpha_{3,5}}(h') = \mathcal{A}_{\alpha_{3,5}}(h) = [2, 3, 4]$, the player A's turn is over. If the player B removes the hook h' corresponding to the box $(2, 1)$ from Y', then B obtains Y'' (with $\alpha_{3,5}|_{Y''}$) below:

$$Y' = \begin{array}{|c|c|c|c|c|}\hline 3&4&3&2&1\\\hline \colorbox{gray}{2}&\colorbox{gray}{3}&\colorbox{gray}{4}&\colorbox{gray}{3}\\\cline{1-4} \colorbox{gray}{1}&2&3\\\cline{1-3}\end{array} \quad\rightarrow\quad Y'' = \begin{array}{|c|c|c|c|c|}\hline 3&4&3&2&1\\\hline 2&3\\\cline{1-2}\end{array}$$

Note that $\mathcal{A}_{\alpha_{3,5}}(h') = [3, 4, 3, 2, 1]$. Notice that the box $(1, 2)$ in Y'' is a unique box in Y'' whose corresponding hook h'' satisfies $\mathcal{A}_{\alpha_{3,5}}(h'') = \mathcal{A}_{\alpha_{3,5}}(h') = [3, 4, 3, 2, 1]$. Because of (M4b), B must remove the hook h'' from Y'', and obtains Y''' (with $\alpha_{3,5}|_{Y'''}$) below:

$$Y'' = \begin{array}{|c|c|c|c|c|}\hline 3&\colorbox{gray}{4}&\colorbox{gray}{3}&\colorbox{gray}{2}&\colorbox{gray}{1}\\\hline 2&\colorbox{gray}{3}\\\cline{1-2}\end{array} \quad\rightarrow\quad Y''' = \begin{array}{|c|}\hline 3\\\hline 2\\\hline\end{array}$$

If the player A removes the hook h''' corresponding to the box $(1, 1)$ from Y''', then A obtains empty Young diagram $\emptyset$:

$$Y''' = \begin{array}{|c|}\hline \colorbox{gray}{3}\\\hline \colorbox{gray}{2}\\\hline\end{array} \quad\rightarrow\quad \emptyset$$

In this case, the winner is player A. We remark that there exists no transition from the starting position $Y_{3,5}$ to Y'' above. In general, not every Young diagram contained in $Y_{m,n}$ is a position of MHRG(m, n). Following this paper, Motegi [6] gave a characterization of the set of all game positions in MHRG(m, n).

3. Sprague-Grundy Values

From computer computation, we obtained the $\mathcal{G}$-value of the starting position in MHRG(m,n) for $1 \le m,n \le 9$ as seen in Table 2.

$m\backslash n$	1	2	3	4	5	6	7	8	9
1	1	1	3	3	5	5	7	7	9
2	1	3	3	1	1	1	1	1	1
3	3	3	0	0	0	0	3	3	10
4	3	1	0	4	4	2	2	5	5
5	5	1	0	4	1	1	14	14	18
6	5	1	0	2	1	7	7	0	0
7	7	1	3	2	14	7	0	0	10
8	7	1	3	5	14	0	0	8	8
9	9	1	10	5	18	0	10	8	1

Table 2: $\mathcal{G}$-value of the starting position $Y_{m,n}$ in MHRG(m,n) for $1 \le m,n \le 9$.

Motivated by Table 2, we made conjectures on the $\mathcal{G}$-value of the starting position $Y_{m,n}$ of MHRG(m,n) for some m,n as follows:

(1) If $m \le n$ and $m+n$ is even, then the $\mathcal{G}$-value of the starting position in MHRG(m,n) is equal to the $\mathcal{G}$-value of the starting position in MHRG$(m,n+1)$.

(2) The sequence $\{\mathcal{G}(Y_{1,n})\}_{n\ge1}$ of the $\mathcal{G}$-values of the starting positions in MHRG$(1,n)$ for $n \ge 1$ is arithmetic periodic.

(3) The sequence $\{\mathcal{G}(Y_{2,n})\}_{n\ge2}$ of the $\mathcal{G}$-values of the starting positions in MHRG$(2,n)$ for $n \ge 2$ is periodic.

(4) The $\mathcal{G}$-value of the starting position in MHRG(n,n) and MHRG$(n,n+1)$ is equal to $\bigoplus_{1\le k\le n} k$, where $\bigoplus_i a_i$ denotes the nim-sum (the addition of numbers in binary form without carrying) of all a_i.

In Section 5, we will prove Theorem 1 which imply that (1) above is true. In Section 6, we will prove Theorems 2, 3 which imply that (2),(3) above are true. In Section 7, we will prove Theorem 4 which imply that (4) above is true.

4. Diagonal Expressions for Young Diagrams and Hooks

4.1. The Diagonal Expression

Recall from Section 2.2 that $\mathcal{F}(Y_{m,n})$ denotes the set of all Young diagrams contained in $Y_{m,n}$. The diagonal expression for $Y \in \mathcal{F}(Y_{m,n})$ is now defined in terms of the following elements.

Let $\boldsymbol{a} \in \mathbb{N}_0^{m+n+1}$ be given by $\boldsymbol{a} = (a_{-m}, a_{-m+1}, \ldots, a_n)$, where we call a_k the *k-th component* of $\boldsymbol{a}$ for $-m \leq k \leq n$. For $-m < i \leq 0$ (resp. $0 < i \leq n$), we say that the pair (a_{i-1}, a_i) satisfies the *adjacency requirement* if $0 \leq a_i - a_{i-1} \leq 1$ (resp. $0 \leq a_{i-1} - a_i \leq 1$). Additionally, we say that $\boldsymbol{a}$ satisfies the *adjacency requirement* if (a_{i-1}, a_i) satisfies the adjacency requirement for all $-m < i \leq n$.

For $m, n \in \mathbb{N}$, let $\mathbb{D}_{m,n} \subset \mathbb{N}_0^{m+n+1}$ denote the set of all elements $\boldsymbol{a} = (a_{-m}, \ldots, a_n) \in \mathbb{N}_0^{m+n+1}$ with $a_{-m} = a_n = 0$ satisfying the adjacency requirement. Finally, set $d_k(Y) := \#\{(i,j) \in Y \mid j - i = k\}$ for $k \in \mathbb{Z}$. Note that if $k \leq -m$ or $k \geq n$, then $d_k(Y) = 0$.

Remark 2. For $i, j \geq 2$, if $(i,j) \in Y$, then $(i-1, j-1) \in Y$. Also, if $(i,j) \notin Y$, then $(i+a, j+a) \notin Y$ for $a \in \mathbb{N}$. Hence we see that $d_k(Y) = \max\{\min\{i,j\} \mid (i,j) \in Y, j - i = k\}$ for $k \in \mathbb{Z}$.

Given the above setting, the following lemma is easily verified.

Lemma 1. *The following statements hold.*

(1) Let $k \geq 0$. Then, $(d_k(Y)+1, d_k(Y)+k+1) \notin Y$. Moreover, if $d_k(Y) > 0$, then $(d_k(Y), d_k(Y)+k) \in Y$.

(2) Let $k < 0$. Then $(d_k(Y)-k+1, d_k(Y)+1) \notin Y$. Moreover, if $d_k(Y) > 0$, then $(d_k(Y)-k, d_k(Y)) \in Y$.

Lemma 2. *The following statements hold.*

(1) If $k > 0$, then $0 \leq d_{k-1}(Y) - d_k(Y) \leq 1$.

(2) If $k \leq 0$, then $0 \leq d_k(Y) - d_{k-1}(Y) \leq 1$.

Proof. (1) Assume that $d_k(Y) = 0$. Then, $(1, k+1) \notin Y$ by Lemma 1, which implies that $(2, k+1) \notin Y$. Hence, $d_{k-1}(Y) = \max\{\min\{i,j\} \mid (i,j) \in Y, j - i = k-1\}$ is equal to 0 or 1 (see Remark 2). Thus we obtain

$$0 \leq d_{k-1}(Y) - d_k(Y) = d_{k-1}(Y) \leq 1.$$

Assume that $d_k(Y) > 0$. By Lemma 1, it follows that

$$(d_k(Y), d_k(Y)+k) \in Y \text{ and } (d_k(Y)+1, d_k(Y)+k+1) \notin Y.$$

Then we have

$$(d_k(Y), d_k(Y)+k-1) \in Y \text{ and } (d_k(Y)+2, d_k(Y)+k+1) \notin Y.$$

Therefore $d_{k-1}(Y) = \max\{\min\{i,j\} \mid (i,j) \in Y, j-i = k-1\}$ is $d_k(Y)$ or $d_k(Y)+1$ by Remark 2. Thus we obtain $0 \leq d_{k-1}(Y) - d_k(Y) \leq 1$.

(2) The proof of (2) is similar to that of (1). $\square$

Definition 7. For every $Y \in \mathcal{F}(Y_{m,n})$, the *diagonal expression* for Y is given by

$$\boldsymbol{d}(Y) = \boldsymbol{d}_{m,n}(Y) = (d_{-m}(Y), d_{-m+1}(Y), \ldots, d_n(Y)).$$

Lemma 3. *The function $\boldsymbol{d}_{m,n}(Y)$ is a bijection of $\mathcal{F}(Y_{m,n}) \to \mathbb{D}_{m,n}$.*

Proof. From Lemma 2, the pair $(d_{i-1}(Y), d_i(Y))$ satisfies the adjacency requirement for all $-m < i \leq n$. Since $d_{-m}(Y) = d_n(Y) = 0$, we have $(d_{-m}(Y), \ldots, d_n(Y)) \in \mathbb{D}_{m,n}$. Also, by the definition of $\boldsymbol{d} = \boldsymbol{d}_{m,n}$, it is obvious that $\boldsymbol{d}$ is an injection.

For $\boldsymbol{a} = (a_{-m}, \ldots, a_n) \in \mathbb{D}_{m,n}$, we define Y as follows. The box (i,j) is contained in Y if and only if $\min\{i,j\} \leq a_{j-i}$. Note that if $j - i \leq -m$ or $n \leq j - i$, then the box (i,j) is not contained in Y. We claim that Y is a Young diagram. It suffices to show that if the box (i,j) is not contained in Y, then neither the box $(i+1,j)$ nor $(i,j+1)$ is contained in Y. If $-m < j - i < 0$, then $\min\{i,j\} = j > a_{j-i}$. By the definition of $\mathbb{D}_{m,n}$, we have

$$0 \leq a_{j-i} - a_{j-i-1} \text{ and } a_{j-i+1} - a_{j-i} \leq 1.$$

Then we get

$$\min\{i+1, j\} = j > a_{j-i-1} \text{ and } \min\{i, j+1\} = j + 1 > a_{j-i+1}.$$

Hence, by the definition of Y, we obtain $(i+1,j), (i,j+1) \notin Y$. The proofs for the cases that $j - i = 0$ and $0 < j - i < n$ are similar. Thus we have shown that Y is a Young diagram. Further, since $(m+1,1), (1,n+1) \notin Y$, it follows that $Y \in \mathcal{F}(Y_{m,n})$.

By the definition of Y, we have $d_k = a_k$ for $-m < k < n$. Hence we obtain $\boldsymbol{d}(Y) = \boldsymbol{a}$, which shows that $\boldsymbol{d}$ is a surjection. Thus we have proved that $\boldsymbol{d}$ is a bijection. □

Example 3. Assume that $m = 3$ and $n = 5$. If $Y \in \mathcal{F}(Y_{3,5})$ is

$$Y = \begin{array}{|c|c|c|c|c|} \hline 3 & 4 & 3 & 2 & 1 \\ \hline 2 & 3 & 4 & 3 & \\ \cline{1-4} 1 & 2 & 3 & & \\ \cline{1-3} \end{array},$$

then $\boldsymbol{d}(Y) = \boldsymbol{d}_{3,5}(Y) = (0, 1, 2, \dot{3}, 2, 2, 1, 1, 0)$ (if necessary, we will accentuate the 0-th component by putting a dot above it as above).

4.2. Diagonal Expressions and Hooks

Let $Y \in \mathcal{F}(Y_{m,n}), (i,j) \in Y$ and set $Y' := Y \setminus h_Y(i,j)$. We set $i' := \max\{x \in \mathbb{N} \mid (x,j) \in Y\}$ and $j' := \max\{x \in \mathbb{N} \mid (i,x) \in Y\}$ For $k \in \mathbb{Z}$, it follows that

$$d_k(Y) - d_k(Y') = \begin{cases} 1 & \text{if } j - i' \leq k \leq j' - i, \\ 0 & \text{otherwise.} \end{cases}$$

Hence the diagonal expression of Y' is

$$\boldsymbol{d}(Y') = (\ldots, d_{j-i'-1}(Y),\, d_{j-i'}(Y)-1, d_{j-i'+1}(Y)-1, \ldots,\\ d_{j'-i-1}(Y)-1, d_{j'-i}(Y)-1, d_{j'-i+1}(Y), \ldots). \qquad (3.1)$$

Let $\boldsymbol{a} = (a_{-m}, \ldots, a_n) \in \mathbb{D}_{m,n}$, $\boldsymbol{a'} = (a'_{-m}, \ldots, a'_n) \in \mathbb{N}_0^{m+n+1}$, and let l, r be such that $-m < l \leq r < n$. If $a'_k = a_k - 1$ for $l \leq k \leq r$, and $a'_k = a_k$ for the other k's, then we write $\boldsymbol{a} \xrightarrow{l,r} \boldsymbol{a'}$ or $\boldsymbol{a'} = \boldsymbol{a}_{[l,r]}$. In this case, the pair (a'_{k-1}, a'_k) satisfies the adjacency requirement for all $-m < k \leq n$ but $k = l$ and $r+1$. Hence, if (a'_{l-1}, a'_l) and (a'_r, a'_{r+1}) satisfy the adjacency requirement, then $\boldsymbol{a'} \in \mathbb{D}_{m,n}$.

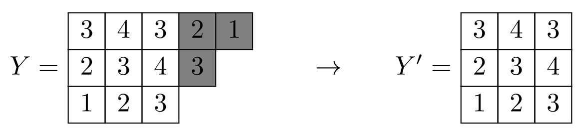
Lemma 4. *Let $Y, Y' \in \mathcal{F}(Y_{m,n})$. The following are equivalent.*

(1) There exists a box $(i,j) \in Y$ such that $Y' = Y \setminus h_Y(i,j)$.

(2) There exist $-m < l \leq r < n$ such that $\boldsymbol{d}(Y) \xrightarrow{l,r} \boldsymbol{d}(Y')$.

In this case, it follows that $l = j - i'$ and $r = j' - i$, where (i', j) is the bottom box in the j-th column of Y, and (i, j') is the rightmost box in the i-th row of Y.

Example 4. Let Y be as in Example 3, and let $Y' = Y \setminus h_Y(1,4)$.



In the diagonal expression, we see that

$$\boldsymbol{d}(Y) = (0,1,2,\dot{3},2,2,1,1,0), \qquad \boldsymbol{d}(Y') = (0,1,2,\dot{3},2,\underline{1,0,0},0),$$

and $\boldsymbol{d}(Y) \xrightarrow{2,4} \boldsymbol{d}(Y')$.

Proof of Lemma 4. The implication (1)⇒(2) and equalities $l = j - i'$ and $r = j' - i$ follow from (3.1). Let us show (2)⇒(1). A proof is given only for the case that $l \leq r \leq 0$. Proofs of the cases $l \leq 0 \leq r$ and $0 \leq l \leq r$ are similar. Notice that $d_l(Y), d_r(Y) > 0$. By Lemma 1 (2), we have both $(d_l(Y) - l, d_l(Y)), (d_r(Y) - r, d_r(Y)) \in Y$. Also, by the adjacency requirement, it follows that $d_l(Y) \leq d_r(Y)$. Since $d_l(Y') = d_l(Y) - 1$ and $d_r(Y') = d_r(Y) - 1$, we see by Lemma 1 (2) that both $(d_l(Y) - l, d_l(Y)), (d_r(Y) - r, d_r(Y)) \notin Y'$, which implied that

$$(d_l(Y) - l + 1, d_l(Y)), (d_r(Y) - r, d_r(Y) + 1) \notin Y'.$$

Since $d_k(Y') = d_k(Y)$ for $k < l$ and $r < k$, we deduce that for i, j such that $j - i < l$ or $r < j - i$, we have $(i,j) \in Y$ if and only if $(i,j) \in Y'$. Hence, both $(d_l(Y) - l + 1, d_l(Y)), (d_r(Y) - r, d_r(Y) + 1) \notin Y$.

Let h be the hook in Y corresponding to the box $(d_r(Y)-r, d_l(Y))$. Since $(d_l(Y)-l, d_l(Y)) \in Y$ and $(d_l(Y)-l+1, d_l(Y)) \notin Y$, the bottom box in the $d_l(Y)$-th column of Y is $(d_l(Y)-l, d_l(Y))$. Also, since $(d_r(Y)-r, d_r(Y)) \in Y$ and $(d_r(Y)-r, d_r(Y)+1) \notin Y$, the rightmost box in the $(d_r(Y)-r)$-th row of Y is $(d_r(Y)-r, d_r(Y))$. We see from (3.1) that the diagonal expression of $Y \setminus h$ is

$$(\ldots, d_{l-1}(Y), d_l(Y)-1, d_{l+1}(Y)-1, \ldots, d_{r-1}(Y)-1, d_r(Y)-1, d_{r+1}(Y), \ldots),$$

which is equal to $\boldsymbol{d}(Y')$. Thus we have proved the lemma. □

The next lemma follows from the proof of Lemma 4.

Lemma 5. *Let $Y \in \mathcal{F}(Y_{m,n})$ and $Y' = Y \setminus h_Y(i,j)$ for $(i,j) \in Y$. Also, let $-m < l \leq r < n$ be such that $\boldsymbol{d}(Y) \xrightarrow{l,r} \boldsymbol{d}(Y')$ in the diagonal expression. Then, $\#(h_Y(i,j)) = \#\mathcal{A}_{\alpha_{m,n}}(h_Y(i,j)) = r-l+1$.*

Definition 8. Let $\boldsymbol{a} = (a_{-m}, \ldots, a_n) \in \mathbb{N}_0^{m+n+1}$. Assume that (a_{k-1}, a_k) satisfies the adjacency requirement for some $-m < k \leq n$. If $(a_{k-1}-1, a_k)$ (resp. (a_{k-1}, a_k-1)) also satisfies the adjacency requirement, then we say that (a_{k-1}, a_k) is a *left* (resp. *right*) *bulge*, and we write $a_{k-1} \searrow a_k$ (resp. $a_{k-1} \nearrow a_k$).

The following lemma can be easily verified.

Lemma 6. *Let $\boldsymbol{a} = (a_{-m}, \ldots, a_n) \in \mathbb{N}_0^{m+n+1}$.*

(1) If (a_{k-1}, a_k) satisfies the adjacency requirement, then (a_{k-1}, a_k) is either a left bulge or a right bulge.

(2) Assume that (a_{k-1}, a_k) satisfies the adjacency requirement. If (a_{k-1}, a_k) is a left bulge, then $(a_{k-1}-1, a_k)$ is a right bulge.

(3) Assume that (a_{k-1}, a_k) satisfies the adjacency requirement. If (a_{k-1}, a_k) is a right bulge, then (a_{k-1}, a_k-1) is a left bulge.

Lemma 7. *Let $\boldsymbol{a} = (a_{-m}, \ldots, a_n), \boldsymbol{a'} = (a'_{-m}, \ldots, a'_n) \in \mathbb{D}_{m,n}$. Assume that $\boldsymbol{a'} = \boldsymbol{a}_{[l,r]} \in \mathbb{D}_{m,n}$ for some $-m < l \leq r < n$. Then,*

$$a_{l-1} \nearrow a_l, a_r \searrow a_{r+1} \text{ and } a'_{l-1} \searrow a'_l, a'_r \nearrow a'_{r+1}.$$

Moreover, for $-m < k \leq n$ with $k \neq l, r+1$, if $a_{k-1} \nearrow a_k$ (resp. $a_{k-1} \searrow a_k$), then $a'_{k-1} \nearrow a'_k$ (resp. $a'_{k-1} \searrow a'_k$).

Proof. Since $\boldsymbol{a'} = \boldsymbol{a}_{[l,r]} \in \mathbb{D}_{m,n}$, it follows that (a_{l-1}, a_l-1) and (a_r-1, a_{r+1}) satisfy the adjacency requirement. Hence, (a_{l-1}, a_l) is a right bulge and (a_r, a_{r+1}) is a left bulge. By Lemma 6 (2) and (3), (a'_{l-1}, a'_l) is a left bulge and (a'_r, a'_{r+1}) is a right bulge. By the definition of $\boldsymbol{a}_{[l,r]}$, we have $a_k - a_{k-1} = a'_k - a'_{k-1}$ for $-m < k \leq n$ with $k \neq l, r+1$. Hence, both (a_{l-1}, a_l) and (a'_{l-1}, a'_l) are both left bulges or rights bulges. Thus we have proved the lemma. □

Let $Y \in \mathcal{F}(Y_{m,n})$ be a Young diagram with the unimodal numbering $\alpha_{m,n}$. By Remark 1 (2), it follows that $\alpha_{m,n}(i', j') = \alpha_{m,n}(i' + a, j' + a)$ for all $(i', j') \in Y$ and $a \in \mathbb{N}$ such that $(i' + a, j' + a) \in Y$. Hence we see that $\mathcal{A}_{\alpha_{m,n}}(Y) = \mathcal{A}_{\alpha_{m,n}}(Y \setminus h_Y(i, j)) \cup \mathcal{A}_{\alpha_{m,n}}(h_Y(i, j))$ for $(i, j) \in Y$.

Lemma 8. *For $Y \in \mathcal{F}(Y_{m,n})$ and $1 \le k \le \hat{\alpha}_{m,n} = \lfloor (n + m)/2 \rfloor$,*

$$\#\{x \in \mathcal{A}_{\alpha_{m,n}}(Y) \mid x = k\} = \begin{cases} d_{-m+k} + d_{n-k} & \text{if } -m + k \ne n - k, \\ d_{-m+k} & \text{if } -m + k = n - k. \end{cases}$$

Proof. Assume that $-m + k \ne n - k$. Then we compute

$$\begin{aligned} &\#\{x \in \mathcal{A}_{\alpha_{m,n}}(Y) \mid x = k\} = \#\{(i, j) \in Y \mid j - i = -m + k \text{ or } n - k\} \\ &= \#\{(i, j) \in Y \mid j - i = -m + k\} + \#\{(i, j) \in Y \mid j - i = n - k\} \\ &= d_{-m+k} + d_{n-k}. \end{aligned}$$

The proof of the case $-m + k = n - k$ is similar. ☐

Lemma 9. *Let $Y \in \mathcal{F}(Y_{m,n})$ and $Y' = Y \setminus h_Y(i, j)$ for $(i, j) \in Y$. Let $-m < l \le r < n$ be such that $\boldsymbol{d}(Y) \xrightarrow{l,r} \boldsymbol{d}(Y')$ in the diagonal expression (see (3.1)). Assume that there exists $(i', j') \in Y'$ such that $\mathcal{A}_{\alpha_{m,n}}(h_{Y'}(i', j')) = \mathcal{A}_{\alpha_{m,n}}(h_Y(i, j))$. Set $Y'' = Y' \setminus h_{Y'}(i', j')$. Then, $\boldsymbol{d}(Y') \xrightarrow{n-m-r,n-m-l} \boldsymbol{d}(Y'')$ in the diagonal expression. Also, there exists no box $(i'', j'') \in Y''$ such that $\mathcal{A}_{\alpha_{m,n}}(h_{Y''}(i'', j'')) = \mathcal{A}_{\alpha_{m,n}}(h_Y(i, j))$.*

Example 5. Let Y be as in Example 3 (note that $m = 3$ and $n = 5$), and set $Y' = Y \setminus h_Y(1, 3)$. Then we have $\mathcal{A}_{\alpha_{3,5}}(h_Y(1, 3)) = [3, 4, 3, 2, 1]$. Notice that $\mathcal{A}_{\alpha_{3,5}}(h_{Y'}(1, 1)) = [1, 2, 3, 4, 3] = \mathcal{A}_{\alpha_{3,5}}(h_Y(1, 3))$. Here we set $Y'' = Y' \setminus h_{Y'}(1, 1)$ and it follows that

$$Y = \begin{array}{|c|c|c|c|c|} \hline 3 & 4 & 3 & 2 & 1 \\ \hline 2 & 3 & 4 & 3 & \\ \cline{1-4} 1 & 2 & 3 & & \\ \cline{1-3} \end{array} \quad \to \quad Y' = \begin{array}{|c|c|c|} \hline 3 & 4 & 3 \\ \hline 2 & 3 & \\ \cline{1-2} 1 & 2 & \\ \cline{1-2} \end{array} \quad \to \quad Y'' = \begin{array}{|c|} \hline 3 \\ \hline 2 \\ \hline \end{array}$$

In this case,

$$\begin{aligned} \boldsymbol{d}(Y) &= (0, 1, 2, \dot{3}, 2, 2, 1, 1, 0), \\ \boldsymbol{d}(Y') &= (0, 1, 2, \underline{\dot{2}, 1, 1, 0, 0}, 0), \\ \boldsymbol{d}(Y'') &= (0, \underline{0, 1, \dot{1}, 0, 0}, 0, 0, 0), \end{aligned}$$

and hence $\boldsymbol{d}(Y) \xrightarrow{0,4} \boldsymbol{d}(Y') \xrightarrow{-2,2} \boldsymbol{d}(Y'')$ where $-2 = 5 - 3 - 4$ and $2 = 5 - 3 - 0$.

Proof of Lemma 9. We set $h := h_Y(i, j)$ and $h' := h_{Y'}(i', j')$. Since $Y'' = Y' \setminus h'$, we see by (3.1) that $\boldsymbol{d}(Y') \xrightarrow{l',r'} \boldsymbol{d}(Y'')$ for some $-m < l' \le r' < n$. Now we show that $l' = m - n - r$ and $r' = n - m - l$. Since $\mathcal{A}_{\alpha_{m,n}}(h') = \mathcal{A}_{\alpha_{m,n}}(h)$ and

$\boldsymbol{d}(Y) \xrightarrow{l,r} \boldsymbol{d}(Y')$, we have $\#\mathcal{A}_{\alpha_{m,n}}(h') = \#\mathcal{A}_{\alpha_{m,n}}(h) = r-l+1$ by Lemma 5. Hence we see that $\boldsymbol{d}(Y') \xrightarrow{a,a+r-l} \boldsymbol{d}(Y'')$ for some $a \in \mathbb{Z}$.

Now it is sufficient to show that $a = n-m-r$. For a contradiction, suppose that $a = l$. Note that $\boldsymbol{d}(Y') \xrightarrow{l,r} \boldsymbol{d}(Y'')$. Hence we see by Lemma 7 that $d_{l-1}(Y') \nearrow d_l(Y')$. Similarly, since $\boldsymbol{d}(Y) \xrightarrow{l,r} \boldsymbol{d}(Y')$, it follows from Lemma 7 that $d_{l-1}(Y') \searrow d_l(Y')$. Thus we get $d_{l-1}(Y') \nearrow d_l(Y')$ and $d_{l-1}(Y') \searrow d_l(Y')$, which contradicts Lemma 6 (1).

Next, suppose that $a \neq l, n-m-r$. For $k \in \mathbb{Z}$, we define $\mu(k) := \min\{k+m, -k+n\}$. Since $\boldsymbol{d}(Y') \xrightarrow{a,a+r-l} \boldsymbol{d}(Y'')$, we have

$$\begin{aligned}\mathcal{A}_{\alpha_{m,n}}(h') &= [\alpha_{m,n}(i',j') \mid (i',j') \in h'] \\ &= [\min\{j'-i'+m, i'-j'+n\} \mid (i',j') \in h'] \\ &= [\mu(k) \mid a \leq k \leq a+r-l].\end{aligned}$$

Note that

$$\begin{aligned}\min \mathcal{A}_{\alpha_{m,n}}(h) &= \min[\min\{j'-i'+m, i'-j'+n\} \mid (i',j') \in h] \\ &= \min\{\min\{l+m, l+n\}, \min\{r+m, r+n\}\} \\ &= \min\{\mu(l), \mu(r)\}.\end{aligned}$$

We give a proof only for the case that $\mu(l) < \mu(r)$. The proofs for the cases in which $\mu(l) = \mu(r)$ and $\mu(l) > \mu(r)$ are similar. If $l \geq n-m-l$, then

$$\begin{aligned}\mu(l) &= \min\{l+m, -l+n\} = m + \min\{l, -l+n-m\} = n-l \\ &\geq \min\{r+m, (n-l) + \underbrace{(l-r)}_{\leq 0}\} = \min\{r+m, -r+n\} = \mu(r),\end{aligned}$$

which is a contradiction. Hence we get $l < n-m-l$ and $\mu(l) = \mu(n-m-l) = l+m$. If $l < a < n-m-r$, then $a+r-l < n-m-l$. Then, we have

$$\begin{aligned}\mu(b) &= \min\{b+m, -b+n\} = \min\{a+m, -a-r+l+n\} \\ &> \min\{l+m, -n+m+l+n\} = l+m = \mu(l)\end{aligned}$$

for $a \leq b \leq a+r-l$. Since $\mathcal{A}_{\alpha_{m,n}}(h') = [\mu(k) \mid a \leq k \leq a+r-l]$, it follows that $\mu(l) \in \mathcal{A}_{\alpha_{m,n}}(h)$ is not contained in $\mathcal{A}_{\alpha_{m,n}}(h')$, which is a contradiction. If $a < l$, then

$$a+m < l+m < n-m-l+m < -a+n$$

and

$$\mu(a) = \min\{a+m, -a+n\} = a+m < l+m = \mu(l) = \min \mathcal{A}_{\alpha_{m,n}}(h).$$

Hence we obtain $\mu(a) \notin \mathcal{A}_{\alpha_{m,n}}(h)$, another contradiction. If $n-m-r<a$, then

$$a+r-l+m > -l+n > l+m > -a-r+l+n$$

and

$$\mu(a+r-l) = \min\{a+r-l+m, -a-r+l+n\} = -a-r+l+n$$
$$< l+m = \mu(l) = \min \mathcal{A}_{\alpha_{m,n}}(h).$$

Hence we get $\mu(a+r-l) \notin \mathcal{A}_{\alpha_{m,n}}(h)$, yet another contradiction. Thus we obtain $a = n-m-r$, as desired.

Suppose that there exists a box $(i'', j'') \in Y''$ such that $\mathcal{A}_{\alpha_{m,n}}(h'') = \mathcal{A}_{\alpha_{m,n}}(h)$, where $h'' := h_{Y''}(i'', j'')$. Note that $\mathcal{A}_{\alpha_{m,n}}(h'') = \mathcal{A}_{\alpha_{m,n}}(h')$. Since $\boldsymbol{d}(Y') \xrightarrow{n-m-r,n-m-l} \boldsymbol{d}(Y'')$, it follows by the argument above that $\boldsymbol{d}(Y'' \setminus h'')$ is equal to $\boldsymbol{d}(Y'')_{[l,r]}$ or $\boldsymbol{d}(Y'')_{[n-m-r,n-m-l]}$.

If $\boldsymbol{d}(Y'' \setminus h'') = \boldsymbol{d}(Y'')_{[l,r]}$, then we see by Lemma 7 that $d_{l-1}(Y'') \nearrow d_l(Y'')$ and $d_r(Y'') \searrow d_{r+1}(Y'')$. Similarly, since $\boldsymbol{d}(Y) \xrightarrow{l,r} \boldsymbol{d}(Y')$, it follows from Lemma 7 that $d_{l-1}(Y') \searrow d_l(Y')$ and $d_r(Y') \nearrow d_{r+1}(Y')$. Note that $\boldsymbol{d}(Y') \xrightarrow{n-m-r,n-m-l} \boldsymbol{d}(Y'')$. If $l = n-m-l+1$, then $r \geq l = n-m-l+1 \geq n-m-r+1 > n-m-r-1$. Thus we see by Lemma 7 that $d_{l-1}(Y'') \searrow d_l(Y'')$ or $d_r(Y'') \nearrow d_{r+1}(Y'')$. Thus we have

$$d_{l-1}(Y'') \nearrow d_l(Y'') \text{ and } d_{l-1}(Y'') \searrow d_l(Y'')$$

or

$$d_r(Y'') \searrow d_{r+1}(Y'') \text{ and } d_r(Y'') \nearrow d_{r+1}(Y''),$$

which contradicts Lemma 6 (1).

If $\boldsymbol{d}(Y'' \setminus h'') = \boldsymbol{d}(Y'')_{[n-m-r,n-m-l]}$, then we see by Lemma 7 that

$$d_{n-m-r-1}(Y'') \nearrow d_{n-m-r}(Y'').$$

Similarly, since $\boldsymbol{d}(Y') \xrightarrow{n-m-r,n-m-l} \boldsymbol{d}(Y'')$, it follows from Lemma 7 that

$$d_{n-m-r-1}(Y'') \searrow d_{n-m-r}(Y'').$$

Thus we get $d_{n-m-r-1}(Y'') \nearrow d_{n-m-r}(Y'')$ and $d_{n-m-r-1}(Y'') \searrow d_{n-m-r}(Y'')$, another contradiction of Lemma 6 (1). □

5. An Isomorphism between Rectangular Diagrams

For fixed $m, n \in \mathbb{N}$, it can be easily shown that $\mathrm{MHRG}(m,n)$ is isomorphic to $\mathrm{MHRG}(n,m)$. In what follows, we assume that $m \leq n$.

Assume that $m+n$ is even. We define $c := (n-m)/2$; note that c is a non-negative integer. Here we will prove that $\mathrm{MHRG}(m,n)$ is isomorphic to $\mathrm{MHRG}(m,n+1)$.

Let $\mathcal{T}(Y_{m,n})$ be the subset of $\mathcal{F}(Y_{m,n})$ consisting of all $Y \in \mathcal{F}(Y_{m,n})$ such that there exists a transition from $Y_{m,n}$ to Y, that is, $\mathcal{T}(Y_{m,n}) = \mathcal{C}(\mathrm{MHRG}(m,n))$.

Remark 3. We see by Lemma 9 that in $\mathrm{MHRG}(m,n)$, the operation (M2b) is performed at most once, and the operation (M2c) is never performed.

Let $Y \in \mathcal{T}(Y_{m,n})$ and $Y' \in \mathcal{O}(Y)$. By Lemmas 4 and 9, there exists $-m < l \leq r < n$ such that

$$\boldsymbol{d}(Y) \xrightarrow{l,r} \boldsymbol{d}(Y')$$

or there exist $-m < l \leq r < n$ and $Y'' \in \mathcal{F}(Y_{m,n})$ such that

$$\boldsymbol{d}(Y) \xrightarrow{l,r} \boldsymbol{d}(Y'') \xrightarrow{n-m-r,n-m-l} \boldsymbol{d}(Y').$$

Definition 9. We define the map $E : \mathbb{N}_0^{m+n+1} \to \mathbb{N}_0^{m+n+2}$ as follows. If $\boldsymbol{a} \in \mathbb{N}_0^{m+n+1}$ is

$$\boldsymbol{a} = (a_{-m}, \ldots, a_{c-1}, \underbrace{a_c}_{c\text{-th}}, a_{c+1}, \ldots, a_n),$$

then

$$E(\boldsymbol{a}) := (a_{-m}, \ldots, a_{c-1}, \underbrace{a_c}_{c\text{-th}}, \underbrace{a_c}_{(c+1)\text{-th}}, a_{c+1}, \ldots, a_n).$$

It can be easily verified that

$$\boldsymbol{a} \in \mathbb{D}_{m,n} \text{ if and only if } E(\boldsymbol{a}) \in \mathbb{D}_{m,n+1}. \tag{5.1}$$

Hence the map $E : \mathbb{N}_0^{m+n+1} \to \mathbb{N}_0^{m+n+2}$ induces the map $E : \mathcal{F}(Y_{m,n}) \to \mathcal{F}(Y_{m,n+1})$ as follows. For $Y \in \mathcal{F}(Y_{m,n})$, we define $E(Y)$ to be the unique element of $\mathcal{F}(Y_{m,n+1})$ whose diagonal expression is

$$E(\boldsymbol{d}(Y)) = (d_{-m}(Y), \ldots, d_{c-1}(Y), d_c(Y), d_c(Y), d_{c+1}(Y), \ldots, d_n(Y)).$$

Note that $\boldsymbol{d}(E(Y)) = E(\boldsymbol{d}(Y))$. Notice, also, that $E : \mathcal{F}(Y_{m,n}) \to \mathcal{F}(Y_{m,n+1})$ is an injection. For $l, r \in \mathbb{Z}$, we define $e_l, e_r : \mathbb{Z} \to \mathbb{Z}$ by

$$e_l(k) := \begin{cases} k & \text{if } k \leq c, \\ k+1 & \text{if } k > c, \end{cases} \qquad e_r(k) := \begin{cases} k & \text{if } k < c, \\ k+1 & \text{if } k \geq c. \end{cases}$$

In particular, note that $e_l(k) \neq c+1$ and $e_r(k) \neq c$. The following lemma can be shown easily.

Lemma 10. *Let $l, r \in \mathbb{Z}$. It follows that $e_l(n-m-k) = n-m+1-e_r(k)$ for $k \in \mathbb{Z}$.*

Lemma 11. *For $l, r \in \mathbb{Z}$ and $\boldsymbol{a} \in \mathbb{N}_0^{m+n+1}$, it follows that $E(\boldsymbol{a}_{[l,r]}) = E(\boldsymbol{a})_{[e_l(l),e_r(r)]}$. Therefore, for $Y \in \mathcal{F}(Y_{m,n})$, it follows that $\boldsymbol{d}(Y)_{[l,r]} \in \mathbb{D}_{m,n}$ if and only if $\boldsymbol{d}(E(Y))_{[e_l(l),e_r(r)]} \in \mathbb{D}_{m,n+1}$.*

Proof. If $c < l \leq r$, then $l+1 = e_l(l), r+1 = e_r(r)$ and

$$E(\boldsymbol{a}_{[l,r]}) = (\ldots, \underbrace{a_c}_{c\text{-th}}, \underbrace{a_c}_{(c+1)\text{-th}}, \ldots, a_{l-1}, \underbrace{a_l - 1}_{(l+1)\text{-th}}, \ldots, \underbrace{a_r - 1}_{(r+1)\text{-th}}, a_{r+1}, \ldots).$$

Thus we obtain $E(\boldsymbol{a}_{[l,r]}) = E(\boldsymbol{a})_{[l+1,r+1]} = E(\boldsymbol{a})_{[e_l(l),e_r(r)]}$.

If $l \leq c \leq r$, then $l = e_l(l), r+1 = e_r(r)$ and

$$E(\boldsymbol{a}_{[l,r]}) = (\ldots, a_{l-1}, \underbrace{a_l - 1}_{l\text{-th}}, \ldots, \underbrace{a_c - 1}_{c\text{-th}}, \underbrace{a_c - 1}_{(c+1)\text{-th}}, \ldots, \underbrace{a_r - 1}_{(r+1)\text{-th}}, a_{r+1}, \ldots).$$

This implies $E(\boldsymbol{a}_{[l,r]}) = E(\boldsymbol{a})_{[l,r+1]} = E(\boldsymbol{a})_{[e_l(l),e_r(r)]}$.

If $l \leq r < c$, then $l = e_l(l), r = e_r(r)$ and

$$E(\boldsymbol{a}_{[l,r]}) = (\ldots, a_{l-1}, \underbrace{a_l - 1}_{l\text{-th}}, \ldots, \underbrace{a_r - 1}_{r\text{-th}}, a_{r+1}, \ldots, \underbrace{a_c}_{c\text{-th}}, \underbrace{a_c}_{(c+1)\text{-th}}, \ldots).$$

And, hence, we obtain $E(\boldsymbol{a}_{[l,r]}) = E(\boldsymbol{a})_{[l,r]} = E(\boldsymbol{a})_{[e_l(l),e_r(r)]}$.

In all cases above, we have $E(\boldsymbol{a}_{[l,r]}) = E(\boldsymbol{a})_{[e_l(l),e_r(r)]}$ for $-m < l \leq r < n$. Hence, by $\boldsymbol{d}(E(Y)) = E(\boldsymbol{d}(Y))$ and (5.1), we obtain

$$\begin{aligned} \boldsymbol{d}(Y)_{[l,r]} \in \mathbb{D}_{m,n} &\overset{(5.1)}{\Leftrightarrow} E(\boldsymbol{d}(Y)_{[l,r]}) \in \mathbb{D}_{m,n+1} \\ &\Leftrightarrow E(\boldsymbol{d}(Y))_{[e_l(l),e_r(r)]} \in \mathbb{D}_{m,n+1} \\ &\Leftrightarrow \boldsymbol{d}(E(Y))_{[e_l(l),e_r(r)]} \in \mathbb{D}_{m,n+1}, \end{aligned}$$

as desired. □

Lemma 12. *Let $Y, Y' \in \mathcal{T}(Y_{m,n})$. Assume that $Y \to Y'$ and $E(Y) \in \mathcal{T}(Y_{m,n+1})$. Then, $E(Y') \in \mathcal{T}(Y_{m,n+1})$ and $E(Y) \to E(Y')$.*

Proof. Since $Y \to Y'$, it follows from definition that

(a) there exists $-m < l \leq r < n$ such that $\boldsymbol{d}(Y) \xrightarrow{l,r} \boldsymbol{d}(Y')$ or

(b) there exist $-m < l \leq r < n$ and $Y'' \in \mathcal{F}(Y_{m,n})$ such that $\boldsymbol{d}(Y) \xrightarrow{l,r} \boldsymbol{d}(Y'') \xrightarrow{n-m-r,n-m-l} \boldsymbol{d}(Y')$.

We give a proof only for the case (b); the proof for the case (a) is easier and entirely similar.

By Lemma 10, we have $e_l(n-m-r) = n-m+1-e_r(r)$ and $e_r(n-m-l) = n-m+1-e_l(l)$. Thus we have

$$\boldsymbol{d}(E(Y)) \xrightarrow{e_l(l),e_r(r)} \boldsymbol{d}(E(Y'')) \xrightarrow{e_l(n-m-r),e_r(n-m-l)} \boldsymbol{d}(E(Y'))$$

by Lemma 11, which implies that $E(Y) \to E(Y')$. Thus we have proved the lemma. □

Let $Y' \in \mathcal{T}(Y_{m,n})$, and let $Y_{m,n} = Y_0 \to Y_1 \to \cdots \to Y_k = Y'$ be a transition from $Y_{m,n}$ to Y' in MHRG(m,n). Note that $E(Y_0) = E(Y_{m,n}) = Y_{m,n+1} \in \mathcal{T}(Y_{m,n+1})$. Also, we see by Lemma 12 that for $0 \leq p < k$, if $E(Y_p) \in \mathcal{T}(Y_{m,n+1})$, then $E(Y_{p+1}) \in \mathcal{T}(Y_{m,n+1})$. Thus we obtain $E(Y') \in \mathcal{T}(Y_{m,n+1})$ by inductive argument. Therefore, we obtain

$$E(\mathcal{T}(Y_{m,n})) \subset \mathcal{T}(Y_{m,n+1}). \tag{5.2}$$

Moreover, it is obvious from Lemma 12 that

$$E(\mathcal{O}(Y)) \subseteq \mathcal{O}(E(Y)) \tag{5.3}$$

for $Y \in \mathcal{T}(Y_{m,n+1})$.

Lemma 13. *It follows that* $d_c(Y) = d_{c+1}(Y)$ *for all* $Y \in \mathcal{T}(Y_{m,n+1})$.

Proof. Suppose, for a contradiction, that there exists $Y \in \mathcal{T}(Y_{m,n+1})$ such that $d_c(Y) \neq d_{c+1}(Y)$. Let $\mathcal{V} \subset \mathcal{T}(Y_{m,n+1})$ be the subset of $\mathcal{T}(Y_{m,n+1})$ consisting of elements $Y \in \mathcal{T}(Y_{m,n+1})$ such that $d_c(Y) \neq d_{c+1}(Y)$; also, let $Y_0 \in \mathcal{V}$ be such that $\#(Y_0) \geq \#(Y)$ for all $Y \in \mathcal{V}$. Since $c \geq 0$ and $(d_c(Y_0), d_{c+1}(Y_0))$ satisfies the adjacency requirement, we have

$$d_c(Y_0) = d_{c+1}(Y_0) + 1 \text{ and } d_c(Y_0) \searrow d_{c+1}(Y_0).$$

Since $Y_0 \neq Y_{m,n+1}$, there exists $Y_1 \in \mathcal{T}(Y_{m,n+1})$ such that $Y_1 \to Y_0$. Note that $\#(Y_1) \geq \#(Y_0)$, which implies that $Y_1 \notin \mathcal{V}$ by the maximality of Y_0. Thus we have $d_c(Y_1) = d_{c+1}(Y_1)$ and $d_c(Y_1) \nearrow d_{c+1}(Y_1)$. By Lemma 8, we set that for $p = 0, 1$, the number t_p of boxes in Y_p having the number $\hat{\alpha}_{m,n} = (m+n)/2$ is equal to $d_c(Y_p) + d_{c+1}(Y_p)$. Thus, $t_1 - t_0$ is odd. If two hooks are removed in $Y_1 \to Y_0$, then the two hooks have the same multiset of numbers. Thus $t_1 - t_0$ is even, but this contradicts the fact that $t_1 - t_0$ is odd. Consequently, one hook is removed in $Y_1 \to Y_0$. Hence

$$d_c(Y_0) = d_c(Y_1) \text{ and } d_{c+1}(Y_0) = d_{c+1}(Y_1) - 1$$

by $0 \leq d_k(Y_1) - d_k(Y_0) \leq 1$ for $-m \leq k \leq n$. Also, there exists $c+1 \leq k = k(Y_1) < n+1$ such that $\boldsymbol{d}(Y_1) \xrightarrow{c+1,k} \boldsymbol{d}(Y_0)$ and $\boldsymbol{d}(Y_0)_{[n+1-m-k,c]} \notin \mathbb{D}_{m,n+1}$. Note that $n+1-m-(c+1) = c$. By Lemma 7, we have

$$d_{n-m-k}(Y_1) \searrow d_{n+1-m-k}(Y_1),\ d_c(Y_1) \nearrow d_{c+1}(Y_1),\ \text{and } d_k(Y_1) \searrow d_{k+1}(Y_1).$$

Now we choose Y_1 such that $k = k(Y_1)$ is a maximum.

Suppose that $Y_1 = Y_{m,n+1}$. In this case, we have $d_p(Y_1) \nearrow d_{p+1}(Y_1)$ for $-m \leq p < n-m$, and $d_p(Y_1) \searrow d_{p+1}(Y_1)$ for $n-m \leq p \leq n$. Since $c \leq n-m$, we have $d_{n-m-k}(Y_1) \nearrow d_{n+1-m-k}(Y_1)$ and $d_{n-m-k}(Y_0) \nearrow d_{n+1-m-k}(Y_0)$ by Lemma 7. Thus we have $\boldsymbol{d}(Y_0)_{[n+1-m-k,c]} \in \mathbb{D}_{m,n+1}$ by Lemma 7, which is a contradiction. Hence we obtain $Y_1 \neq Y_{m,n+1}$. Then, there exists $Y_2 \in \mathcal{T}(Y_{m,n+1})$ such that $Y_2 \to Y_1$. Note that $d_c(Y_2) = d_{c+1}(Y_2)$ and $d_c(Y_2) \nearrow d_{c+1}(Y_2)$.

Suppose that $d_{n-m-k}(Y_2) \searrow d_{n+1-m-k}(Y_2)$ and $d_k(Y_2) \searrow d_{k+1}(Y_2)$. By Lemma 7, we have $\boldsymbol{d}(Y_2)_{[c+1,k]} \in \mathbb{D}_{m,n+1}$. Let $Y_1' \in \mathcal{F}(Y_{m,n+1})$ be the Young diagram whose diagonal expression is equal to $\boldsymbol{d}(Y_2)_{[c+1,k]}$; also, notice that $d_c(Y_1') \neq d_{c+1}(Y_1')$ and $d_{n-m-k}(Y_1') \searrow d_{n+1-m-k}(Y_1')$. Since $\boldsymbol{d}(Y_1')_{[n+1-m-k,c]} \notin \mathbb{D}_{m,n+1}$, it follows that $Y_1' \in \mathcal{O}(Y_2)$ and hence $Y_1' \in \mathcal{V}$. Since $\#(Y_1) - \#(Y_0) = \#(Y_2) - \#(Y_1')$, we have $\#(Y_1') = \#(Y_2) - \#(Y_1) + \#(Y_0) > \#(Y_0)$ which contradicts the maximality of Y_0.

Suppose that $d_{n-m-k}(Y_2) \searrow d_{n+1-m-k}(Y_2)$ and $d_k(Y_2) \nearrow d_{k+1}(Y_2)$. If two hooks are removed in $Y_2 \to Y_1$, then there exist $-m < l \leq r < n$ and $Y' \in \mathcal{F}(Y_{m,n})$ such that

$$\boldsymbol{d}(Y_2) \xrightarrow{l,r} \boldsymbol{d}(Y') \xrightarrow{n+1-m-r,n+1-m-l} \boldsymbol{d}(Y_1).$$

Since $d_k(Y_2) \nearrow d_{k+1}(Y_2)$ and $d_k(Y_1) \searrow d_{k+1}(Y_1)$, we have

$$\boldsymbol{d}(Y_2) \xrightarrow{k+1,r} \boldsymbol{d}(Y') \xrightarrow{n+1-m-r,n-m-k} \boldsymbol{d}(Y_1)$$

or

$$\boldsymbol{d}(Y_2) \xrightarrow{l,n-m-k} \boldsymbol{d}(Y') \xrightarrow{k+1,n+1-m-l} \boldsymbol{d}(Y_1).$$

Thus we have $d_{n-m-k}(Y_1) \nearrow d_{n+1-m-k}(Y_1)$, another contradiction. Hence one hook is removed in $Y_2 \to Y_1$. Then there exist $p \geq k+1$ such that

$$\boldsymbol{d}(Y_2) \xrightarrow{k+1,p} \boldsymbol{d}(Y_1).$$

Note that $\boldsymbol{d}(Y_1)_{[n+1-m-p,n-m-k]} \notin \mathbb{D}_{m,n+1}$, also, that

$$d_{n-m-p}(Y_2) \searrow d_{n+1-m-p}(Y_2) \text{ and } d_p(Y_2) \searrow d_{p+1}(Y_2).$$

By Lemma 7, we have $\boldsymbol{d}(Y_2)_{[c+1,p]} \in \mathbb{D}_{m,n+1}$. Let $Y_1' \in \mathcal{F}(Y_{m,n+1})$ be the Young diagram whose diagonal expression is equal to $\boldsymbol{d}(Y_2)_{[c+1,p]}$; notice that $d_c(Y_1') \neq d_{c+1}(Y_1')$. Since $\boldsymbol{d}(Y_1')_{[n+1-m-p,c]} \notin \mathbb{D}_{m,n+1}$, it follows that $Y_1' \in \mathcal{O}(Y_2)$. Hence $Y_1' = \boldsymbol{d}(Y_2)_{[c+1,p]} = (\boldsymbol{d}(Y_2)_{[k+1,p]})_{[c+1,k]} = Y_0$ which contradicts the maximality of k.

Suppose that $d_{n-m-k}(Y_2) \nearrow d_{n+1-m-k}(Y_2)$ and $d_k(Y_2) \searrow d_{k+1}(Y_2)$. If two hooks are removed in $Y_2 \to Y_1$, then there exist $-m < l \leq r < n$ and $Y' \in \mathcal{F}(Y_{m,n})$ such that

$$\boldsymbol{d}(Y_2) \xrightarrow{l,r} \boldsymbol{d}(Y') \xrightarrow{n+1-m-r,n+1-m-l} \boldsymbol{d}(Y_1).$$

Since $d_{n-m-k}(Y_2) \nearrow d_{n+1-m-k}(Y_2)$ and $d_{n-m-k}(Y_1) \searrow d_{n+1-m-k}(Y_1)$, we have

$$\boldsymbol{d}(Y_2) \xrightarrow{n+1-m-k,r} \boldsymbol{d}(Y') \xrightarrow{n+1-m-r,k} \boldsymbol{d}(Y_1)$$

or

$$\boldsymbol{d}(Y_2) \xrightarrow{l,k} \boldsymbol{d}(Y') \xrightarrow{n+1-m-k,n+1-m-l} \boldsymbol{d}(Y_1).$$

Thus we have $d_k(Y_1) \nearrow d_{k+1}(Y_1)$, another contradiction, hence one hook is removed in $Y_2 \to Y_1$. Consequently, there exist $p \geq n+1-m-k$ such that

$$\boldsymbol{d}(Y_2) \xrightarrow{n+1-m-k,p} \boldsymbol{d}(Y_1).$$

Note that $\boldsymbol{d}(Y_1)_{[n+1-m-p,k]} \notin \mathbb{D}_{m,n+1}$ and $d_{n-m-p}(Y_2) \searrow d_{n+1-m-p}(Y_2)$, $d_p(Y_2) \searrow d_{p+1}(Y_2)$. Since $d_c(Y_2) \nearrow d_{c+1}(Y_2)$, we have $p \neq c$. By Lemma 7, we have $\boldsymbol{d}(Y_2)_{[c+1,\max(p,n+1-m-p)]} \in \mathbb{D}_{m,n+1}$. Also, notice that $c+1 \leq \max(p, n+1-m-p)$ since $p \neq c$. Let $Y_1' \in \mathcal{F}(Y_{m,n+1})$ be the Young diagram whose diagonal expression is equal to $\boldsymbol{d}(Y_2)_{[c+1,\max(p,n+1-m-p)]}$; note that $d_c(Y_1') \neq d_{c+1}(Y_1')$. Since $\boldsymbol{d}(Y_1')_{[\min\{p,n+1-m-p\},c]} \notin \mathbb{D}_{m,n+1}$, it follows that $Y_1' \in \mathcal{O}(Y_2)$ and hence $Y_1' \in \mathcal{V}$. If $\max(p, n+1-m-p) \leq k$, then $\#(Y_1) - \#(Y_0) \geq \#(Y_2) - \#(Y_1')$ and hence $\#(Y_1') \geq \#(Y_2) - \#(Y_1) + \#(Y_0) > \#(Y_0)$. If $\max(p, n+1-m-p) > k$, then $\#(Y_2) - \#(Y_1) > \#(Y_2) - \#(Y_1')$ and, hence, $\#(Y_1') > \#(Y_1) > \#(Y_0)$. In any case, we obtain $\#(Y_1') > \#(Y_0)$, which contradicts the maximality of Y_0.

Suppose that $d_{n-m-k}(Y_2) \nearrow d_{n+1-m-k}(Y_2)$ and $d_k(Y_2) \nearrow d_{k+1}(Y_2)$. Let $Y_1' \in \mathcal{O}(Y_2)$. If $d_{n-m-k}(Y_1') \searrow d_{n+1-m-k}(Y_1')$ and $d_k(Y_1') \searrow d_{k+1}(Y_1')$, then by Lemma 7, we have

$$\boldsymbol{d}(Y_2) \xrightarrow{n+1-m-k,n-m-k} \boldsymbol{d}(Y') \xrightarrow{k+1,k} \boldsymbol{d}(Y_1')$$

or

$$\boldsymbol{d}(Y_2) \xrightarrow{k+1,k} \boldsymbol{d}(Y') \xrightarrow{n+1-m-k,n-m-k} \boldsymbol{d}(Y_1')$$

for $Y' \in \mathcal{F}(Y_{m,n})$, which is a contradiction. Thus there exists no option $Y_1' \in \mathcal{O}(Y_2)$ such that $d_{n-m-k}(Y_1') \searrow d_{n+1-m-k}(Y_1')$ and $d_k(Y_1') \searrow d_{k+1}(Y_1')$, which contradicts $Y_2 \to Y_1$. Thus we have proved Lemma 13. □

Theorem 1. *Let $m, n \in \mathbb{N}$ be such that $m \leq n$ and $m+n$ is even. Then the map E gives an isomorphism from MHRG(m,n) to MHRG$(m,n+1)$. Therefore, for each $Y \in \mathcal{T}(Y_{m,n})$, it follows that $\mathcal{G}(Y) = \mathcal{G}(E(Y))$. In particular, $\mathcal{G}(Y_{m,n})$ in MHRG(m,n) is equal to $\mathcal{G}(Y_{m,n+1})$ in MHRG$(m,n+1)$.*

Proof. We have shown that the map $E : \mathcal{T}(Y_{m,n}) \to \mathcal{T}(Y_{m,n+1})$ is injective (see (5.2)) and $E(\mathcal{O}(Y)) \subseteq \mathcal{O}(E(Y))$ for $Y \in \mathcal{T}(Y_{m,n})$ (see (5.3)). Hence it remains to show that $E(\mathcal{O}(Y)) \supseteq \mathcal{O}(E(Y))$ for $Y \in \mathcal{T}(Y_{m,n})$ and $E(\mathcal{T}(Y_{m,n})) = \mathcal{T}(Y_{m,n+1})$.

We first show that $E(\mathcal{O}(Y)) \supseteq \mathcal{O}(E(Y))$. Let $Y \in \mathcal{T}(Y_{m,n})$, and let $X \in \mathcal{O}(E(Y))$. There exists $-m < l \leq r < n$ such that

$$\boldsymbol{d}(Y) \xrightarrow{l,r} \boldsymbol{d}(X) \tag{a}$$

or there exist $-m < l \leq r < n$ and $X' \in \mathcal{F}(Y_{m,n})$ such that

$$\boldsymbol{d}(Y) \xrightarrow{l,r} \boldsymbol{d}(X') \xrightarrow{n-m-r,n-m-l} \boldsymbol{d}(X). \tag{b}$$

By Lemma 13, we have $d_c(E(Y)) \nearrow d_{c+1}(E(Y))$ and $r \neq c$.

In the first case (a), we get $\boldsymbol{d}(X)_{[n+1-m-r,n+1-m-l]} \notin \mathbb{D}_{m,n+1}$. If $l = c+1$, then $d_c(X) \searrow d_{c+1}(X)$ and $d_c(X) > d_{c+1}(X)$, which contradicts Lemma 13. If $l \neq c+1$, then there exist $-m < l_0 \leq r_0 < n$ such that $e_l(l_0) = l, e_r(r_0) = r$. By Lemma 11, we have

$$\boldsymbol{d}(Y)_{[l_0,r_0]} \in \mathbb{D}_{m,n} \text{ and } (\boldsymbol{d}(Y)_{[l_0,r_0]})_{[n-m-r_0,n-m-l_0]} \notin \mathbb{D}_{m,n}.$$

Thus, the Young diagram $Y' \in \mathcal{T}(Y_{m,n})$, whose diagonal expression is equal to $\boldsymbol{d}(Y)_{[l_0,r_0]} \in \mathbb{D}_{m,n}$, is an option of Y. By the proof of Lemma 12, we obtain $X = E(Y') \in E(\mathcal{O}(Y))$.

Consider the second case (b). If $l = c+1$, then

$$\boldsymbol{d}(X) = (\boldsymbol{d}(E(Y))_{[c+1,r]})_{[n+1-m-r,c]} = \boldsymbol{d}(E(Y))_{[n+1-m-r,r]}.$$

Then there exist $-m < l_0 \leq r_0 < n$ such that $e_l(l_0) = n+1-m-r, e_r(r_0) = r$. By Lemma 10, we have

$$e_l(n-m-r_0) = e_l(n-m-r_0) + e_r(r_0) - r = n-m+1-r = e_l(l_0)$$

and hence $l_0 = n-m-r_0$. By Lemma 11, we have

$$\boldsymbol{d}(Y)_{[l_0,r_0]} \in \mathbb{D}_{m,n} \text{ and } (\boldsymbol{d}(Y)_{[l_0,r_0]})_{[n-m-r_0,n-m-l_0]} = (\boldsymbol{d}(Y)_{[l_0,r_0]})_{[l_0,r_0]} \notin \mathbb{D}_{m,n}.$$

Thus, the Young diagram $Y' \in \mathcal{T}(Y_{m,n})$, whose diagonal expression is equal to $\boldsymbol{d}(Y)_{[l_0,r_0]} \in \mathbb{D}_{m,n}$, is an option of Y. By the proof of Lemma 12, we obtain $X = E(Y') \in E(\mathcal{O}(Y))$. If $l \neq c+1$, then there exist $-m < l_0 \leq r_0 < n$ such that $e_l(l_0) = l, e_r(r_0) = r$. By Lemma 11, we have

$$\boldsymbol{d}(Y)_{[l_0,r_0]} \in \mathbb{D}_{m,n} \text{ and } (\boldsymbol{d}(Y)_{[l_0,r_0]})_{[n-m-r_0,n-m-l_0]} \in \mathbb{D}_{m,n}.$$

Thus, the Young diagram $Y' \in \mathcal{T}(Y_{m,n})$, whose diagonal expression is equal to $(\boldsymbol{d}(Y)_{[l_0,r_0]})_{[n-m-r_0,n-m-l_0]} \in \mathbb{D}_{m,n}$, is an option of Y. By the proof of Lemma 12, we obtain $X = E(Y') \in E(\mathcal{O}(Y))$. In any case, we obtain $X \in E(\mathcal{O}(Y))$, as desired.

We next show that $E(\mathcal{T}(Y_{m,n})) = \mathcal{T}(Y_{m,n+1})$. Let $X' \in \mathcal{T}(Y_{m,n+1})$, and let $Y_{m,n+1} = X_0 \to X_1 \to \cdots \to X_k = X'$ be a transition from $Y_{m,n+1}$ to X' in MHRG$(m, n+1)$. We show by induction on k that $X' \in E(\mathcal{T}(Y_{m,n}))$. If $k = 0$, then $X' = Y_{m,n+1} = E(Y_{m,n}) \in E(\mathcal{T}(Y_{m,n}))$. Assume that $k > 0$; note that $X_{k-1} \in E(\mathcal{T}(Y_{m,n}))$ by the induction hypothesis. Let $X'_{k-1} \in \mathcal{T}(Y_{m,n})$ be such that $X_{k-1} = E(X'_{k-1})$. Since $E(\mathcal{O}(X'_{k-1})) = \mathcal{O}(E(X'_{k-1})) = \mathcal{O}(X_{k-1})$ as shown above, we get

$$X' = X_k \in \mathcal{O}(X_{k-1}) = E(\mathcal{O}(X'_{k-1})) \subset E(\mathcal{T}(Y_{m,n})),$$

as desired. Therefore, we conclude that $E(\mathcal{T}(Y_{m,n})) \supset \mathcal{T}(Y_{m,n+1})$ and hence $E(\mathcal{T}(Y_{m,n})) = \mathcal{T}(Y_{m,n+1})$. This completes the proof of Theorem 1. □

6. Sprague-Grundy Values for MHRG for $m = 1$ and $m = 2$

6.1. Values for MHRG$(1, n)$

Theorem 2. *Let $m = 1$ and $n \in \mathbb{N}$. In MHRG$(1, n)$,*

$$\mathcal{T}(Y_{1,n}) = \begin{cases} \mathcal{F}(Y_{1,n}) & \text{if } n \text{ is odd,} \\ \mathcal{F}(Y_{1,n}) \setminus \{Y_{1,\frac{n}{2}}\} & \text{if } n \text{ is even.} \end{cases}.$$

Moreover, for $0 \le l \le n$ such that $Y_{1,l} \in \mathcal{T}(Y_{1,n})$,

$$\mathcal{G}(Y_{1,l}) = \begin{cases} l & \text{if } n \text{ is odd,} \\ l & \text{if } n \text{ is even and } l < n/2, \\ l-1 & \text{if } n \text{ is even and } n/2 < l. \end{cases}.$$

In particular,

$$\mathcal{G}(Y_{1,n}) = \begin{cases} n & \text{if } n \text{ is odd,} \\ n-1 & \text{if } n \text{ is even.} \end{cases}.$$

Proof. By Theorem 1, it suffices to show the assertion for the case that n is odd.

We set $k = (n+1)/2 \in \mathbb{N}$. We see that for $0 \le l \le n$, the unimodal numbering of $Y_{1,l} \in \mathcal{F}(Y_{1,n})$ is as follows:

1	2	$\cdots$	$l-1$	l

if $0 \le l \le k$,

1	2	$\cdots$	$k-1$	k	$k-1$	$\cdots$	$n+2-l$	$n+1-l$

if $k < l \le n$.

By this fact, we deduce that in MHRG$(1, n)$ (with odd n), the operation removing two hooks never takes place. Hence, we obtain $\mathcal{O}(Y_{1,l}) = \{Y_{1,i} \mid 0 \le i < l\}$ for all $0 \le l \le n$. The assertion of the theorem follows immediately from the last statement and the definition of the $\mathcal{G}$-value. □

6.2. Values for $\mathrm{MHRG}(2, n)$

Let $m = 2$ and $n \geq 2$. Recall that $Y = (\lambda_1, \lambda_2)$ denotes the Young diagram having λ_1 boxes in the first row and λ_2 boxes in the second row. If n is even, then $\mathrm{MHRG}(2, n)$ is isomorphic to $\mathrm{MHRG}(2, n+1)$ (see Theorem 1). Thus it suffices to study the case in which n is even; we set $n' := n/2 \in \mathbb{N}$.

Lemma 14. *Let $(\lambda_1, \lambda_2) \in \mathcal{F}(Y_{2,2n'})$ and $\mu_1, \mu_2 \in \mathbb{N}_0$ with $2n' \geq \mu_1 \geq \mu_2 \geq 0$. Then $(\lambda_1, \lambda_2) = (\mu_1, \mu_2)$ if and only if*

$$d_{\mu_1 - 1}(Y) \searrow d_{\mu_1}(Y),\ d_{\mu_2 - 2}(Y) \searrow d_{\mu_2 - 1}(Y),$$

and

$$d_{k-1}(Y) \nearrow d_k(Y)$$

for $-2 < k \leq 2n' = n$ with $k \neq \mu_1, \mu_2 - 1$.

Proof. If $\lambda_2 = 0$, then $(\lambda_1, 0) = (\mu_1, \mu_2)$ if and only if $\mu_2 = 0$, $d_{-1}(Y) = 0$, $d_k(Y) = 1$ for $0 \leq k < \mu_1$, and $d_k(Y) = 0$ for $\mu_1 \leq k < 2n'$. The latter is equivalent to $d_{\mu_1 - 1}(Y) \searrow d_{\mu_1}(Y)$, $d_{\mu_2 - 2}(Y) \searrow d_{\mu_2 - 1}(Y)$, and $d_{k-1}(Y) \nearrow d_k(Y)$ for $-2 < k \leq 2n' = n$ with $k \neq \mu_1, \mu_2 - 1$.

If $\lambda_2 > 0$, then $(\lambda_1, \lambda_2) = (\mu_1, \mu_2)$ if and only if $d_{-1}(Y) = 1$, $d_k(Y) = 2$ for $0 \leq k < \mu_2 - 1$, $d_k(Y) = 1$ for $\mu_2 - 1 \leq k < \mu_1$, and $d_k(Y) = 0$ for $\mu_1 \leq k < 2n'$. The latter is equivalent to $d_{\mu_1 - 1}(Y) \searrow d_{\mu_1}(Y)$, $d_{\mu_2 - 2}(Y) \searrow d_{\mu_2 - 1}(Y)$, and $d_{k-1}(Y) \nearrow d_k(Y)$ for $-2 < k \leq 2n' = n$ with $k \neq \mu_1, \mu_2 - 1$.

Thus we have proved the lemma. □

Lemma 15. *Let $Y = (\lambda_1, \lambda_2) \in \mathcal{F}(Y_{2,2n'})$ and $(i, j) \in Y$. Also, set*

$$Y' = (\lambda'_1, \lambda'_2) = Y \setminus h_Y(i, j).$$

Then, $\lambda'_1 + \lambda'_2 = 2n'$ if and only if there exists a box $(i', j') \in Y'$ such that $\mathcal{A}_{\alpha_{2,n}}(h_Y(i, j)) = \mathcal{A}_{\alpha_{2,n}}(h_{Y'}(i', j'))$. In this case, $Y'' := Y' \setminus h'_Y(i', j')$ is equal to $(2n' - \lambda_2, 2n' - \lambda_1)$.

Proof. We first show the "if" part. By Lemma 4, there exist $-2 < l, r < 2n'$ such that $\boldsymbol{d}(Y) \xrightarrow{l,r} \boldsymbol{d}(Y')$. If there exists a box $(i', j') \in Y'$ such that $\mathcal{A}_{\alpha_{2,n}}(h_Y(i, j)) = \mathcal{A}_{\alpha_{2,n}}(h_{Y'}(i', j'))$, then it follows from Lemma 9 that $\boldsymbol{d}(Y')_{[2n'-2-r, 2n'-2-l]} \in \mathbb{D}_{2,2n'}$. Note that $2n' - 2 - l + 1 \neq l$. By Lemmas 7 and 14, the pair $(r, 2n' - 2 - l)$ is equal to $(\lambda_1 - 1, \lambda_2 - 2)$ or $(\lambda_2 - 2, \lambda_1 - 1)$, and hence

$$\begin{aligned} \lambda'_1 + \lambda'_2 &= (\lambda_1 + \lambda_2) - (r - l + 1) = (\lambda_1 + \lambda_2) - (2 - 2n' + \lambda_1 - 1 + \lambda_2 - 2 + 1) \\ &= (\lambda_1 + \lambda_2) - (2n' + \lambda_1 + \lambda_2) = 2n'. \end{aligned}$$

We next show the "only if" part. As above, assume that $\boldsymbol{d}(Y) \xrightarrow{l,r} \boldsymbol{d}(Y')$. If $\lambda'_1 + \lambda'_2 = 2n'$, then

$$d_{\lambda'_1-1}(Y') \searrow d_{\lambda'_1}(Y'),\ d_{2n'-\lambda'_1-2}(Y') \searrow d_{2n'-\lambda_1-1}(Y'),$$

and

$$d_{k-1}(Y') \nearrow d_k(Y')$$

for $-2 < k \leq 2n'$ with $k \neq \lambda'_1, 2n' - \lambda'_1 - 1$. By Lemma 7, we have $l = \lambda'_1$ or $l = 2n' - \lambda'_1 - 1$. If $l = \lambda'_1$, then $r \neq 2n' - \lambda'_1 - 2$ and hence $2n' - 2 - r \neq \lambda'_1$. Thus,

$$\boldsymbol{d}(Y')_{[2n'-2-r,2n'-2-l]} = \boldsymbol{d}(Y')_{[2n'-2-r,2n'-2-\lambda'_1]} \in \mathbb{D}_{2,2n'}.$$

If $l = 2n' - \lambda'_1 - 1$, then $r \neq \lambda'_1 - 1$ and hence $2n' - 2 - r \neq 2n' - 1 - \lambda'_1$. Thus,

$$\boldsymbol{d}(Y')_{[2n'-2-r,2n'-2-l]} = \boldsymbol{d}(Y')_{[2n'-2-r,\lambda'_1-1]} \in \mathbb{D}_{2,2n'}.$$

In both cases, we have $\boldsymbol{d}(Y')_{[2n'-2-r,2n'-2-l]} \in \mathbb{D}_{2,2n'}$, which implies that there exists a box $(i', j') \in Y'$ such that $\mathcal{A}_{\alpha_{2,n}}(h_Y(i,j)) = \mathcal{A}_{\alpha_{2,n}}(h_{Y'}(i',j'))$ (see Lemma 9).

Finally, let us show that $Y'' := Y' \setminus h_{Y'}(i', j')$ is equal to $(2n' - \lambda_2, 2n' - \lambda_1)$. By Lemma 7, we have

$$\boldsymbol{d}(Y) \xrightarrow{l,r} \boldsymbol{d}(Y') \xrightarrow{2n'-2-r,2n'-2-l} \boldsymbol{d}(Y''),$$

and

$$d_{l-1}(Y'') \searrow d_l(Y''),\ d_{2n'-2-r-1}(Y'') \searrow d_{2n'-2-r}(Y''),$$

and

$$d_k(Y'') \nearrow d_k(Y'')$$

for $-2 < k \leq 2n'$ with $k \neq l, 2n' - 2 - r$. As seen above, the pair $(r, 2n' - 2 - l)$ is equal to $(\lambda_1 - 1, \lambda_2 - 2)$ or $(\lambda_2 - 2, \lambda_1 - 1)$. If $(r, 2n' - 2 - l) = (\lambda_1 - 1, \lambda_2 - 2)$, then

$$l = 2n' - \lambda_2 > 2n' - 1 - \lambda_1 = 2n' - 2 - l.$$

Otherwise, if $(r, 2n' - 2 - l) = (\lambda_2 - 2, \lambda_1 - 1)$, then

$$l = 2n' - 1 - \lambda_1 < 2n' - \lambda_2 = 2n' - 2 - l.$$

In both cases, we get

$$d_{2n'-2-\lambda_1}(Y'') \searrow d_{2n'-1-\lambda_1}(Y''),\ d_{2n'-\lambda_2-1}(Y'') \searrow d_{2n'-\lambda_2}(Y''),$$

and

$$d_k(Y'') \nearrow d_k(Y'')$$

for $-2 < k \leq 2n'$ with $k \neq 2n' - 1 - \lambda_1$ and $k \neq 2n' - \lambda_2$. Hence we obtain $Y'' = (2n' - \lambda_2, 2n' - \lambda_1)$ by Lemma 14, as desired. □

For $Y \in \mathcal{F}(Y_{2,2n'})$, we set $OH(Y) := \{Y \setminus h_Y(i,j) \mid (i,j) \in Y\}$. If $Y = (\lambda_1, \lambda_2)$, then

$$\begin{aligned} OH(Y) &= \{(\lambda_1', \lambda_2) \mid \lambda_2 \leq \lambda_1' < \lambda_1\} \cup \{(\lambda_1, \lambda_2') \mid 0 \leq \lambda_2' < \lambda_2\} \\ &\cup \{(\lambda_2 - 1, \lambda_1') \mid 0 \leq \lambda_1' < \lambda_2\}. \end{aligned}$$

By Lemma 15, we can easily show the following lemma.

Lemma 16. *In MHRG$(2, 2n')$,*

$$\mathcal{T}(Y_{2,2n'}) = \mathcal{F}(Y_{2,2n'}) \setminus \{(\lambda_1', \lambda_2') \in \mathcal{F}(Y_{2,2n'}) \mid \lambda_1' + \lambda_2' = 2n'\}.$$

Moreover, for $Y = (\lambda_1, \lambda_2) \in \mathcal{F}(Y_{2,2n'})$,

(1) if $\lambda_1 + \lambda_2 < 2n'$, then $\mathcal{O}(Y) = OH(Y)$;

(2) if $\lambda_1 + \lambda_2 > 2n'$, then $\mathcal{O}(Y) = OH(Y) \setminus \{(\lambda_1', \lambda_2') \in \mathcal{F}(Y_{2,2n'}) \mid \lambda_1' + \lambda_2' = 2n'\} \cup \{(2n' - \lambda_2, 2n' - \lambda_1)\}$.

By Lemma 16, the $\mathcal{G}$-value of $Y = (\lambda_1, \lambda_2) \in \mathcal{T}(Y_{2,2n'})$ with $\lambda_1 + \lambda_2 < n = 2n'$ is equal to the $\mathcal{G}$-value of the game position corresponding to Y in the Sato–Welter game (see, e.g., [5, Theorem 2]). For later use, we list those $Y = (\lambda_1, \lambda_2) \in \mathcal{T}(Y_{2,2n'})$ with $\lambda_1 + \lambda_2 < 2n'$ whose $\mathcal{G}$-values are 0, 1, or 2.

$\mathcal{G}(Y) = 0$	$\mathcal{G}(Y) = 1$	$\mathcal{G}(Y) = 2$
$(2i, 2i)$	$(1 + 4i, 4i)$ $(2 + 4i, 1 + 4i)$	$(2 + 4i, 4i)$ $(1 + 4i, 1 + 4i)$

Table 3: $Y = (\lambda_1, \lambda_2) \in \mathcal{F}(Y_{2,2n'})$ with $\lambda_1 + \lambda_2 < 2n'$ whose $\mathcal{G}$-values are 0, 1, or 2.

Theorem 3. *As above, assume that n is even, and set $n' = n/2$. In MHRG$(2, 2n')$, the list of those $Y = (\lambda_1, \lambda_2) \in \mathcal{F}(Y_{2,2n'})$ with $\lambda_1 + \lambda_2 > 2n'$ whose $\mathcal{G}$-values are* 0,1 *or* 2 *is given by Table 4.*

Proof. We give a proof only for the case of $n' = 4n''$ for $n'' \in \mathbb{N}$; the proofs of the cases $n' = 4n'' + 1, 4n'' + 2, 4n'' + 3$ for $n'' \in \mathbb{N}_0$ are similar. We set

$$G_k := \{(\lambda_1, \lambda_2) \in \mathcal{T}(Y_{2,2n'}) \mid \lambda_1 + \lambda_2 > 2n', \mathcal{G}((\lambda_1, \lambda_2)) = k\}$$

for $k \in \mathbb{N}_0$.

First, we determine G_0. Let $Y = (\lambda_1, \lambda_2) \in \mathcal{T}(Y_{2,2n'})$ with $\lambda_1 + \lambda_2 > 2n'$. If $\lambda_2 < n'$ and λ_2 is even (resp. odd), then we deduce that $Y' = (\lambda_2, \lambda_2)$ (resp. $Y' = (\lambda_2 - 1, \lambda_2 - 1)$) is contained in $\mathcal{O}(Y)$. Since $\mathcal{G}(Y') = 0$ by Table 3, we obtain $Y \notin G_0$.

n'	$\mathcal{G}(Y)=0$	$\mathcal{G}(Y)=1$	$\mathcal{G}(Y)=2$
$4n''$	$(n'+1+4i, n'+4i)$ $(n'+2+4i, n'+1+4i)$	$(n'+2, n')$ $(n'+1, n'+1)$ $(n'+4+2i, n'+4+2i)$	$(n'+2, n'+2)$ $(n'+3, n')$ $(n'+4, n'+1)$ $(n'+7+4i, n'+6+4i)$ $(n'+8+4i, n'+7+4i)$
$4n''+1$	$(n'+2+4i, n'+1+4i)$ $(n'+3+4i, n'+2+4i)$	$(n'+2+2i, n'+2i)$	$(n'+1, n')$ $(n'+2, n'-1)$ $(n'+3, n'+1)$ $(n'+5+2i, n'+5+2i)$
$4n''+2$	$(n'+1+4i, n'+4i)$ $(n'+2+4i, n'+1+4i)$	$(n'+2+2i, n'+2+2i)$	$(n'+3+4i, n'+2+4i)$ $(n'+4+4i, n'+3+4i)$
$4n''+3$	$(n'+2+4i, n'+1+4i)$ $(n'+3+4i, n'+2+4i)$	$(n'+1+2i, n'+1+2i)$	$(n'+4+8i, n'+1+8i)$ $(n'+5+8i, n'+2+8i)$ $(n'+6+8i, n'+3+8i)$ $(n'+7+8i, n'+4+8i)$

Table 4: $Y=(\lambda_1,\lambda_2)\in\mathcal{F}(Y_{2,2n'})$ with $\lambda_1+\lambda_2>2n'$ whose $\mathcal{G}$-values are 0, 1, or 2.

Now, we see by Lemma 16 that

$$\begin{aligned}\mathcal{O}((n'+1,n')) &= \Big(\{(n',n')\}\cup\{(n'+1,\lambda_2')\mid 0\le\lambda_2'<n'\}\\ &\qquad\cup\{(n'-1,\lambda_1')\mid 0\le\lambda_1'<n'\}\Big)\\ &\qquad\setminus\{(\lambda_1',\lambda_2')\mid\lambda_1'+\lambda_2'=2n'\}\cup\{(n',n'-1)\}\\ &=\{(n'+1,\lambda_2')\mid 0\le\lambda_2'<n'-1\}\\ &\qquad\cup\{(n'-1,\lambda_1')\mid 0\le\lambda_1'<n'\}\cup\{(n',n'-1)\}.\end{aligned}$$

Note that $n'=4n''$ is even. By Table 3 and the argument above, it can be seen that $\mathcal{O}((n'+1,n'))$ has no position whose $\mathcal{G}$-value is 0. Thus we get $\mathcal{G}((n'+1,n'))=0$. If

$$\begin{aligned}Y\in\{(n'+1,n'+1)\}&\cup\{(\lambda_1',n')\mid n'+2\le\lambda_1'\le 2n'\}\\ &\cup\{(\lambda_1',n'+2)\mid n'+2\le\lambda_1'\le 2n'\},\end{aligned}$$

then $(n'+1,n')\in\mathcal{O}(Y)$, which implies that $Y\notin G_0$.

Similarly, we see by Lemma 16 that

$$\begin{aligned}\mathcal{O}((n'+2,n'+1)) &= \Big(\{(n'+1,n'+1)\}\cup\{(n'+2,\lambda_2')\mid 0\le\lambda_2'<n'+1\}\\ &\qquad\cup\{(n',\lambda_1')\mid 0\le\lambda_1'<n'+1\}\Big)\\ &\qquad\setminus\{(\lambda_1',\lambda_2')\mid\lambda_1'+\lambda_2'=2n'\}\cup\{(n'-1,n'-2)\}\\ &=\{(n'+1,n'+1)\}\\ &\qquad\cup\Big(\{(n'+2,\lambda_2')\mid 0\le\lambda_2'<n'+1\}\setminus\{(n'+2,n'-2)\}\Big)\\ &\qquad\cup\{(n',\lambda_1')\mid 0\le\lambda_1'<n'\}\cup\{(n'-1,n'-2)\}.\end{aligned}$$

By Table 3 and the argument above, we deduce that $\mathcal{O}((n'+2, n'+1))$ has no position whose $\mathcal{G}$-value is 0. Thus we obtain $\mathcal{G}((n'+2, n'+1)) = 0$. If

$$Y \in \{(n'+2, n'+2)\} \cup \{(\lambda_1', n'+1) \mid n'+3 \le \lambda_1' \le 2n'\} \\ \cup \{(\lambda_1', n'+3) \mid n'+3 \le \lambda_1' \le 2n'\},$$

then $(n'+2, n'+1) \in \mathcal{O}(Y)$, which implies that $Y \notin G_0$. Therefore, for $Y = (\lambda_1, \lambda_2) \in \mathcal{F}(Y_{2,2n'})$ with $n' \le \lambda_2 \le n'+3$ and $\lambda_2 \le \lambda_1 \le 2n'$,

$$Y \in G_0 \text{ if and only if } Y = [(n'+1, n'), (n'+2, n'+1)]. \tag{6.1}$$

Let $i \in \mathbb{N}$ with $n'+4+4i \le 2n'$. By Lemma 16,

$$(n'+4+4i, n'+4+4i) \to (n'-4-4i, n'-4-4i).$$

Since

$$\mathcal{G}((n'-4-4i, n'-4-4i)) = \mathcal{G}((4n''-4-4i, 4n''-4-4i)) = 0$$

by Table 3, we obtain $\mathcal{G}((n'+4+4i, n'+4+4i)) \neq 0$. Furthermore, in the same way that (6.1) was obtained, it can be verified that for $Y = (\lambda_1, \lambda_2) \in \mathcal{F}(Y_{2,2n'})$ with $n'+4i \le \lambda_2 \le n'+3+4i$ and $\lambda_2 \le \lambda_1 \le 2n'$, $Y \in G_0$ if and only if $Y = [(n'+1+4i, n'+4i), (n'+2+4i, n'+1+4i)]$. Therefore, we obtain

$$G_0 = \Big(\{(n'+1+4i, n'+4i) \mid i \ge 0\} \cup \{(n'+2+4i, n'+1+4i) \mid i \ge 0\}\Big) \\ \cap \mathcal{F}(Y_{2,2n'}),$$

as desired.

Next, we determine G_1. Let $Y = (\lambda_1, \lambda_2) \in \mathcal{T}(Y_{2,2n'})$ with $\lambda_1 + \lambda_2 > 2n'$. Similar to the determination of G_0, if $\lambda_2 < n'$, then $Y \notin G_1$. By Table 3 and $\mathcal{G}((n'+1, n')) = 0$, we deduce that $\mathcal{O}((n'+2, n'))$ and $\mathcal{O}((n'+1, n'+1))$ have no position whose $\mathcal{G}$-value is 1, but we have a position $(n'+1, n')$, whose $\mathcal{G}$-value is 0. Thus we get

$$\mathcal{G}((n'+2, n')) = \mathcal{G}((n'+1, n'+1)) = 1.$$

If

$$Y \in \{(\lambda_1', n') \mid n'+2 \le \lambda_1' \le 2n'\} \cup \{(\lambda_1', n'+1) \mid n'+1 \le \lambda_1' \le 2n'\} \\ \cup \{(\lambda_1', n'+2) \mid n'+1 \le \lambda_1' \le 2n'\} \cup \{(\lambda_1', n'+3) \mid n'+2 \le \lambda_1' \le 2n'\},$$

then $(n'+2, n') \in \mathcal{O}(Y)$ or $(n'+1, n'+1) \in \mathcal{O}(Y)$, which implies that $Y \notin G_1$. Therefore, for $Y = (\lambda_1, \lambda_2) \in \mathcal{F}(Y_{2,2n'})$ with $n' \le \lambda_2 \le n'+3$ and $\lambda_2 \le \lambda_1 \le 2n'$, $Y \in G_1$ if and only if $Y = [(n'+2, n'), (n'+1, n'+1)]$.

We see by Lemma 16 that

$$\begin{aligned}\mathcal{O}((n'+4,n'+4)) = \Big(&\{(n'+4,\lambda_2') \mid 0 \le \lambda_2' < n'+3\}\\ &\cup \{(n'+3,\lambda_1') \mid 0 \le \lambda_1' < n'+3\}\Big)\\ &\setminus \{(\lambda_1',\lambda_2') \mid \lambda_1'+\lambda_2' = 2n'\} \cup \{(n'-4,n'-4)\}.\end{aligned}$$

By Table 3 and the argument above, we deduce that $\mathcal{O}((n'+2,n'+1))$ has no position whose $\mathcal{G}$-value is 1, but we have a position $(n'-4,n'-4)$, whose $\mathcal{G}$-value is 0. Thus we get $\mathcal{G}((n'+4,n'+4)) = 1$. If

$$\begin{aligned}Y \in &\{(\lambda_1',n'+4) \mid n'+5 \le \lambda_1' \le 2n'\}\\ &\cup \{(\lambda_1',n'+5) \mid n'+5 \le \lambda_1' \le 2n'\},\end{aligned}$$

then $(n'+4,n'+4) \in \mathcal{O}(Y)$, which implies that $Y \notin G_1$. Therefore, for $Y = (\lambda_1,\lambda_2) \in \mathcal{F}(Y_{2,2n'})$ with $n'+4 \le \lambda_2 \le n'+5$ and $\lambda_2 \le \lambda_1 \le 2n'$, $Y \in G_1$ if and only if $Y = (n'+4,n'+4)$. Similarly, for each $i \in \mathbb{N}$ (with $n'+4+2i \le 2n'$), it can be verified that for $Y = (\lambda_1,\lambda_2) \in \mathcal{F}(Y_{2,2n'})$ with $n'+4+2i \le \lambda_2 \le n'+5+2i$ and $\lambda_2 \le \lambda_1 \le 2n'$, $Y \in G_1$ if and only if $Y = (n'+4+2i,n'+4+2i)$. Therefore, we obtain

$$\begin{aligned}G_1 = &\Big(\{(n'+2,n'),(n'+1,n'+1)\} \cup \{(n'+4+2i,n'+4+2i) \mid i \ge 0\}\Big)\\ &\cap \mathcal{F}(Y_{2,2n'})\end{aligned}$$

as desired.

Finally, we determine G_2. Let $Y = (\lambda_1,\lambda_2) \in \mathcal{T}(Y_{2,2n'})$ with $\lambda_1+\lambda_2 > 2n'$. Similar to G_0 and G_1, we determine G_2 as follows.

- If $\lambda_2 < n'$, then $Y \notin G_2$.
- If $n' \le \lambda_2 \le n'+5$ and $\lambda_2 \le \lambda_1 \le 2n'$, then $Y \in G_1$ if and only if $Y = [(n'+2,n'+2),(n'+3,n'),(n'+4,n'+1)]$.
- For each $i \in \mathbb{N}_0$ (with $n'+6+4i \le 2n'$), if $n'+6+4i \le \lambda_2 \le n'+9+4i$ and $\lambda_2 \le \lambda_1 \le 2n'$, then $Y \in G_0$ if and only if $Y = [(n'+7+4i,n'+6+4i),(n'+8+4i,n'+7+4i)]$.

Therefore, we obtain

$$\begin{aligned}G_2 = \Big(&\{(n'+2,n'+2),(n'+3,n'),(n'+4,n'+1)\}\\ &\cup \{(n'+7+4i,n'+6+4i) \mid i \ge 0\}\\ &\cup \{(n'+8+4i,n'+7+4i) \mid i \ge 0\}\Big) \cap \mathcal{F}(Y_{2,2n'}),\end{aligned}$$

as desired. This completes the proof of Theorem 3. □

The following is an immediate consequence of Theorem 3, together with Theorem 1.

Corollary 1. *Let $n \geq 2$. In MHRG$(2, n)$, the $\mathcal{G}$-value of the starting position $Y_{2,n}$ is given as follows:*

$$\mathcal{G}(Y_{2,n}) = \begin{cases} 3 & \text{if } n = 2, 3, \\ 2 & \text{if } n \equiv 2, 3 \text{ mod } 8, \\ 1 & \text{otherwise.} \end{cases}$$

Proof. We can easily calculate the $\mathcal{G}$-value of the starting position in the cases that $n = 2, 3$. In the other case, we can prove it by Theorem 3 and Theorem 1. □

7. Relation between MHRG and HRG in Terms of Shifted Young Diagrams

7.1. Shifted Young Diagrams

Shifted Young diagrams are described as follows (see [7] for additional details). Let $m \in \mathbb{N}$, and let $\lambda_1, \ldots, \lambda_m \in \mathbb{N}$ be such that $\lambda_1 > \cdots > \lambda_m > 0$. The set $S = (\lambda_1, \ldots, \lambda_m) := \{(i, j) \in \mathbb{N}^2 \mid i \leq j,\ 1 \leq i \leq m,\ 1 \leq j \leq \lambda_i\}$ is called the *shifted Young diagram* corresponding to $(\lambda_1, \ldots, \lambda_m)$. An element of the shifted Young diagram is called a *box*, and the shifted Young diagram is described in terms of boxes as follows.

$$S = (7,6,4,3,2) = \begin{array}{|c|c|c|c|c|c|c|} \hline (1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6) & (1,7) \\ \hline & (2,2) & (2,3) & (2,4) & (2,5) & (2,6) & (2,7) \\ \hline & & (3,3) & (3,4) & (3,5) & (3,6) & \\ \hline & & & (4,4) & (4,5) & (4,6) & \\ \hline & & & & (5,5) & (5,6) & \\ \hline \end{array}$$

For $i \in \mathbb{N}$, the subset $\{(i, j) \mid j \in \mathbb{N}\} \cap S$ of S is called the *i-th row* of S. Similarly, for $j \in \mathbb{N}$, the subset $\{(i, j) \mid i \in \mathbb{N}\} \cap S$ of S is called the *j-th column* of S. We call $h(S) := \max\{i \mid (i, j) \in S\}$ the height of S.

For a shifted Young diagram S, let $\mathcal{F}(S)$ denote the set of all shifted Young diagrams contained in S.

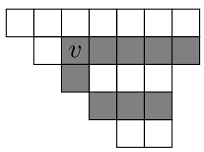
7.2. Hooks of a Shifted Young Diagram

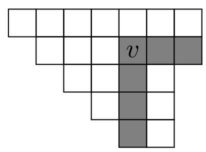
Definition 10. For a box (i,j) of a shifted Young diagram S, we define

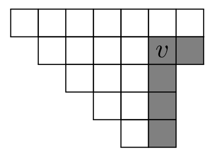
$$\begin{aligned} \mathrm{arm}_S(i,j) &:= \{(i',j') \in S \mid i = i', j < j'\}, \\ \mathrm{leg}_S(i,j) &:= \{(i',j') \in S \mid i < i', j = j'\}, \\ \mathrm{tail}_S(i,j) &:= \{(i',j') \in S \mid j+1 = i', j < j'\}, \\ h_S(i,j) &:= \{(i,j)\} \sqcup \mathrm{arm}_S(i,j) \sqcup \mathrm{leg}_S(i,j) \sqcup \mathrm{tail}_S(i,j). \end{aligned}$$

The set $h_S(i,j)$ is called the *hook* corresponding to the box (i,j).

Example 6. In the figures below, the shaded boxes form the hook corresponding to the box $v = (i,j)$.

(a): 

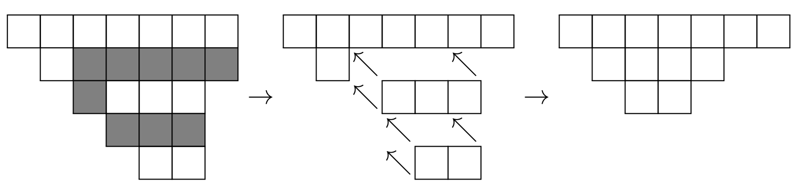
(b): 

(c): 

Definition 11. For a box (i,j) of a shifted Young diagram S, we remove the hook $h_S(i,j)$ corresponding to the box (i,j) as follows:

1. Remove all boxes in the hook $h_S(i,j)$.
2. Move each box (i',j') satisfying $j+1 > i' > i$ and $j' > j$ to $(i'-1, j'-1)$.
3. Move each box (i',j') satisfying $i' > j+1$ to $(i'-2, j'-2)$.

Example 7. If we remove the hook corresponding to the box $(2,3)$ from the shifted Young diagram $S = (7,6,4,3,2)$, then we get $S' = (7,4,2)$.



Definition 12. A *Hook Removing Game* (HRG for short) in terms of shifted Young diagrams is an impartial game. The rules of this game are as follows:

(HS1) Given a shifted Young diagram S, each player chooses a box $(i,j) \in S$, and remove the hook $h_S(i,j)$ corresponding to the box (i,j) from S on his/her turn.

(HS2) The player who makes the empty shifted Young diagram $\emptyset$ wins.

We denote HRG (in terms of shifted Young diagrams) whose starting position is a shifted Young diagram S by HRG(S). It is clear from the definition of HRG(S) that $\mathcal{F}(S)$ is identical to the set of all positions in HRG(S).

Proposition 3. *Let $S = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ be a shifted Young diagram, and let T be a shifted Young diagram containing S. The $\mathcal{G}$-value of S in HRG(T) is equal to*

$$\mathcal{G}(S) = \bigoplus_{1 \le i \le n} \lambda_i,$$

where $\bigoplus_i a_i$ denotes the nim-sum (the addition of numbers in binary form without carrying) of all a_i.

While this formula is (apparently) well-known by experts, will deduce it from the results of [5], or by the fact that HRG(S) is isomorphic to Turning Turtles (for Turning Turtles, see, e.g., [11, page 182]).

7.3. Diagonal Expression of a Shifted Young Diagram

We now describe the diagonal expression for shifted Young diagrams. Fix $n \in \mathbb{N}$. An element $\boldsymbol{b} \in \mathbb{N}_0^{n+1}$ is written as $\boldsymbol{b} = [b_0, \ldots, b_n]$. Also, we denote by $\mathbb{SD}_n \subset \mathbb{N}_0^{n+1}$ the set of all elements $\boldsymbol{b} = [b_0, \ldots, b_n] \in \mathbb{N}_0^{n+1}$ with $b_n = 0$ satisfying

$$0 \le b_k - b_{k+1} \le 1$$

for $0 \le k < n$.

Denote by S_n the shifted Young diagram corresponding to $(n, n-1, n-2, \ldots, 2, 1)$, and let $S \in \mathcal{F}(S_n)$. We set

$$d_k = d_k(S) := \#\{(i,j) \in S \mid j - i = k\}$$

for $k \in \mathbb{Z}$. Note that if $k \ge n$, then $d_k = 0$. As in Lemma 3, we deduce that

$$\boldsymbol{sd}(S) = \boldsymbol{sd}_n(S) := [d_0(S), \ldots, d_n(S)] \in \mathbb{SD}_n$$

for $S \in \mathcal{F}(S_n)$ and the fact that the map $\boldsymbol{sd} = \boldsymbol{sd}_n : \mathcal{F}(S_n) \to \mathbb{SD}_n$, $S \mapsto \boldsymbol{sd}(S)$ is bijective.

Definition 13. We call $\boldsymbol{sd}(S) = \boldsymbol{sd}_n(S)$ the diagonal expression of $S \in \mathcal{F}(S_n)$.

Let $\boldsymbol{b} = [b_0, \ldots, b_n] \in \mathbb{SD}_n$, $\boldsymbol{b'} = [b'_0, \ldots, b'_n] \in \mathbb{N}_0^{n+1}$, and $0 \le l \le r < n$. If

$$b'_k = \begin{cases} b_k - 1 & \text{if } l \le k \le r, \\ b_k & \text{otherwise,} \end{cases}$$

then we write $\boldsymbol{b} \xrightarrow{l,r} \boldsymbol{b'}$. If

$$b'_k = \begin{cases} b_k - 2 & \text{if } 0 \leq k \leq r' < r, \\ b_k - 1 & \text{if } r' < k \leq r, \\ b_k & \text{otherwise,} \end{cases}$$

then we write $\boldsymbol{b} \xrightarrow{0,r}\xrightarrow{0,r'} \boldsymbol{b'}$ (or $\boldsymbol{b} \xrightarrow{0,r'}\xrightarrow{0,r} \boldsymbol{b'}$). Otherwise, if

$$b'_k = \begin{cases} b_k - 2 & \text{if } 0 \leq k \leq r' < r, \\ b_k - 1 & \text{if } r' < k \leq r, \\ b_k & \text{otherwise,} \end{cases}$$

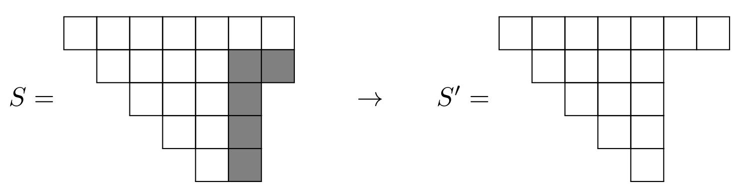
then $\boldsymbol{b'} \in \mathbb{SD}_n$.

Lemma 17. *Let $S, S' \in \mathcal{F}(S_n)$. The following are equivalent.*

(1) There exists a box $(i,j) \in S$ such that $S' = S \setminus h_S(i,j)$.

(2) There exists $0 \leq l \leq r < n$ such that $\boldsymbol{sd}(S) \xrightarrow{l,r} \boldsymbol{sd}(S')$ or we have $0 \leq r' < r < n$ such that $\boldsymbol{sd}(S) \xrightarrow{0,r}\xrightarrow{0,r'} \boldsymbol{sd}(S')$.

Let us explain the key point of a proof of the lemma by using some examples. Let $S \in \mathcal{F}(S_n)$, and write $\boldsymbol{sd}(S)$ as $\boldsymbol{sd}(S) = [d_0, \ldots, d_n]$ for $S \in \mathcal{F}(S_n)$. Let us consider (1) $\Longrightarrow$ (2). If $h(S) \leq j$, then the removed hook $h_s(i,j)$ is of the form either (b) or (c) in Example 6. Thus, there exist $0 \leq l \leq r < n$ such that $\boldsymbol{sd}(S) \xrightarrow{l,r} \boldsymbol{sd}(S')$. For example, let S be as in Example 6, and let $S' = S \setminus h_S(2,6)$. Note that the right-half of S is an (ordinary) Young diagram. Removing the hook $h_s(i,j)$ of this form from S naturally corresponds to removing a hook from the Young diagram (see [7, Chapter 4]).

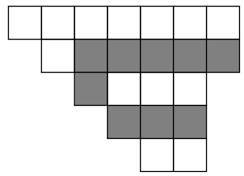
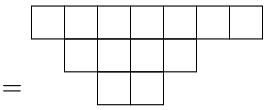


In the diagonal expression, we see that

$$\boldsymbol{sd}(S) = [5,5,4,3,2,2,1,0], \qquad \boldsymbol{sd}(S') = [5,4,3,2,1,1,1,0],$$

and hence $\boldsymbol{sd}(S) \xrightarrow{1,5} \boldsymbol{sd}(S')$.

If $j < h(S)$, then the removed hook is of the form (a) in Example 6. In this case, we deduce that $\boldsymbol{sd}(S) \xrightarrow{0,r}\xrightarrow{0,r'} \boldsymbol{sd}(S')$ for some $0 \leq r' < r < n$. For example, let S be as in Example 6, and let $S' = S \setminus h_S(2,3)$.

$S =$  $\rightarrow$ $S' =$ 

In the diagonal expression, we see that

$$\boldsymbol{sd}(S) = [5,5,4,3,2,2,1,0], \qquad \boldsymbol{sd}(S') = [3,3,2,2,1,1,1,0],$$

and hence $\boldsymbol{sd}(S) \xrightarrow{0,5}\xrightarrow{0,2} \boldsymbol{sd}(S')$.

The implication (2) $\Longrightarrow$ (1) can be verified as in Lemma 4.

Definition 14. A sequence $(a_{-m}, \ldots, a_n) \in \mathbb{D}_{m,n}$ is said to be *symmetric* if $a_i = a_{n-m-i}$ for all $-m \leq i \leq n$.

Lemma 18.

(1) Let $Y \in \mathcal{F}(Y_{n,n})$. The sequence $\boldsymbol{d}(Y) \in \mathbb{D}_{n,n}$ is symmetric if and only if $Y \in \mathcal{T}(Y_{n,n})$.

(2) Let $Y \in \mathcal{F}(Y_{n,n+1})$. The sequence $\boldsymbol{d}(Y) \in \mathbb{D}_{n,n+1}$ is symmetric if and only if $Y \in \mathcal{T}(Y_{n,n+1})$.

Proof. By Theorem 1, we need only to show part (1) since it is clear that for $Y \in \mathcal{T}(Y_{n,n})$, $\boldsymbol{d}(Y)$ is symmetric if and only if $\boldsymbol{d}(E(Y))$ is symmetric. We show by induction on $\#Y$ that if $Y \in \mathcal{T}(Y_{n,n})$, then $\boldsymbol{d}(Y) = (d_{-n}(Y), \ldots, d_n(Y)) \in \mathbb{D}_{n,n}$ is symmetric. If $Y = Y_{n,n}$, then

$$\boldsymbol{d}(Y_{n,n}) = (0, 1, \ldots, n-1, n, n-1, \ldots, 1, 0)$$

is symmetric. Assume that $Y \neq Y_{n,n}$. Then there exists $\hat{Y} \in \mathcal{T}(Y_{n,n})$ such that $\hat{Y} \to Y$. Note that $d(\hat{Y}) = (d_{-n}(\hat{Y}), \ldots, d_n(\hat{Y}))$ is symmetric by the induction hypothesis, and $d_{k-1}(\hat{Y}) \nearrow d_k(\hat{Y})$ if and only if $d_{-k}(\hat{Y}) \searrow d_{-k+1}(\hat{Y})$ for $-n < k \leq n$. Then,

(i) there exist $-n < l \leq r < n$ such that $\boldsymbol{d}(\hat{Y}) \xrightarrow{l,r} \boldsymbol{d}(Y)$, or

(ii) there exist $-n < l \leq r < n$ such that $\boldsymbol{d}(\hat{Y}) \xrightarrow{l,r} \boldsymbol{d}(\hat{Y}') \xrightarrow{l'=-r, r'=-l} \boldsymbol{d}(Y)$.

Let us consider case (i). Suppose that $l \neq -r$. Note that $d_{l-1}(\hat{Y}) \nearrow d_l(\hat{Y})$, $d_{-l}(\hat{Y}) \searrow d_{-l+1}(\hat{Y})$, $d_r(\hat{Y}) \searrow d_{r+1}(\hat{Y})$, and $d_{-r-1}(\hat{Y}) \nearrow d_{-r}(\hat{Y})$. By Lemma 7, we have

$$d_{-r-1}(Y) \nearrow d_{-r}(Y) \text{ and } d_{-l}(Y) \searrow d_{-l+1}(Y).$$

Thus $\boldsymbol{d}(Y)_{[-r,-l]} \in \mathbb{D}_{n,n}$ by Lemma 9, but this is a contradiction. Hence we deduce that $l = -r$. Then we have $\boldsymbol{d}(\hat{Y}) \xrightarrow{-r,r} \boldsymbol{d}(Y)$. In this case, it is obvious that $\boldsymbol{d}(Y) \in \mathbb{D}_{n,n}$ is symmetric.

Let us consider case (ii). We will show that $d_k(Y) = d_{-k}(Y)$ for any $-n < k < n$. Assume that $l \leq k \leq r$ and $-r \leq k \leq -l$. In this case, we have

$$d_k(\hat{Y}) = d_k(\hat{Y}') + 1 = d_k(Y) + 2.$$

Since $l \leq -k \leq r$ and $-r \leq -k \leq -l$, we have

$$d_{-k}(\hat{Y}) = d_{-k}(\hat{Y}') + 1 = d_{-k}(Y) + 2.$$

Thus we have

$$d_k(Y) = d_k(\hat{Y}) - 2 = d_{-k}(\hat{Y}) - 2 = d_{-k}(Y).$$

The proofs for the other cases are similar. Hence $\boldsymbol{d}(Y) \in \mathbb{D}_{n,n}$ is symmetric.

Next, we show that if $\boldsymbol{d}(Y) = (d_{-n}(Y), \ldots, d_n(Y)) \in \mathbb{D}_{n,n}$ is symmetric, then $Y \in \mathcal{T}(Y_{n,n})$. Let

$$\mathbb{A} := \{0 \leq i \leq n-1 \mid d_i = d_{i+1} + 1\}$$

and write it as $\mathbb{A} = \{i_1, i_2, \ldots, i_k\}$. Then there exists a transition

$$Y_{n,n} = Y_0 \to Y_1 \to Y_2 \to \cdots \to Y_{k-1} \to Y_k = Y$$

such that $\boldsymbol{d}(Y_{l-1}) \xrightarrow{-i_l,i_l} \boldsymbol{d}(Y_l)$ for $1 \leq l \leq k$. Thus we obtain $Y \in \mathcal{T}(Y_{n,n})$, as desired. □

Let $\boldsymbol{a} = (a_{-n}, a_{n-1}, \ldots, a_{-1}, \dot{a_0}, a_1, \ldots, a_n, a_{n+1}) \in \mathbb{D}_{n,n+1}$. Assume that

$$\hat{\boldsymbol{a}} := [a_1, a_2, \ldots, a_n, a_{n+1}] \in \mathbb{N}_0^{n+1}.$$

By the definition of $\mathbb{D}_{n,n+1}$, we thus have $\hat{\boldsymbol{a}} \in \mathbb{SD}_n$.

Definition 15. The map $A : \mathcal{T}(Y_{n,n+1}) \to \mathcal{F}(S_n)$ is defined as follows. If the diagonal expression of $Y \in \mathcal{T}(Y_{n,n+1})$ is

$$\boldsymbol{d}(Y) = (a_{-n}, a_{n-1}, \ldots, a_{-1}, \dot{a_0}, a_1, \ldots, a_n, a_{n+1}),$$

then we define $A(Y) \in \mathcal{F}(S_n)$ to be the shifted Young diagram in $\mathcal{F}(S_n)$ whose diagonal expression is equal to

$$\boldsymbol{sd}(A(Y)) = [a_1, a_2, \ldots, a_n, a_{n+1}].$$

Lemma 19. *Let $Y \in \mathcal{T}(Y_{n,n+1})$, and let $Y' \in \mathcal{O}(Y)$. Also, set $S := A(Y) \in \mathcal{F}(S_n)$. Then there exists $S' \in \mathcal{O}(S)$ such that $A(Y') = S'$.*

Proof. Since $Y' \in \mathcal{O}(Y)$, we see that

(i) there exist $-n < l \le r < n+1$ such that $\boldsymbol{d}(Y) \xrightarrow{l,r} \boldsymbol{d}(Y')$, or

(ii) there exist $-n < l \le r < n+1$ and $Y'' \in \mathcal{F}(Y_{n,n+1})$ such that $\boldsymbol{d}(Y) \xrightarrow{l,r} \boldsymbol{d}(Y'') \xrightarrow{-r+1,-l+1} \boldsymbol{d}(Y')$.

First, we consider case (i). By the proof of Lemma 18, we see that $l = -r+1$ and hence $\boldsymbol{d}(Y) \xrightarrow{-r+1,r} \boldsymbol{d}(Y')$. In this case, we have $d_{r-1}(S) = d_r(S)+1$. Let $S' \in \mathcal{F}(S_n)$ be such that $\boldsymbol{sd}(S) \xrightarrow{0,r-1} \boldsymbol{sd}(S')$. Then we deduce that $A(Y') = S'$.

Next, we consider case (ii). By the proof of Lemma 18, we see that $l \neq -r+1$ and hence $\boldsymbol{d}(Y)_{[l,r]}, (\boldsymbol{d}(Y)_{[l,r]})_{[-r+1,-l+1]} \in \mathbb{D}_{n,n+1}$.

Assume that $0 \le l \le r$. In this case, we have $d_{l-2}(S) = d_{l-1}(S)$ and $d_{r-1}(S) = d_r(S)+1$. Let $S' \in \mathcal{F}(S_n)$ be such that $\boldsymbol{sd}(S) \xrightarrow{l-1,r-1} \boldsymbol{sd}(S')$. Then we deduce that $A(Y') = S'$.

Assume that $l \le r \le 0$. In this case, we have $d_{-r-1}(S) = d_{-r}(S)$ and $d_{-l}(S) = d_{-l+1}(S)+1$. Let $S' \in \mathcal{F}(S_n)$ be such that $\boldsymbol{sd}(S) \xrightarrow{-r,-l} \boldsymbol{sd}(S')$. Then we deduce that $A(Y') = S'$.

Assume that $l \le 0 < r$. In this case, we have $d_{r-1}(S) = d_r(S)+1$ and $d_{-l}(S) = d_{-l+1}(S)+1$. Let $S' \in \mathcal{F}(S_n)$ be such that $\boldsymbol{sd}(S) \xrightarrow{0,-l} \xrightarrow{0,r-1} \boldsymbol{sd}(S')$. Then we deduce that $A(Y') = S'$.

Thus we have proved the lemma. □

Let $\boldsymbol{b} = [b_0, b_1, \dots, b_{n-1}, b_n] \in \mathbb{SD}_n$. Assume that

$$\hat{\boldsymbol{b}} := (b_{-n}, b_{n-1}, \dots, b_{-1}, \dot{b_0}, b_0, b_1, \dots, b_{n-1}, b_n) \in \mathbb{N}_0^{2n+2}.$$

By the definition of $\mathbb{SD}_n$, we have $\hat{\boldsymbol{b}} \in \mathbb{D}_{n,n+1}$.

Definition 16. The map $B : \mathcal{F}(S_n) \to \mathcal{T}(Y_{n,n+1})$ is defined as follows. If the diagonal expression of $Y \in \mathcal{F}(S_n)$ is

$$\boldsymbol{sd}(S) = [a_0, a_1, \dots, a_{n-1}, a_n].$$

then we define $B(S) \in \mathcal{T}(Y_{n,n+1})$ to be the rectangular Young diagram in $\mathcal{T}(Y_{n,n+1})$ whose diagonal expression is equal to

$$\boldsymbol{d}(B(S)) = (a_n, a_{n-1}, \dots, \dot{a_0}, \underbrace{a_0}_{\text{1st}}, a_1, \dots, a_{n-1}, \underbrace{a_n}_{(n+1)\text{-th}}).$$

Lemma 20. *Let $S \in \mathcal{F}(S_n)$, and let $S' \in \mathcal{O}(S)$. Also, set $Y := B(S) \in \mathcal{T}(Y_{n,n+1})$. Then there exists $Y' \in \mathcal{O}(Y)$ such that $B(S') = Y'$.*

Proof. Since $S' \in \mathcal{O}(S)$, we see that

(i) there exist $0 \le l \le r < n$ such that $\boldsymbol{sd}(S) \xrightarrow{l,r} \boldsymbol{sd}(S')$, or

(ii) there exist $0 \leq r' < r < n$ such that $\boldsymbol{sd}(S) \xrightarrow{0,r} \xrightarrow{0,r'} \boldsymbol{sd}(S')$.

First, we consider case (i). Assume that $l = 0$. In this case, $d_r(S) = d_{r+1}(S) + 1$. Then, we have

$$d_{r+1}(B(S)) = d_{r+2}(B(S)) + 1, d_{-r-1}(B(S)) + 1 = d_{-r}(B(S)),$$

and hence

$$\boldsymbol{d}(B(S))_{[-r,r+1]} \in \mathbb{D}_{n,n+1}$$

by Lemma 7. Let $Y' \in \mathcal{O}(Y)$ be such that $\boldsymbol{d}(Y) \xrightarrow{-r,r+1} \boldsymbol{d}(Y')$. Then we deduce that $B(S') = Y'$. Assume that $0 < l \leq r$. In this case, $d_{l-1}(S) = d_l(S)$ and $d_r(S) = d_{r+1}(S) + 1$. Then, we have

$$d_l(B(S)) = d_{l+1}(B(S)), d_{-l}(B(S)) = d_{-l+1}(B(S)),$$
$$d_{r+1}(B(S)) = d_{r+2}(B(S)) + 1, d_{-r-1}(B(S)) + 1 = d_{-r}(B(S)),$$

and hence

$$\boldsymbol{d}(B(S))_{[l+1,r+1]}, (\boldsymbol{d}(B(S))_{[l+1,r+1]})_{[-r,-l]} \in \mathbb{D}_{n,n+1}$$

by Lemma 7. Let $Y' \in \mathcal{O}(Y)$ be such that $\boldsymbol{d}(Y) \xrightarrow{l+1,r+1} \boldsymbol{d}(Y'') \xrightarrow{-r,-l} \boldsymbol{d}(Y')$. Then we deduce that $B(S') = Y'$.

Next, we consider case (ii). In this case, $d_r(S) = d_{r+1}(S) + 1$ and $d_{r'}(S) = d_{r'+1}(S) + 1$. Then, we have

$$d_{r+1}(B(S)) = d_{r+2}(B(S)) + 1, d_{-r-1}(B(S)) + 1 = d_{-r}(B(S)),$$
$$d_{r'+1}(B(S)) = d_{r'+2}(B(S)) + 1, d_{-r'-1}(B(S)) + 1 = d_{-r'}(B(S)),$$

and hence, by Lemma 7, we have

$$\boldsymbol{d}(B(S))_{[-r',r+1]}, (\boldsymbol{d}(B(S))_{[-r',r+1]})_{[-r,r'+1]} \in \mathbb{D}_{n,n+1}.$$

Let $Y' \in \mathcal{O}(Y)$ be such that $\boldsymbol{d}(Y) \xrightarrow{-r',r+1} \boldsymbol{d}(Y'') \xrightarrow{-r,r'+1} \boldsymbol{d}(Y')$. This implies that $B(S') = Y'$.

Thus we have proved the lemma. □

The next theorem follows from Lemmas 19 and 20.

Theorem 4. *For $n \in \mathbb{N}$, MHRG$(n, n+1)$ and HRG(S_n) are isomorphic. In particular, $\mathcal{G}(Y_{n,n+1})$ in MHRG$(n, n+1)$ is equal to $\mathcal{G}(S_n)$ in HRG(S_n).*

Combining Proposition 3, Theorem 1, and Theorem 4, we obtain the following corollary.

Corollary 2. *In MHRG(n,n) (resp. MHRG$(n,n+1)$), the $\mathcal{G}$-value of the starting position $Y_{n,n}$ (resp. $Y_{n,n+1}$) is equal to*

$$\mathcal{G}(Y_{n,n}) = \mathcal{G}(Y_{n,n+1}) = \bigoplus_{1\le k\le n} k.$$

Example 8. Assume that $n = 3$. The $\mathcal{G}$-value of $Y_{3,4} = \begin{array}{|c|c|c|c|}\hline 3&3&2&1\\\hline 2&3&3&2\\\hline 1&2&3&3\\\hline\end{array}$ is equal to $1 \oplus 2 \oplus 3 = 0$.

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