SOME NEW CONGRUENCES FOR ℓ-REGULAR CUBIC PARTITIONS WITH ODD PARTS OVERLINED

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Abstract

Naika and Harishkumar (2022) defined the restricted cubic partition function $\overline{a}_\ell(n)$, which counts the number of ℓ-regular cubic partitions of any positive integer n where the first occurrence of each distinct odd part may be overlined, and proved infinite families of congruences for the particular cases $\overline{a}_3(n)$ and $\overline{a}_5(n)$. In this paper, we extend and generalize some of the results of Naika and Harishkumar, and also prove new congruences of $\overline{a}_\ell(n)$ for $\ell = 2k, 2k+1, 4k, 4k+1, 3, 4, 5, 8, 9, 16$, where $\ell > 2$ and $k > 0$ are integers.

1. Introduction

For any complex numbers ω and q (with $|q| < 1$), define

$$(\omega; q)_\infty := \prod_{k=0}^{\infty} (1 - \omega q^k). \tag{1}$$

To be concise, for any positive integers u, we write

$$g_u := (q^u; q^u)_\infty.$$

A partition of a positive integer n is a sequence of positive integers $m_1 \geq m_2 \geq \cdots \geq m_k$ with $n = \sum_{i=1}^{k} m_i$. If $\lambda(n)$ denotes the number of partitions of a positive

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integer n, then the generating function of $\lambda(n)$ is given by

$$\sum_{n=0}^{\infty} \lambda(n) q^n = \frac{1}{g_1}.$$

For any integer $\ell \geq 2$, a partition of a positive integer n is said to be ℓ-regular if none of its parts are divisible by ℓ. If $\lambda_\ell(n)$ denotes the number of ℓ-regular partitions of n, then

$$\sum_{n=0}^{\infty} \lambda_\ell(n) q^n = \frac{g_\ell}{g_1}.$$

In [3], Chan studied the cubic partition of a positive integer n in which even parts appear in two colors. If $a(n)$ denotes the number of cubic partitions of a positive integer n, then the generating function of $a(n)$ [3] is given by

$$\sum_{n=0}^{\infty} a(n) q^n = \frac{1}{g_1 g_2}.$$

For example, the number of cubic partitions of $n = 4$ is 9, namely

$$4_r,\ 4_b,\ 3+1,\ 2_r+2_r,\ 2_b+2_b,\ 2_r+2_b,\ 2_r+1+1,\ 2_b+1+1,\ 1+1+1+1,$$

where suffixes r and b denote red and blue colors, respectively. For congruences and other arithmetic properties of $a(n)$ see [3, 4, 5, 6].

Recently, Naika and Harishkumar [10] defined the restricted cubic partition function $\overline{a}_\ell(n)$, which counts the number of ℓ-regular cubic partitions of n where the first occurrence of each distinct odd part may be overlined. For example, $\overline{a}_3(5) = 18$ with the relevant partitions $5,\ \overline{5},\ 4_r+1,\ 4_r+\overline{1},\ 4_b+1,\ 4_b+\overline{1},\ 2_r+2_r+1,\ 2_r+2_r+\overline{1},\ 2_b+2_b+1,\ 2_b+2_b+\overline{1},\ 2_r+2_b+1,\ 2_r+2_b+\overline{1},\ 2_r+1+1+1,\ 2_r+\overline{1}+1+1,\ 2_b+1+1+1,\ 2_b+\overline{1}+1+1,\ 1+1+1+1+1,\ \overline{1}+1+1+1+1$, where suffixes r and b denote red and blue colors, respectively. For any integer $\ell > 2$, the generating function for $\overline{a}_\ell(n)$ [10] is given by

$$\sum_{n=0}^{\infty} \overline{a}_\ell(n) q^n = \frac{g_2 g_\ell^2 g_{4\ell}}{g_1^2 g_4 g_{2\ell}}. \tag{2}$$

Naika and Harishkumar [10] proved many infinite families of congruences modulo powers of 2 and 3 for $\overline{a}_3(n)$, and modulo powers of 2 for $\overline{a}_5(n)$. In this paper, we extend and generalize some of the results of [10], and also establish some new congruences of $\overline{a}_\ell(n)$. In Section 3, we prove some new congruences of $\overline{a}_\ell(n)$ for $\ell = 2k, 2k+1, 4k, 4k+1, 3, 4, 5, 8, 9, 16$, where $\ell > 2$ and $k > 0$ are integers. To prove our congruences, we will employ some theta-function and q-series identities which are listed in Section 2.

2. Some Theta-Function and q-Series Identities

This section is devoted to record some q-series and theta-function identities. Ramanujan's general theta-function $\mathfrak{f}(x_1, x_2)$ [2, Page 34, Equation 18.1] is defined by

$$\mathfrak{f}(x_1, x_2) = \sum_{n=-\infty}^{\infty} x_1^{n(n+1)/2} x_2^{n(n-1)/2}, \ |x_1 x_2| < 1.$$

The three significant special cases of $\mathfrak{f}(x_1, x_2)$ are the theta-functions $\phi(q)$, $\psi(q)$, and $f(-q)$ [2, Page 36, Entry 22 (i), (ii), (iii)] defined by

$$\begin{aligned}\phi(q) &:= \mathfrak{f}(q, q) = \sum_{n=-\infty}^{\infty} q^{n^2} = \frac{g_2^5}{g_1^2 g_4^2},\\ \psi(q) &:= \mathfrak{f}(q, q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{g_2^2}{g_1},\\ f(-q) &:= \mathfrak{f}(-q, -q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n+1)/2} = g_1.\end{aligned} \tag{3}$$

One can use basic q-operations to show that

$$(-q; -q)_\infty = \frac{g_2^3}{g_1 g_4}. \tag{4}$$

Lemma 1 ([7, Theorem 2.2])**.** *For any prime $p \geq 5$, we have*

$$g_1 = \sum_{\substack{r=-(p-1)/2\\ r\neq(\pm p-1)/6}}^{(p-1)/2} (-1)^r q^{(3r^2+r)/2} \mathfrak{f}\left(-q^{(3p^2+(6r+1)p)/2}, -q^{(3p^2-(6r+1)p)/2}\right) + (-1)^{(\pm p-1)/6} q^{(p^2-1)/24} g_{p^2},$$

where

$$\frac{\pm p - 1}{6} := \begin{cases} \dfrac{(p-1)}{6}, & if\ p \equiv 1 \pmod 6,\\ \dfrac{(-p-1)}{6}, & if\ p \equiv -1 \pmod 6.\end{cases}$$

Furthermore, if

$$\frac{-(p-1)}{2} \leq r \leq \frac{(p-1)}{2} \ and \ r \neq \frac{(\pm p - 1)}{6},$$

then

$$\frac{3r^2 + r}{2} \not\equiv \frac{p^2 - 1}{24} \pmod p.$$

Lemma 2 ([7, Theorem 2.1])**.** *For any odd prime p, we have*

$$\psi(q)=\sum_{s=0}^{(p-3)/2} q^{(s^2+s)/2}\mathfrak{f}\left(q^{(p^2+(2s+1)p)/2},q^{(p^2-(2s+1)p)/2}\right)+q^{(p^2-1)/8}\psi(q^{p^2}).$$

Furthermore, $\dfrac{(s^2+s)}{2}\not\equiv\dfrac{(p^2-1)}{8} \pmod{p}$, *when* $\quad 0\le s\le\dfrac{(p-3)}{2}$.

Lemma 3. *We have*

$$\frac{1}{g_1^2}=\frac{g_8^5}{g_2^5g_{16}^2}+2q\frac{g_4^2g_{16}^2}{g_2^5g_8}, \tag{5}$$

$$\frac{1}{g_1^4}=\frac{g_4^{14}}{g_2^{14}g_8^4}+4q\frac{g_4^2g_8^4}{g_2^{10}}, \tag{6}$$

$$g_1^2=\frac{g_2g_8^5}{g_4^2g_{16}^2}-2q\frac{g_2g_{16}^2}{g_8}, \tag{7}$$

$$\frac{g_2^2}{g_1}=\frac{g_6g_9^2}{g_3g_{18}}+q\frac{g_{18}^2}{g_9}, \tag{8}$$

$$\frac{g_3}{g_1}=\frac{g_4g_6g_{16}g_{24}^2}{g_2^2g_8g_{12}g_{48}}+q\frac{g_6g_8^2g_{48}}{g_2^2g_{16}g_{24}}, \tag{9}$$

$$\frac{g_1^3}{g_3}=\frac{g_4^3}{g_{12}}-3q\frac{g_2^2g_{12}^3}{g_4g_6^2}, \tag{10}$$

$$\frac{g_3}{g_1^3}=\frac{g_4^6g_6^3}{g_2^9g_{12}^2}+3q\frac{g_4^2g_6g_{12}^2}{g_2^7}, \tag{11}$$

$$\frac{g_4}{g_1}=\frac{g_{12}g_{18}^4}{g_3^3g_{36}^2}+q\frac{g_6^2g_9^3g_{36}}{g_3^4g_{18}^2}+2q^2\frac{g_6g_{18}g_{36}}{g_3^3}, \tag{12}$$

$$\frac{g_5}{g_1}=\frac{g_8g_{20}^2}{g_2^2g_{40}}+q\frac{g_4^3g_{10}g_{40}}{g_2^3g_8g_{20}}, \tag{13}$$

$$\frac{g_1}{g_5}=\frac{g_2g_8g_{20}^3}{g_4g_{10}^3g_{40}}-q\frac{g_4^2g_{40}}{g_8g_{10}^2}, \tag{14}$$

$$\frac{g_9}{g_1}=\frac{g_{12}^3g_{18}}{g_2^2g_6g_{36}}+q\frac{g_4^2g_6g_{36}}{g_2^3g_{12}}. \tag{15}$$

Identities (5), (6), (8), and (10) follow from (1.9.4), (1.10.1), (14.3.3), and (22.1.13) of [9], respectively. Identity (13) is from [8]. Replacing q by $-q$ in (5), (10), and (13) and then employing (4), we arrive at (7), (11), and (14), respectively. Identity (9) is from [12], (12) is Lemma 2.6 of [1], and (15) is from [11].

The next lemma is a consequence of the binomial expansion and (1).

Lemma 4. *For any positive integers k and m, and a prime p, we have*

$$g_{pm}^{p^{k-1}}\equiv g_m^{p^k} \pmod{p^k}. \tag{16}$$

3. Congruences for $\overline{a}_\ell(n)$

This section is devoted to proving congruences for $\overline{a}_\ell(n)$. First, we define the Legendre symbol $\left(\frac{\alpha}{p}\right)$. Let p be any odd prime and α be any integer relatively prime to p. Then

$$\left(\frac{\alpha}{p}\right) = \begin{cases} 1, & \text{if } \alpha \text{ is a quadratic residue of } p, \\ -1, & \text{if } \alpha \text{ is a quadratic non-residue of } p. \end{cases}$$

Theorem 5. *We have the following statements.*

(*i*) *If $\ell = 2k$ and $k > 1$ are any integers, then*

$$\overline{a}_\ell(4n+3) \equiv 0 \pmod 4. \tag{17}$$

(*ii*) *If $\ell = 2k+1$ and $k \geq 1$ are any integers, then*

$$\overline{a}_\ell(4n+2) \equiv 0 \pmod 4. \tag{18}$$

(*iii*) *If $\ell = 4k$ and $k > 0$ are any integers, then*

$$\overline{a}_\ell(4n+2) \equiv 0 \pmod 4. \tag{19}$$

(*iv*) *If $\ell = 4k+1$ and $k > 0$ are any integers, then*

$$\overline{a}_\ell(4n+3) \equiv 0 \pmod 4. \tag{20}$$

Proof. Employing (5) and (7) in (2), we obtain

$$\sum_{n=0}^{\infty} \overline{a}_\ell(n) q^n \equiv \frac{g_8 g_{8\ell}}{g_4^3 g_{4\ell}} + 2q g_4^5 g_{4\ell} + 2q^\ell \frac{g_{4\ell}^7}{g_4} \pmod 4. \tag{21}$$

If $\ell = 2k$, where $k > 1$ is any integer, then collecting the terms involving powers of q that are congruent to 1 modulo 2 from both sides of (21) and simplifying the resulting equality yields

$$\sum_{n=0}^{\infty} \overline{a}_\ell(2n+1) q^n \equiv 2 g_2^5 g_{2\ell} \pmod 4. \tag{22}$$

Collecting the terms involving odd powers of q from (22), we arrive at (17).

If $\ell = 2k+1$, where $k \geq 1$ is any integer, then collecting the terms involving powers of q that are multiples of 2 from both sides of (21) and simplifying the resulting equality yields

$$\sum_{n=0}^{\infty} \overline{a}_\ell(2n) q^n \equiv \frac{g_4 g_{4\ell}}{g_2^3 g_{2\ell}} \pmod 4. \tag{23}$$

Collecting the terms involving odd powers of q from (23), we arrive at (18).

If $\ell = 4k$, where $k > 0$ is any integer, then collecting the terms involving powers of q that are multiples of 2 from both sides of (21) and simplifying the resulting equality yields

$$\sum_{n=0}^{\infty} \overline{a}_\ell(2n)q^n \equiv \frac{g_4 g_{4\ell}}{g_2^3 g_{2\ell}} + 2q^{2k}\frac{g_{2\ell}^7}{g_2} \pmod 4. \tag{24}$$

Collecting the terms involving odd powers of q from (24), we arrive at (19).

If $\ell = 4k+1$, where $k > 0$ is any integer, then collecting the terms involving powers of q that are congruent to 1 modulo 2 from both sides of (21) and simplifying the resulting equality yields

$$\sum_{n=0}^{\infty} \overline{a}_\ell(2n+1)q^n \equiv 2g_2^5 g_{2\ell} + 2q^{2k}\frac{g_{2\ell}^7}{g_2} \pmod 4. \tag{25}$$

Collecting the terms involving odd powers of q from (25), we arrive at (20). $\square$

Theorem 6. *Suppose $p \geq 5$ is any prime with $\left(\frac{-6}{p}\right) = -1$. Then for any integers $\delta \geq 0$, $n \geq 0$, and $1 \leq t \leq p-1$, we have*

$$\sum_{n=0}^{\infty} \overline{a}_3\left(8p^{2\delta}n + \frac{10p^{2\delta}-1}{3}\right)q^n \equiv 6g_4g_6 \pmod{12}, \tag{26}$$

$$\overline{a}_3\left(8p^{2\delta+2}n + 8p^{2\delta+1}t + \frac{10p^{2\delta+2}-1}{3}\right) \equiv 0 \pmod{12}. \tag{27}$$

Proof. Setting $\ell = 3$ in (2), we obtain

$$\sum_{n=0}^{\infty} \overline{a}_3(n)q^n = \frac{g_2 g_3^2 g_{12}}{g_1^2 g_4 g_6}. \tag{28}$$

Employing (9) in (28) and collecting the terms involving powers of q that are congruent to 1 modulo 2 from both sides of the resulting equation, we obtain

$$\sum_{n=0}^{\infty} \overline{a}_3(2n+1)q^n = 2\frac{g_3 g_4 g_{12}}{g_1^3}. \tag{29}$$

Utilizing (11) in (29), we obtain

$$\sum_{n=0}^{\infty} \overline{a}_3(2n+1)q^n = 2\frac{g_4^7 g_6^3}{g_2^9 g_{12}} + 6q\frac{g_4^3 g_6 g_{12}^3}{g_2^7}. \tag{30}$$

Collecting the terms involving powers of q that are congruent to 1 modulo 2 from both sides of (30) and simplifying the resulting equality yields

$$\sum_{n=0}^{\infty} \overline{a}_3(4n+3)q^n = 6\frac{g_2^3 g_3 g_6^3}{g_1^7}. \tag{31}$$

With the help of (16), Equation (31) can be written as

$$\sum_{n=0}^{\infty}\overline{a}_3(4n+3)q^n \equiv 6\frac{g_2g_3g_6^3}{g_1^3} \pmod{12}. \tag{32}$$

With the help of (11), Congruence (32) can be written as

$$\sum_{n=0}^{\infty}\overline{a}_3(4n+3)q^n \equiv 6\frac{g_4^6g_6^6}{g_2^8g_{12}^2}+6q\frac{g_4^2g_6^4g_{12}^2}{g_2^6} \pmod{12}. \tag{33}$$

Collecting the terms involving even powers of q from (33), we obtain

$$\sum_{n=0}^{\infty}\overline{a}_3(8n+3)q^n \equiv 6\frac{g_2^6g_3^6}{g_1^8g_6^2} \pmod{12}. \tag{34}$$

With the help of (16), Congruence (34) can be written as

$$\sum_{n=0}^{\infty}\overline{a}_3(8n+3)q^n \equiv 6g_4g_6 \pmod{12}, \tag{35}$$

which is the $\delta=0$ case of (26). Assume that (26) is true for $\delta\geq 0$. Employing Lemma 1 in (26), we obtain

$$\begin{aligned}
&\sum_{n=0}^{\infty}\overline{a}_3\left(8p^{2\delta}n+\frac{10p^{2\delta}-1}{3}\right)q^n \\
&\equiv 6\Bigg[\sum_{\substack{x=-(p-1)/2\\ x\neq(\pm p-1)/6}}^{(p-1)/2}(-1)^x q^{2(3x^2+x)}\mathfrak{f}\left(-q^{2(3p^2+(6x+1)p)},-q^{2(3p^2-(6x+1)p)}\right) \\
&\qquad\qquad +(-1)^{(\pm p-1)/6}q^{(p^2-1)/6}g_{4p^2}\Bigg] \\
&\quad\times\Bigg[\sum_{\substack{y=-(p-1)/2\\ y\neq(\pm p-1)/6}}^{(p-1)/2}(-1)^y q^{3(3y^2+y)}\mathfrak{f}\left(-q^{3(3p^2+(6y+1)p)},-q^{3(3p^2-(6y+1)p)}\right) \\
&\qquad\qquad +(-1)^{(\pm p-1)/6}q^{(p^2-1)/4}g_{6p^2}\Bigg] \pmod{12}.
\end{aligned} \tag{36}$$

Consider the congruence

$$2\left(3x^2+x\right)+3\left(3y^2+y\right)\equiv 5\left(\frac{p^2-1}{12}\right) \pmod{p},$$

which is identical to

$$(12x+2)^2+6\left(6y+1\right)^2\equiv 0 \pmod{p}. \tag{37}$$

Since $\left(\dfrac{-6}{p}\right) = -1$, the only solution of (37) is $x = y = (\pm p - 1)/6$. Therefore, collecting the terms involving $q^{pn+5(p^2-1)/12}$ from (36), dividing by $q^{5(p^2-1)/12}$, and replacing q by $q^{1/p}$, we obtain

$$\sum_{n=0}^{\infty} \overline{a}_3\left(8p^{2\delta+1}n + \frac{10p^{2\delta+2}-1}{3}\right)q^n \equiv 6g_{4p}g_{6p} \pmod{12}. \tag{38}$$

Collecting the terms involving q^{pn} from (38) and replacing q by $q^{1/p}$, we obtain

$$\sum_{n=0}^{\infty} \overline{a}_3\left(8p^{2\delta+2}n + \frac{10p^{2\delta+2}-1}{3}\right)q^n \equiv 6g_4g_6 \pmod{12},$$

which is the $\delta + 1$ case of (26). Thus, by employing induction, we complete the proof of (26). Collecting the terms involving q^{pn+t} for $1 \leq t \leq p-1$ from (38), we arrive at (27). □

Corollary 1. *We have*

$$\overline{a}_3(8n+5) \equiv 0 \pmod{18}, \tag{39}$$

$$\overline{a}_3(16n+11) \equiv 0 \pmod{12}, \tag{40}$$

$$\overline{a}_3(16n+15) \equiv 0 \pmod{12}. \tag{41}$$

Proof. Collecting the terms involving even powers of q from (30), we obtain

$$\sum_{n=0}^{\infty} \overline{a}_3(4n+1)q^n = 2\frac{g_2^7g_3^3}{g_1^9g_6}. \tag{42}$$

With the help of (16), Equation (42) can be written as

$$\sum_{n=0}^{\infty} \overline{a}_3(4n+1)q^n \equiv 2\frac{g_2^7}{g_6} \pmod{18}. \tag{43}$$

Collecting the terms involving powers of q that are congruent to 1 modulo 2 from both sides of (43), we arrive at (39). Collecting the terms involving odd powers of q from (35), we arrive at (40). Collecting the terms involving odd powers of q from (33), we obtain

$$\sum_{n=0}^{\infty} \overline{a}_3(8n+7)q^n \equiv 6\frac{g_2^2g_3^4g_6^2}{g_1^6} \pmod{12}. \tag{44}$$

With the help of (16), Congruence (44) can be written as

$$\sum_{n=0}^{\infty} \overline{a}_3(8n+7)q^n \equiv 6\frac{g_6^4}{g_2} \pmod{12}. \tag{45}$$

Collecting the terms involving odd powers of q from (45), we arrive at (41). □

Remark 1. Congruences (40) and (41) are extensions of the congruences $\overline{a}_3(16n+11) \equiv 0 \pmod 4$ and $\overline{a}_3(16n+15) \equiv 0 \pmod 4$ due to Naika and Harishkumar [10], respectively.

Theorem 7. *Suppose $p \geq 5$ is any prime with $\left(\frac{-3}{p}\right) = -1$. Then for any integers $\delta \geq 0$, $n \geq 0$, and $1 \leq t \leq p-1$, we have*

$$\sum_{n=0}^{\infty} \overline{a}_3\left(8p^{2\delta}n + \frac{4p^{2\delta}-1}{3}\right)q^n \equiv 2g_1g_3 \pmod 6, \tag{46}$$
$$\overline{a}_3\left(8p^{2\delta+2}n + 8p^{2\delta+1}t + \frac{4p^{2\delta+2}-1}{3}\right) \equiv 0 \pmod 6.$$

Proof. Collecting the terms involving powers of q that are multiples 2 from both sides of (43), we obtain

$$\sum_{n=0}^{\infty} \overline{a}_3(8n+1)q^n \equiv 2g_1^4 \pmod 6. \tag{47}$$

With the help of (16), Congruence (47) can be written as

$$\sum_{n=0}^{\infty} \overline{a}_3(8n+1)q^n \equiv 2g_1g_3 \pmod 6. \tag{48}$$

Congruence (48) is the $\delta = 0$ case of (46). As one can now proceed via the same argument used in our proof of Theorem 6, we omit the remaining details. □

Remark 2. Congruence (46) is a generalization of the following result of Naika and Harishkumar [10]: for all $n \geq 0$ and $\delta \geq 0$,

$$\sum_{n=0}^{\infty} \overline{a}_3\left(8 \cdot 5^{2\delta}n + \frac{4 \cdot 5^{2\delta}-1}{3}\right)q^n \equiv 2g_1g_3 \pmod 3.$$

Theorem 8. *Suppose $p \geq 5$ is any prime with $\left(\frac{-2}{p}\right) = -1$. Then for any integers $\delta \geq 0$, $n \geq 0$, and $1 \leq t \leq p-1$, we have*

$$\sum_{n=0}^{\infty} \overline{a}_4\left(4p^{2\delta}n + \frac{3p^{2\delta}-1}{2}\right)q^n \equiv 2g_1g_8 \pmod 4, \tag{49}$$
$$\overline{a}_4\left(4p^{2\delta+2}n + 4p^{2\delta+1}t + \frac{3p^{2\delta+2}-1}{2}\right) \equiv 0 \pmod 4.$$

Proof. Setting $\ell = 4$ in (2), we obtain

$$\sum_{n=0}^{\infty} \overline{a}_4(n)q^n = \frac{g_2g_4g_{16}}{g_1^2g_8}. \tag{50}$$

Employing (5) in (50), we obtain

$$\sum_{n=0}^{\infty} \overline{a}_4(n)q^n = \frac{g_4 g_8^4}{g_2^4 g_{16}} + 2q\frac{g_4^3 g_{16}^3}{g_2^4 g_8^2}. \tag{51}$$

Collecting the terms involving powers of q that are congruent to 1 modulo 2 from (51) and simplifying the resulting equality yields

$$\sum_{n=0}^{\infty} \overline{a}_4(2n+1)q^n = 2\frac{g_2^3 g_8^3}{g_1^4 g_4^2}. \tag{52}$$

Using (6) in (52), we obtain

$$\sum_{n=0}^{\infty} \overline{a}_4(2n+1)q^n = 2\frac{g_4^{12}}{g_2^{11} g_8} + 8q\frac{g_8^7}{g_2^7}. \tag{53}$$

Collecting the terms involving powers of q that are divisible by 2 from (53) and simplifying the resulting equality yields

$$\sum_{n=0}^{\infty} \overline{a}_4(4n+1)q^n = 2\frac{g_2^{12}}{g_1^{11} g_4}. \tag{54}$$

With the help of (16), Equation (54) can be written as

$$\sum_{n=0}^{\infty} \overline{a}_4(4n+1)q^n \equiv 2g_1 g_8 \pmod{4},$$

which is the $\delta = 0$ case of (49). As one can now proceed via the same argument used in our proof of Theorem 6, we omit the remaining details. □

Corollary 2. *We have*

$$\overline{a}_4(4n+3) \equiv 0 \pmod{8}, \tag{55}$$

$$\overline{a}_4(28n+4j+3) \equiv 0 \pmod{56}, \quad \textit{for} \quad j = 1,2,3,4,5,6, \tag{56}$$

$$\overline{a}_4(36n+19) \equiv 0 \pmod{24}. \tag{57}$$

Proof. Collecting the terms involving powers of q that are congruent to 1 modulo 2 from both sides of (53) and simplifying the resulting equality yields

$$\sum_{n=0}^{\infty} \overline{a}_4(4n+3)q^n = 8\frac{g_4^7}{g_1^7}. \tag{58}$$

Now (55) follows easily from (58). Using (16) with $\{p = 7, k = 1\}$ in (58), we obtain

$$\sum_{n=0}^{\infty} \overline{a}_4(4n+3)q^n \equiv 8\frac{g_{28}}{g_7} \pmod{56}. \tag{59}$$

Collecting the terms involving powers of q that are not divisible by 7 from (59), we arrive at (56). Using (16) with $\{p=3, k=1\}$ in (58), we obtain

$$\sum_{n=0}^{\infty} \overline{a}_4(4n+3)q^n \equiv 8\frac{g_4 g_{12}^2}{g_1 g_3^2} \pmod{24}. \tag{60}$$

Using (12) in (60) and collecting the terms involving powers of q that are congruent to 1 modulo 3 from both sides of the resulting equation, we obtain

$$\sum_{n=0}^{\infty} \overline{a}_4(12n+7)q^n \equiv 8\frac{g_2^2 g_3^3 g_4^2 g_{12}}{g_1^6 g_6^2} \pmod{24}. \tag{61}$$

Using (16) in (61), we obtain

$$\sum_{n=0}^{\infty} \overline{a}_4(12n+7)q^n \equiv 8\frac{g_3 g_4^2 g_{12}}{g_2 g_6} \pmod{24}. \tag{62}$$

Utilizing (8) in (62), we obtain

$$\sum_{n=0}^{\infty} \overline{a}_4(12n+7)q^n \equiv 8\frac{g_3 g_{12}^2 g_{18}^2}{g_6^2 g_{36}} + 8q^2 \frac{g_3 g_{12} g_{36}^2}{g_6 g_{18}} \pmod{24}. \tag{63}$$

Collecting the terms involving powers of q that are congruent to 1 modulo 3 from both sides of (63), we arrive at (57). □

Theorem 9. *Suppose $p \geq 5$ is any prime with $\left(\frac{-3}{p}\right) = -1$. Then for any integers $\delta \geq 0$, $n \geq 0$, and $1 \leq t \leq p-1$, we have*

$$\sum_{n=0}^{\infty} \overline{a}_5\left(10p^{2\delta}n + \frac{5p^{2\delta}-2}{3}\right)q^n \equiv 2(-1)^{\delta(\pm p-1)/6} g_1 \psi(q) \pmod{8}, \tag{64}$$

$$\overline{a}_5\left(10p^{2\delta+2}n + 10p^{2\delta+1}t + \frac{5p^{2\delta+2}-2}{3}\right) \equiv 0 \pmod{8}, \tag{65}$$

$$\sum_{n=0}^{\infty} \overline{a}_5\left(40p^{2\delta}n + \frac{65p^{2\delta}-2}{3}\right)q^n \equiv 4(-1)^{\delta(\pm p-1)/6} g_1 \psi(q^4) \pmod{8}, \tag{66}$$

$$\overline{a}_5\left(40p^{2\delta+2}n + 40p^{2\delta+1}t + \frac{65p^{2\delta+2}-2}{3}\right) \equiv 0 \pmod{8}. \tag{67}$$

Proof. Setting $\ell = 5$ in (2), we obtain

$$\sum_{n=0}^{\infty} \overline{a}_5(n)q^n = \frac{g_2 g_5^2 g_{20}}{g_1^2 g_4 g_{10}}. \tag{68}$$

Using (13) in (68) and collecting the terms involving powers of q that are congruent to 1 modulo 2 from both sides of the resulting equation, we obtain

$$\sum_{n=0}^{\infty} \overline{a}_5(2n+1)q^n = 2\frac{g_2^2 g_{10}^2}{g_1^4}. \tag{69}$$

With the help of (16), Equation (69) can be written as

$$\sum_{n=0}^{\infty} \overline{a}_5(2n+1)q^n \equiv 2g_{10}^2 \pmod{8}. \tag{70}$$

Collecting the terms involving powers of q that are multiples of 5 from both sides of (70) and simplifying the resulting equality yields

$$\sum_{n=0}^{\infty} \overline{a}_5(10n+1)q^n \equiv 2g_2^2 \equiv 2g_1 \cdot \frac{g_2^2}{g_1} \pmod{8}. \tag{71}$$

With the help of (3), Congruence (71) can be written as

$$\sum_{n=0}^{\infty} \overline{a}_5(10n+1)q^n \equiv 2g_1\psi(q) \pmod{8},$$

which is the $\delta = 0$ case of (64). The remaining proofs of Congruences (64) and (65) are similar to the proofs of Congruences (26) and (27), respectively, and the desired result can be obtained by appealing to Lemmas 1 and 2. As a result, we omit the details.

Collecting the terms involving powers of q that are multiples of 10 from both sides of (70) and simplifying the resulting equality yields

$$\sum_{n=0}^{\infty} \overline{a}_5(20n+1)q^n \equiv 2g_1^2 \pmod{8}. \tag{72}$$

Substituting (7) in (72) and then collecting the terms involving powers of q that are congruent to 1 modulo 2 from both sides of the resulting equation, we obtain

$$\sum_{n=0}^{\infty} \overline{a}_5(40n+21)q^n \equiv 4\frac{g_1 g_8^2}{g_4} \pmod{8}.$$

Then, using (3), we get

$$\sum_{n=0}^{\infty} \overline{a}_5(40n+21)q^n \equiv 4g_1\psi(q^4) \pmod{8},$$

which is the $\delta = 0$ case of (66). Now suppose that (66) holds for some $\delta \geq 0$. Employing Lemmas 1 and 2 in (66), we obtain

$$\begin{aligned}
&\sum_{n=0}^{\infty} \overline{a}_5\left(40p^{2\delta}n + \frac{65p^{2\delta}-2}{3}\right)q^n \\
&\equiv 4(-1)^{\delta(\pm p-1)/6}\Bigg[\sum_{\substack{x=-(p-1)/2\\ x\neq(\pm p-1)/6}}^{(p-1)/2} (-1)^x q^{(3x^2+x)/2}\mathfrak{f}\left(-q^{(3p^2+(6x+1)p)/2}, -q^{(3p^2-(6x+1)p)/2}\right) \\
&\qquad + (-1)^{(\pm p-1)/6} q^{(p^2-1)/24} g_{p^2}\Bigg] \\
&\quad \times\left[\sum_{y=0}^{(p-3)/2} q^{2(y^2+y)}\mathfrak{f}\left(q^{2(p^2+(2y+1)p)}, q^{2(p^2-(2y+1)p)}\right) + q^{(p^2-1)/2}\psi(q^{4p^2})\right] \pmod 8.
\end{aligned} \tag{73}$$

Consider the congruence

$$\left(\frac{3x^2+x}{2}\right) + 2\left(y^2+y\right) \equiv 13\left(\frac{p^2-1}{24}\right) \pmod p,$$

which is equivalent to

$$(6x+1)^2 + 3\left(4y+2\right)^2 \equiv 0 \pmod p. \tag{74}$$

Since $\left(\dfrac{-3}{p}\right) = -1$, the only solution of (74) is $x = (\pm p-1)/6$ and $y = (p-1)/2$. Therefore, collecting the terms involving powers of q that are congruent to $13(p^2-1)/24$ modulo p from both sides of (73) and simplifying the resulting equality yields

$$\sum_{n=0}^{\infty} \overline{a}_5\left(40p^{2\delta+1}n + \frac{65p^{2\delta+2}-2}{3}\right)q^n \equiv 4(-1)^{(\delta+1)(\pm p-1)/6} g_p\psi(q^{4p}) \pmod 8. \tag{75}$$

Collecting the terms involving powers of q that are multiples p from both sides of (75) and simplifying the resulting equality yields

$$\sum_{n=0}^{\infty} \overline{a}_5\left(40p^{2\delta+2}n + \frac{65p^{2\delta+2}-2}{3}\right)q^n \equiv 4(-1)^{(\delta+1)(\pm p-1)/6} g_1\psi(q^4) \pmod 8,$$

which is the $\delta+1$ case of (66). Thus, by employing induction, we complete the proof of (66). Finally, collecting the terms involving q^{pn+t} for $1 \leq t \leq p-1$ from both sides of (75) yields (67). □

Corollary 3. *We have*

$$\overline{a}_5(8n+7) \equiv 0 \pmod{16}, \tag{76}$$

$$\overline{a}_5(20n+2j+1) \equiv 0 \pmod{8}, \quad \textit{for} \quad j = 1,2,3,\ldots,9. \tag{77}$$

Proof. Employing (6) in (69) and then collecting the terms involving powers of q that are congruent to 1 modulo 2 from both sides of the resulting equation, we obtain

$$\sum_{n=0}^{\infty} \overline{a}_5(4n+3)q^n = 8\frac{g_2^2 g_4^4 g_5^2}{g_1^8}. \tag{78}$$

With the help of (16), Equation (78) can be written as

$$\sum_{n=0}^{\infty} \overline{a}_5(4n+3)q^n \equiv 8\frac{g_4^4 g_{10}}{g_2^2} \pmod{16}. \tag{79}$$

Collecting the terms involving odd powers of q from (79), we arrive at (76). Collecting the terms involving powers of q that are congruent to j modulo 10, where $j = 1,2,3,\ldots,9$ from both sides of (70), we arrive at (77). □

Remark 3. Congruence (77) is an extension of congruence $\overline{a}_5(40n+8j+1) \equiv 0 \pmod{8}$, *where* $j = 1,2,3,4$ due to Naika and Harishkumar [10].

Theorem 10. *Suppose $p \geq 5$ is any prime with $\left(\frac{-15}{p}\right) = -1$. Then for any integers $\delta \geq 0$, $n \geq 0$, and $1 \leq t \leq p-1$, we have*

$$\sum_{n=0}^{\infty} \overline{a}_5\left(16p^{2\delta+2}n + \frac{122p^{2\delta+2}-2}{3}\right)q^n \equiv 2(-1)^{(\delta+1)(\pm p-1)/6} g_1\psi(q^{20}) \pmod{4}, \tag{80}$$

$$\overline{a}_5\left(16p^{2\delta+2}n + 16p^{2\delta+1}t + \frac{122p^{2\delta+2}-2}{3}\right) \equiv 0 \pmod{4}. \tag{81}$$

Proof. Setting $\ell = 5$ in (23), we obtain

$$\sum_{n=0}^{\infty} \overline{a}_5(2n)q^n \equiv \frac{g_4 g_{20}}{g_2^3 g_{10}} \pmod{4}. \tag{82}$$

Collecting the terms involving powers of q that are multiples of 2 from (82), we obtain

$$\sum_{n=0}^{\infty} \overline{a}_5(4n)q^n \equiv \frac{g_2 g_{10}}{g_1^3 g_5} \pmod{4}. \tag{83}$$

Employing (16) in (83), we obtain

$$\sum_{n=0}^{\infty} \overline{a}_5(4n)q^n \equiv \frac{g_1 g_{10}}{g_2 g_5} \pmod{4}. \tag{84}$$

Employing (14) in (84) and then collecting the terms involving powers of q that are congruent to 0 modulo 2 from both sides of the resulting equation, we obtain

$$\sum_{n=0}^{\infty} \overline{a}_5(8n)q^n \equiv \frac{g_4 g_{10}^3}{g_2 g_5^2 g_{20}} \pmod{4}. \tag{85}$$

Utilizing (5) in (85) and then collecting the terms involving powers of q that are congruent to 1 modulo 2 from both sides of the resulting equation, we obtain

$$\sum_{n=0}^{\infty} \overline{a}_5(16n+8)q^n \equiv 2q^2 \frac{g_2 g_{10} g_{40}^2}{g_1 g_5^2 g_{20}} \pmod{4}. \tag{86}$$

Utilizing (16) with $\{p=2, k=1\}$ in (86) and then applying (3), we obtain

$$\sum_{n=0}^{\infty} \overline{a}_5(16n+8)q^n \equiv 2q^2 \frac{g_1 g_{40}^2}{g_{20}} \equiv 2q^2 g_1 \psi(q^{20}) \pmod{4}. \tag{87}$$

Applying Lemmas 1 and 2 into (87), we obtain

$$\begin{aligned}
&\sum_{n=0}^{\infty} \overline{a}_5(16n+8)q^n \\
&\equiv 2q^2 \Bigg[\sum_{\substack{x=-(p-1)/2 \\ x \neq (\pm p-1)/6}}^{(p-1)/2} (-1)^x q^{(3x^2+x)/2} \mathfrak{f}\left(-q^{(3p^2+(6x+1)p)/2}, -q^{(3p^2-(6x+1)p)/2}\right) \\
&\qquad\qquad + (-1)^{(\pm p-1)/6} q^{(p^2-1)/24} g_{p^2} \Bigg] \\
&\quad \times \Bigg[\sum_{y=0}^{(p-3)/2} q^{10(y^2+y)} \mathfrak{f}\left(q^{10(p^2+(2y+1)p)}, q^{10(p^2-(2y+1)p)}\right) \\
&\qquad\qquad + q^{20(p^2-1)/8} \psi(q^{20p^2}) \Bigg] \pmod{4}.
\end{aligned} \tag{88}$$

Consider the congruence

$$\left(\frac{3x^2+x}{2}\right) + 10\left(y^2+y\right) + 2 \equiv 61\left(\frac{p^2-1}{24}\right) + 2 \pmod{p},$$

which is identical to

$$(6x+1)^2 + 15\left(4y+2\right)^2 \equiv 0 \pmod{p}. \tag{89}$$

Since $\left(\dfrac{-15}{p}\right) = -1$, the only solution of (89) is $x = (\pm p-1)/6$ and $y = (p-1)/2$. Therefore, collecting the terms containing $q^{p^2 n + 61(p^2-1)/24+2}$ from (88), dividing by

$q^{61(p^2-1)/24+2}$, and replacing q^{p^2} by q, we obtain

$$\sum_{n=0}^{\infty} \overline{a}_5\left(16p^2n+\frac{122}{3}\left(p^2-1\right)+40\right)q^n \equiv 2(-1)^{(\pm p-1)/6}g_1\psi(q^{20}) \pmod{4}. \tag{90}$$

Iterating (90) by using Lemmas 1 and 2, bringing out the terms containing $q^{p^2n+61(p^2-1)/24}$, dividing by $q^{61(p^2-1)/24}$, and replacing q^{p^2} by q, we deduce that, for integer $\delta \geq 0$

$$\sum_{n=0}^{\infty} \overline{a}_5\left(16p^{2\delta}n+\frac{122}{3}\left(p^{2\delta}-1\right)+40\right)q^n \equiv 2(-1)^{\delta(\pm p-1)/6}g_1\psi(q^{20}) \pmod{4}. \tag{91}$$

Utilizing Lemmas 1 and 2 in (91) and collecting the terms containing $q^{pn+61(p^2-1)/24}$, dividing by $q^{61(p^2-1)/24}$, and replacing q by $q^{1/p}$, we obtain

$$\sum_{n=0}^{\infty} \overline{a}_5\left(16p^{2\delta+1}n+\frac{122}{3}\left(p^{2\delta+2}-1\right)+40\right)q^n \equiv 2(-1)^{(\delta+1)(\pm p-1)/6}g_p\psi(q^{20p}) \pmod{4}. \tag{92}$$

Collecting the terms containing q^{pn} from (92) and replacing q by $q^{1/p}$, we arrive at (80). Collecting the terms involving q^{pn+t} for $1 \leq t \leq p-1$ from (92), we arrive at (81). □

Theorem 11. *Suppose $p \geq 5$ is any prime with $\left(\frac{-3}{p}\right) = -1$. Then for any integers $\delta \geq 0$, $n \geq 0$, and $1 \leq t \leq p-1$, we have*

$$\sum_{n=0}^{\infty} \overline{a}_8\left(4p^{2\delta}n+\frac{13p^{2\delta}-7}{6}\right)q^n \equiv 2(-1)^{\delta(\pm p-1)/6}g_1\psi(q^4) \pmod{4}, \tag{93}$$

$$\overline{a}_8\left(4p^{2\delta+2}n+4p^{2\delta+1}t+\frac{13p^{2\delta+2}-7}{6}\right) \equiv 0 \pmod{4}, \tag{94}$$

$$\sum_{n=0}^{\infty} \overline{a}_8\left(4p^{2\delta}n+\frac{19p^{2\delta}-7}{6}\right)q^n \equiv 4(-1)^{\delta(\pm p-1)/6}\psi(q)g_{16} \pmod{8}, \tag{95}$$

$$\overline{a}_8\left(4p^{2\delta+2}n+4p^{2\delta+1}t+\frac{19p^{2\delta+2}-7}{6}\right) \equiv 0 \pmod{8}. \tag{96}$$

Proof. Setting $\ell = 8$ in (2), we obtain

$$\sum_{n=0}^{\infty} \overline{a}_8(n)q^n = \frac{g_2g_8^2g_{32}}{g_1^2g_4g_{16}}. \tag{97}$$

Utilizing (5) in (97), we obtain

$$\sum_{n=0}^{\infty} \overline{a}_8(n)q^n = \frac{g_8^7g_{32}}{g_2^4g_4g_{16}^3}+2q\frac{g_4g_8g_{16}g_{32}}{g_2^4}. \tag{98}$$

Collecting the terms involving powers of q that are not divisible by 2 from (98), we obtain

$$\sum_{n=0}^{\infty} \overline{a}_8(2n+1)q^n = 2\frac{g_2 g_4 g_8 g_{16}}{g_1^4}. \tag{99}$$

Employing (6) in (99), we find that

$$\sum_{n=0}^{\infty} \overline{a}_8(2n+1)q^n = 2\frac{g_4^{15} g_{16}}{g_2^{13} g_8^3} + 8q\frac{g_4^3 g_8^5 g_{16}}{g_2^9}. \tag{100}$$

Collecting the terms involving powers of q that are divisible by 2 from (100) and then simplifying, we obtain

$$\sum_{n=0}^{\infty} \overline{a}_8(4n+1)q^n = 2\frac{g_2^{15} g_8}{g_1^{13} g_4^3}. \tag{101}$$

Employing (16) with $\{p=2, k=1\}$ in (101) and then using (3), we find that

$$\sum_{n=0}^{\infty} \overline{a}_8(4n+1)q^n \equiv 2\frac{g_1 g_8^2}{g_4} \equiv 2g_1\psi(q^4) \pmod{4},$$

which is the $\delta = 0$ case of (93). The remaining proofs of Congruences (93) and (94) are similar to the proofs of Congruences (66) and (67), respectively, and the desired result can be obtained by appealing to Lemmas 1 and 2. As a result, we omit the details. Collecting the terms involving powers of q that are multiples of 2 from (98), we obtain

$$\sum_{n=0}^{\infty} \overline{a}_8(2n)q^n = \frac{g_4^7 g_{16}}{g_1^4 g_2 g_8^3}. \tag{102}$$

Employing (6) in (102) and then collecting the terms involving powers of q that are not multiples of 2 from both sides of the resulting equation, we obtain

$$\sum_{n=0}^{\infty} \overline{a}_8(4n+2)q^n = 4\frac{g_2^9 g_4 g_8}{g_1^{11}}. \tag{103}$$

Employing (16) in (103), we obtain

$$\sum_{n=0}^{\infty} \overline{a}_8(4n+2)q^n \equiv 4\frac{g_2^2 g_4^2 g_8}{g_1} \pmod{8}. \tag{104}$$

Using (16) in (104) and then employing (3), we obtain

$$\sum_{n=0}^{\infty} \overline{a}_8(4n+2)q^n \equiv 4\frac{g_2^2 g_{16}}{g_1} \equiv 4\psi(q)g_{16} \pmod{8},$$

which is the $\delta = 0$ case of (95). The remaining proofs of Congruences (95) and (96) are similar to the proofs of Congruences (66) and (67), respectively, and the desired result can be obtained by appealing to Lemmas 1 and 2. As a result, we omit the details. □

Theorem 12. *Suppose $p \geq 5$ is any prime with $\left(\frac{-6}{p}\right) = -1$. Then for any integers $\delta \geq 0$, $n \geq 0$, and $1 \leq t \leq p-1$, we have*

$$\sum_{n=0}^{\infty} \overline{a}_8\left(4p^{2\delta}n + \frac{25p^{2\delta}-7}{6}\right)q^n \equiv 8(-1)^{\delta(\pm p-1)/6} g_1 \psi(q^8) \pmod{16}, \tag{105}$$

$$\overline{a}_8\left(4p^{2\delta+2}n + 4p^{2\delta+1}t + \frac{25p^{2\delta+2}-7}{6}\right) \equiv 0 \pmod{16}. \tag{106}$$

Proof. Collecting the terms involving powers of q that are not divisible by 2 from (100), we obtain

$$\sum_{n=0}^{\infty} \overline{a}_8(4n+3)q^n = 8\frac{g_2^3 g_4^5 g_8}{g_1^9}. \tag{107}$$

Employing (16) with $\{p = 2, k = 1\}$ in (107), we obtain

$$\sum_{n=0}^{\infty} \overline{a}_8(4n+3)q^n \equiv 8\frac{g_4^5 g_8}{g_1^3} \pmod{16}. \tag{108}$$

Using (16) with $\{p = 2, k = 1\}$ in (108), we obtain

$$\sum_{n=0}^{\infty} \overline{a}_8(4n+3)q^n \equiv 8 g_1 g_8^3 \pmod{16}. \tag{109}$$

Again, using (16) with $\{p = 2, k = 1\}$ in (109) and then applying (3), we obtain

$$\sum_{n=0}^{\infty} \overline{a}_8(4n+3)q^n \equiv 8\frac{g_1 g_{16}^2}{g_8} \equiv 8 g_1 \psi(q^8) \pmod{16},$$

which is the $\delta = 0$ case of (105). The remaining proofs of Congruences (105) and (106) are similar to the proofs of Congruences (66) and (67), respectively, and the desired result can be obtained by appealing to Lemmas 1 and 2. As a result, we omit the details. □

Theorem 13. *Suppose $p \geq 5$ is any prime with $\left(\frac{-2}{p}\right) = -1$. Then for any integers $\delta \geq 0$, $n \geq 0$, and $1 \leq t \leq p-1$, we have*

$$\sum_{n=0}^{\infty} \overline{a}_9\left(8p^{2\delta}n + \frac{19p^{2\delta}-4}{3}\right)q^n \equiv 2 g_1 g_{18} \pmod{4}, \tag{110}$$

$$\overline{a}_9\left(8p^{2\delta+2}n + 8p^{2\delta+1}t + \frac{19p^{2\delta+2}-4}{3}\right) \equiv 0 \pmod{4}.$$

Proof. Setting $\ell = 9$ in (2), we obtain

$$\sum_{n=0}^{\infty} \overline{a}_9(n)q^n = \frac{g_2 g_9^2 g_{36}}{g_1^2 g_4 g_{18}}. \tag{111}$$

Using (15) in (111) and then collecting the terms involving powers of q that are congruent to 1 modulo 2 from both sides of the resulting equation, we obtain

$$\sum_{n=0}^{\infty} \overline{a}_9(2n+1)q^n = 2\frac{g_2 g_6^2 g_{18}}{g_1^4}. \tag{112}$$

Using (6) in (112) and simplifying, we obtain

$$\sum_{n=0}^{\infty} \overline{a}_9(2n+1)q^n = 2\frac{g_4^{14} g_6^2 g_{18}}{g_2^{13} g_8^4} + 8q\frac{g_4^2 g_6^2 g_8^4 g_{18}}{g_2^9}. \tag{113}$$

Collecting the terms involving powers of q that are multiples of 2 from (113), we obtain

$$\sum_{n=0}^{\infty} \overline{a}_9(4n+1)q^n = 2\frac{g_2^{14} g_3^2 g_9}{g_1^{13} g_4^4}. \tag{114}$$

Using (16) in (114), we obtain

$$\sum_{n=0}^{\infty} \overline{a}_9(4n+1)q^n \equiv 2\frac{g_6 g_9}{g_1} \pmod 4. \tag{115}$$

Using (15) in (115) and then collecting the terms involving powers of q that are not divisible by 2 from both sides of the resulting equation, we obtain

$$\sum_{n=0}^{\infty} \overline{a}_9(8n+5)q^n \equiv 2\frac{g_2^2 g_3^2 g_{18}}{g_1^3 g_6} \pmod 4. \tag{116}$$

With the help of (16), Congruence (116) can be written as

$$\sum_{n=0}^{\infty} \overline{a}_9(8n+5)q^n \equiv 2g_1 g_{18} \pmod 4. \tag{117}$$

Congruence (117) is the $\delta = 0$ case of (110). As one can now proceed via the same argument used in our proof of Theorem 6, we omit the remaining details. □

Corollary 4. *We have*

$$\overline{a}_9(4n+3) \equiv 0 \pmod 8. \tag{118}$$

Proof. Collecting the terms involving odd powers of q from (113), we arrive at (118). □

Theorem 14. *Suppose $p \geq 5$ is any prime with $\left(\frac{-2}{p}\right) = -1$. Then for any integers $\delta \geq 0$, $n \geq 0$, and $1 \leq t \leq p-1$, we have*

$$\sum_{n=0}^{\infty} \overline{a}_{16}\left(4p^{2\delta}n + \frac{11p^{2\delta}-5}{2}\right)q^n \equiv 8g_1g_{32} \pmod{16}, \tag{119}$$

$$\overline{a}_{16}\left(4p^{2\delta+2}n + 4p^{2\delta+1}t + \frac{11p^{2\delta+2}-5}{2}\right) \equiv 0 \pmod{16}.$$

Proof. Setting $\ell = 16$ in (2), we obtain

$$\sum_{n=0}^{\infty} \overline{a}_{16}(n)q^n = \frac{g_2g_{16}^2g_{64}}{g_1^2g_4g_{32}}. \tag{120}$$

Employing (5) in (120) and then collecting the terms involving powers of q that are not divisible by 2 from both sides of the resulting equation, we obtain

$$\sum_{n=0}^{\infty} \overline{a}_{16}(2n+1)q^n = 2\frac{g_2g_8^4g_{32}}{g_1^4g_4g_{16}}. \tag{121}$$

Employing (6) in (121) and then collecting the terms involving powers of q that are not divisible by 2 from both sides of the resulting equation, we obtain

$$\sum_{n=0}^{\infty} \overline{a}_{16}(4n+3)q^n = 8\frac{g_2g_4^8g_{16}}{g_1^9g_8}. \tag{122}$$

Utilizing (16) with $\{p=2, k=1\}$, Equation (122) can be written as

$$\sum_{n=0}^{\infty} \overline{a}_{16}(4n+3)q^n \equiv 8g_1g_4^8 \pmod{16}. \tag{123}$$

Utilizing (16) with $\{p=2, k=1\}$, Congruence (123) can be written as

$$\sum_{n=0}^{\infty} \overline{a}_{16}(4n+3)q^n \equiv 8g_1g_{32} \pmod{16}. \tag{124}$$

Congruence (124) is the $\delta = 0$ case of (119). As one can now proceed via the same argument used in our proof of Theorem 6, we omit the remaining details. □

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