MODULAR RELATIONS FOR THE ROGERS-RAMANUJAN-SLATER TYPE FUNCTIONS OF ORDER EIGHTEEN WITH APPLICATIONS TO PARTITIONS

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Abstract

In this paper, we establish several modular relations involving the Rogers-Ramanujan-Slater type functions of order eighteen that are analogues to Ramanujan's well known forty identities. Furthermore, we give partition theoretic interpretations of two modular relations.

1. Introduction

Throughout the paper, we assume $|q| < 1$ and we use the standard notation

$$(a;q)_0 := 1, \quad (a;q)_n := \prod_{j=0}^{n-1}(1-aq^j) \quad \text{and} \quad (a;q)_\infty = \prod_{n=0}^{\infty}(1-aq^n).$$

The well-known Rogers-Ramanujan functions are defined for $|q| < 1$ by

$$G(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} \quad \text{and} \quad H(q) := \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q;q)_n}.$$

These functions satisfy the famous Rogers-Ramanujan identities

$$G(q) = \frac{1}{(q;q^5)_\infty (q^4;q^5)_\infty} \quad \text{and} \quad H(q) = \frac{1}{(q^2;q^5)_\infty (q^3;q^5)_\infty}.$$

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In [24], Ramanujan remarks, "I have now found an algebraic relation between $G(q)$ and $H(q)$", viz. $H(q)\{G(q)\}^{11} - q^2G(q)\{H(q)\}^{11} = 1 + 11q\{G(q)H(q)\}^6$. Another interesting formula is $H(q)G(q^{11}) - q^2G(q)H(q^{11}) = 1$.

These two identities are from a list of forty identities involving the Rogers-Ramanujan functions presented by Ramanujan. Ramanujan's forty identities for $G(q)$ and $H(q)$ were first brought to the mathematical world by Birch [16] in 1975. Many of these identities were established by Rogers [27], Watson [30], Bressoud [17] and Biagioli [15]. Recently, Berndt et al. [14] offered proofs of thirty-five of the forty identities. A number of mathematicians have tried to find new identities for the Rogers-Ramanujan functions similar to those found by Ramanujan [24], including Berndt and Yesilyurt [13], Gugg [21], and Bulkhali and Ranganatha [19].

In [10], Baruah and Bora established several modular relations for the nonic analogues of the Rogers-Ramanujan functions which are defined as

$$A(q) := \sum_{n=0}^{\infty} \frac{(q;q)_{3n}\ q^{3n^2}}{(q^3;q^3)_n(q^3;q^3)_{2n}} = \frac{(q^4;q^9)_\infty(q^5;q^9)_\infty(q^9;q^9)_\infty}{(q^3;q^3)_\infty},$$

$$B(q) := \sum_{n=0}^{\infty} \frac{(q;q)_{3n}\ (1-q^{3n+2})\ q^{3n(n+1)}}{(q^3;q^3)_n(q^3;q^3)_{2n+1}} = \frac{(q^2;q^9)_\infty(q^7;q^9)_\infty(q^9;q^9)_\infty}{(q^3;q^3)_\infty},$$

and

$$C(q) := \sum_{n=0}^{\infty} \frac{(q;q)_{3n+1}\ q^{3n(n+1)}}{(q^3;q^3)_n(q^3;q^3)_{2n+1}} = \frac{(q;q^9)_\infty(q^8;q^9)_\infty(q^9;q^9)_\infty}{(q^3;q^3)_\infty}.$$

They also established several other modular relations that are connected with the Rogers-Ramanujan functions, Göllnitz-Gordon functions, and septic analogues of Rogers-Ramanujan type functions. Motivated by Rogers-Ramanujan functions, the modular relations of Rogers-Ramanujan type and Rogers-Ramanujan-Slater type functions are studied in Table 1.

Various Rogers-Ramanujan type functions	**Order**	**References**
Cubic functions	6	[8, 9]
Septic functions	7	[22]
Göllnitz-Gordon functions	8	[11, 20, 23, 31]
Nonic functions	9	[10]
Rogers-Ramanujan type functions of order 10	10	[2]
Rogers-Ramanujan type functions of order 11	11	[4]
Rogers-Ramanujan type functions of order 12	12	[29]
Rogers-Ramanujan type functions of order 13	13	[18]

Table 1: Various Rogers-Ramanujan type functions

Recently, Adiga et al. [5, 6, 7] established several modular relations for the Rogers-Ramanujan-Slater type function of order fifteen. These relations are analogues to Ramanujan's famous forty identities for the Rogers-Ramanujan functions. Further, they gave interesting partition theoretic interpretations of these relations. In [3], Gugg studied the sextodecic analogous of the Rogers-Ramanujan functions. Motivated by Gugg [3] works, the sextodecic analogous of the Rogers-Ramanujan functions were developed by Adiga and Bulkhali [21]. Almost all of these functions which have been studied so far are due to Rogers [26] and Slater [28].

In [25, p. 33], Ramanujan stated the following identity:

$$\frac{f(aq^3,\ a^{-1}q^3)}{f(-q^2)} = \sum_{n=0}^{\infty} \frac{q^{2n^2}\ (-a^{-1}q;\ q^2)_n\ (-aq;\ q^2)_n}{(q^2;\ q^2)_{2n}}, \tag{1}$$

where

$$f(a,b) = \sum_{n=-\infty}^{\infty} a^{n(n+1)/2}\ b^{n(n-1)/2}, \quad |ab| < 1,$$

is the general theta function of Ramanujan. The above result of Ramanujan yields infinitely many identities of Rogers-Ramanujan-Slater type when a is set to $\pm q^r$ for $r \in \mathbb{Q}$.

Now, we define the following Rogers-Ramanujan-Slater type identities of order eighteen using Identity (1) with $q = q^3$ and $a = -q^2, a = -q^4, a = -q^8$, respectively, and we obtain

$$P(q) := \frac{f(-q^7, -q^{11})}{f(-q^6)} = \sum_{n=0}^{\infty} \frac{q^{6n^2}\ (q;q^6)_n\ (q^5;q^6)_n}{(q^6;q^6)_{2n}}, \tag{2}$$

$$Q(q) := \frac{f(-q^5, -q^{13})}{f(-q^6)} = 1 - \sum_{n=1}^{\infty} \frac{q^{6n^2-1}\ (q^5;q^6)_{n-1}\ (q;q^6)_{n+1}}{(q^6;q^6)_{2n}}, \tag{3}$$

and

$$R(q) := \frac{f(-q, -q^{17})}{f(-q^6)} = 1 - \sum_{n=1}^{\infty} \frac{q^{6n^2-5}\ (q;q^6)_{n-1}\ (q^5;q^6)_{n+1}}{(q^6;q^6)_{2n}}. \tag{4}$$

The main purpose of this paper is to establish several modular relations involving $P(q)$, $Q(q)$, and $R(q)$, which are analogues to Ramanujan's forty identities. Further, we extract partition theoretic interpretations of our modular relations.

2. Definitions and Preliminary Results

In this section, we present some basic definitions and preliminary results on Ramanujan's theta functions.

The function $f(a,b)$ satisfies the following basic properties [1, Entry 18]:

$$f(a,b) = f(b,a), \qquad f(1,a) = 2f(a,a^3), \qquad \text{and} \qquad f(-1,a) = 0.$$

If n is an integer,

$$f(a,b) = a^{n(n+1)/2}\, b^{n(n-1)/2}\, f(a(ab)^n, b(ab)^{-n}).$$

The well-known Jacobi triple product identity [1, Entry 19] is given by

$$f(a,b) = (-a;ab)_\infty\, (-b;ab)_\infty\, (ab;ab)_\infty. \tag{5}$$

The three most interesting special cases of $f(a,b)$ are [1, Entry 22]

$$\varphi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = (-q;q^2)_\infty^2 (q^2;q^2)_\infty,$$

$$\psi(q) := f(q,q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty},$$

and

$$f(-q) := f(-q,-q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q;q)_\infty.$$

Also, using Ramanujan's notation, we have

$$\chi(q) := (-q;q^2)_\infty.$$

For convenience, we define

$$f_n := f(-q^n) = (q^n;q^n)_\infty,$$

for positive integers n. The following lemma is a consequence of Identity (5) and Entry 24 of [1, p. 39].

Lemma 1 ([1]). *We have*

$$\varphi(q) = \frac{f_2^5}{f_1^2\, f_4^2}, \qquad \psi(q) = \frac{f_2^2}{f_1}, \qquad \varphi(-q) = \frac{f_1^2}{f_2}, \qquad \psi(-q) = \frac{f_1\, f_4}{f_2},$$

$$f(q) = \frac{f_2^3}{f_1\, f_4}, \qquad \chi(q) = \frac{f_2^2}{f_1\, f_4}, \qquad \textit{and} \qquad \chi(-q) = \frac{f_1}{f_2}.$$

Lemma 2 ([1]). *If $ab = cd$, then*

$$f(a,b)f(c,d) + f(-a,-b)f(-c,-d) = 2f(ac,bd)f(ad,bc), \tag{6}$$

and

$$f(a,b)f(c,d) - f(-a,-b)f(-c,-d) = 2af\left(\frac{b}{c}, \frac{c}{b}abcd\right) f\left(\frac{b}{d}, \frac{d}{b}abcd\right). \tag{7}$$

Remark 1. Adding (6) and (7) together, we deduce

$$f(a,b)f(c,d) = f(ac,bd)f(ad,bc) + af\left(\frac{b}{c}, \frac{c}{b}abcd\right) f\left(\frac{b}{d}, \frac{d}{b}abcd\right). \tag{8}$$

Lemma 3 ([1]). *We have*

$$f(q,q^5) = \chi(q)\psi(-q^3). \tag{9}$$

Lemma 4 ([12, Chapter 20]). *We have*

$$f(-q,-q^8)f(-q^2,-q^7)f(-q^4,-q^5) = \frac{f(-q)f^3(-q^9)}{f(-q^3)}. \tag{10}$$

Lemma 5 ([12, Chapter 20]). *We have*

$$f(q,q^2) = \frac{\varphi(-q^3)}{\chi(-q)}. \tag{11}$$

3. Main Results

In this section, we present several new modular relations for the functions $P(q)$, $Q(q)$, and $R(q)$. For simplicity, we use the notations $P_n := P(q^n)$, $Q_n := Q(q^n)$, and $R_n := R(q^n)$ for a positive integer n.

We prove the following theorem using ideas similar to those of Watson [30]. In Watson's method, one expresses the left sides of the identities in terms of theta functions by using Identity (5). After clearing fractions, we see that the right side can be expressed as a product of two theta functions, say with summation indices m and n. One then tries to find variable substitutions of the form

$$\alpha m + \beta n = 9M + a \quad \textit{and} \quad \gamma m + \delta n = 9N + b,$$

so that the product on the right side decomposes into the requisite sum of two products of theta functions on the left side.

Theorem 1. *We have*

$$P_1P_8+q^3Q_1Q_8+q^{15}R_1R_8 = \frac{1}{2qf_6f_{48}}\left\{\frac{f_2^5f_8^2}{f_1^2f_4^2f_{16}}-\frac{f_9^2f_{72}^2}{f_{18}f_{144}}\right\}-q^8\frac{f_3f_{18}^2f_{24}f_{144}^2}{f_6^2f_9f_{48}^2f_{72}}, \tag{12}$$

$$P_2P_7+q^3Q_2Q_7+q^{15}R_2R_7 = \frac{1}{2qf_{12}f_{42}}\left\{\frac{f_2^5f_{14}^2}{f_1^2f_4^2f_{28}}-\frac{f_{18}^2f_{63}^2}{f_{36}f_{126}}\right\}-q^8\frac{f_6f_{21}f_{36}^2f_{126}^2}{f_{12}^2f_{18}f_{42}^2f_{63}}, \tag{13}$$

$$P_3P_6+q^3Q_3Q_6+q^{15}R_3R_6 = \frac{1}{2q}\left\{\frac{f_2^5f_{18}}{f_1^2f_4^2f_{36}^2}-\frac{f_{27}^2f_{54}^2}{f_{18}f_{36}f_{54}f_{108}}\right\}-q^8\frac{f_9f_{54}^2f_{108}^2}{f_{18}f_{27}f_{36}^2f_{48}}, \tag{14}$$

$$P_4P_5+q^3Q_4Q_5+q^{15}R_4R_5 = \frac{1}{2qf_{24}f_{30}}\left\{\frac{f_2^5f_{20}^2}{f_1^2f_4^2f_{40}}-\frac{f_{36}^2f_{45}^2}{f_{72}f_{90}}\right\}-q^8\frac{f_{12}f_{15}f_{72}^2f_{90}^2}{f_{24}^2f_{30}^2f_{36}f_{45}}, \tag{15}$$

$$P_1Q_2-q^3Q_1R_2+qR_1P_2 = \frac{1}{2qf_6f_{12}}\left\{\frac{f_9^2f_{18}}{f_{36}}-\frac{f_1^2f_2}{f_4}\right\}+q^2\frac{f_3f_{18}f_{36}^2}{f_6f_9f_{12}^2}, \tag{16}$$

$$q^5P_1Q_{20}-q^{32}Q_1R_{20}+R_1P_{20} = \frac{1}{2q^4f_6f_{120}}\left\{\frac{f_9^2f_{180}^2}{f_{18}f_{360}}-\frac{f_4^2f_{10}^5}{f_5^2f_8f_{20}^2}\right\}+q^{17}\frac{f_3f_{18}^2f_{60}f_{360}^2}{f_6^2f_9f_{120}^2f_{180}}, \tag{17}$$

and

$$q^{23}P_1R_{14}+Q_1P_{14}-q^6R_1Q_{14} = \frac{1}{2q^2f_6f_{84}}\left\{\frac{f_9^2f_{126}^2}{f_{18}f_{252}}-\frac{f_2^2f_{14}^5}{f_4f_7^2f_{28}^2}\right\}+q^{13}\frac{f_3f_{18}^2f_{42}f_{252}^2}{f_6^2f_9f_{84}^2f_{126}}. \tag{18}$$

Proof. Using Identities (2)-(4), we may rewrite Identity (12) in the form

$$\begin{aligned} &q^{15}f(-q^8,-q^{136})\ f(-q,-q^{17})+q^3f(-q^{40},-q^{104})\ f(-q^{13},-q^5)\\ &\quad+f(-q^{56},-q^{88})\ f(-q^{11},-q^7)=\frac{1}{2q}\left\{f(q,q)\ f(-q^8,-q^8)\right.\\ &\quad\left.-\ f(-q^{72},-q^{72})\ f(-q^9,-q^9)\right\}-q^9f(-q^{24},-q^{120})\ f(-q^3,-q^{15}). \end{aligned} \tag{19}$$

In this representation, we make variable substitutions by setting

$$m+n=9M+a \quad \text{and} \quad -m+8n=9N+b,$$

where a and b have values selected from the set $\{0, \pm 1, \pm 2, \pm 3, \pm 4\}$. Then

$$m = 8M - N + (8a - b)/9 \quad \text{and} \quad n = M + N + (a + b)/9.$$

It follows that the values of a and b are associated as in the following table:

a	0	± 1	± 2	± 3	± 4
b	0	∓ 1	∓ 2	∓ 3	∓ 4

When a assumes the values, $0, \pm 1, \pm 2, \pm 3, \pm 4$ in succession, it is easy to see that the corresponding values of $2m^2 + 16n^2$ are, respectively, as follows:

$$\begin{aligned}
&144m^2 + 18n^2 - 128m + 16n + 32,\\
&144m^2 + 18n^2 - 96m + 12n + 18,\\
&144m^2 + 18n^2 - 64m + 8n + 8,\\
&144m^2 + 18n^2 - 32m + 4n + 8,\\
&144m^2 + 18n^2,\\
&144m^2 + 18n^2 + 32m - 4n + 2,\\
&144m^2 + 18n^2 + 64m - 8n + 8,\\
&144m^2 + 18n^2 + 96m - 12n + 18,\\
&144m^2 + 18n^2 + 128m - 16n + 32.
\end{aligned}$$

It is evident from the equation connecting m and n with M and N that, there is a one-to-one correspondence between all pairs of integers (m, n) and all sets of integers (M, N, a). From these correspondences, we deduce that

$$\begin{aligned}
f(q,q)f(-q^8,-q^8) = \sum_{m,n=-\infty}^{\infty} (-1)^n \, q^{\frac{2m^2+16n^2}{2}} &= 2q^{16} f(-q^8,-q^{136}) f(-q,-q^{17})\\
&+ 2q^9 f(-q^{24},-q^{120}) f(-q^3,-q^{15}) + 2q^4 f(-q^{40},-q^{104}) f(-q^{13},-q^5)\\
&+ 2q f(-q^{56},-q^{88}) f(-q^{11},-q^7) + f(-q^{72},-q^{72}) f(-q^9,-q^9),
\end{aligned}$$

which is nothing but Identity (19). The proofs of the Identities (13)-(15) are similar to Identity (12), with its variable substitutions as follows:

Identity	**Variable substitutions**
13	$7m + n = 9M + a$ and $2m - n = 9N + b$
14	$-m + 3n = 9M + a$ and $m + 6n = 9N + b$
15	$-m + 4n = 9M + a$ and $m + 5n = 9N + b$

where a and b have values selected from the set $\{0, \pm 1, \pm 2, \pm 3, \pm 4\}$. So we omit the proof.

Using Identities (2)-(4), we may rewrite Identity (16) in the form

$$\begin{aligned} q^3 f(-q^2,-q^{34})\ f(-q^5,-q^{13}) - q^2 f(-q^{14},-q^{22})\ f(-q,-q^{17}) \\ - f(-q^{10},-q^{26})\ f(-q^{11},-q^7) = \frac{1}{2q}\left\{f(-q,-q)\ f(-q^2,-q^2)\right. \\ \left. - f(-q^{18},-q^{18})\ f(-q^9,-q^9)\right\} - q^2 f(-q^6,-q^{30})\ f(-q^3,-q^{15}). \end{aligned} \tag{20}$$

In this representation, we make variable substitutions by setting

$$2m+n = 9M+a \quad \text{and} \quad m-4n = 9N+b,$$

where a and b have values from the set $0, \pm1, \pm2, \pm3, \pm4$. Then

$$m = 4M+N+(4a+b)/9 \quad \text{and} \quad n = M-2N+(a-2b)/9.$$

It follows that the values of a and b are associated as in the following table:

a	0	±1	±2	±3	±4
b	0	∓4	±1	∓3	±2

When a assumes the values, 0, ±1, ±2, ±3, ±4 in succession, it is easy to see that the corresponding values of $2m^2+4n^2$ are, respectively, as follows:

$$\begin{aligned} &36m^2+18n^2-32m-8n+8, \\ &36m^2+18n^2-24m+12n+6, \\ &36m^2+18n^2-16m-4n+2, \\ &36m^2+18n^2-8m+16n+4, \\ &36m^2+18n^2, \\ &36m^2+18n^2+8m-16n+4, \\ &36m^2+18n^2+16m+4n+2, \\ &36m^2+18n^2+24m-12n+6, \\ &36m^2+18n^2+32m+8n+8. \end{aligned}$$

It is evident from the equation connecting m and n with M and N that there is a one-to-one correspondence between all pairs of integers (m,n) and all sets of integers (M,N,a). From these correspondences, we deduce that

$$\begin{aligned} f(-q,-q)f(-q^2,-q^2) = \sum_{m,n=-\infty}^{\infty} (-1)^{m+n} q^{\frac{2m^2+4n^2}{2}} \\ =2q^4 f(-q^2,-q^{34})f(-q^5,-q^{13}) + 2q^3 f(-q^6,-q^{30})f(-q^3,q^{15}) \\ - 2q f(-q^{10},-q^{26})f(-q^7,-q^{11}) - 2q^2 f(-q^{14},-q^{22})f(-q,-q^{17}) \\ + f(-q^{18},-q^{18})\ f(-q^9,-q^9), \end{aligned}$$

which is nothing but Identity (20). The proofs of the Identities (17)-(18) are similar to Identity (16), with its variable substitutions as follows:

Identity	Variable substitutions
17	$m+n=9M+a$ and $4m-5n=9N+b$
18	$m-n=9M+a$ and $2m+7n=9N+b$

where a and b have values selected from the set $\{0, \pm 1, \pm 2, \pm 3, \pm 4\}$. So we omit the proof. □

The following identities are relations involving some combinations of $P(q)$, $Q(q)$, and $R(q)$ and the nonic analogues of the Rogers-Ramanujan functions $A(q)$, $B(q)$, and $C(q)$. For simplicity, we use the notations $A_n := A(q^n)$, $B_n := B(q^n)$, and $C_n := C(q^n)$, for a positive integer n.

Theorem 2. *We have*

$$P_1Q_1R_1 = \frac{\chi(-q)f^3(-q^{18})}{\chi(-q^3)f^3(-q^6)}, \tag{21}$$

$$\frac{P_1A_2 + q^3R_1C_2}{A_1} = \frac{f_9^2}{f_6f_{18}}, \tag{22}$$

$$\frac{Q_1A_2 - q^2R_1B_2}{B_1} = \frac{f_9^2}{f_6f_{18}}, \tag{23}$$

$$\frac{P_1B_2 - qQ_1C_2}{C_1} = \frac{f_9^2}{f_6f_{18}}, \tag{24}$$

$$\frac{P_1A_2 + Q_1B_2 + q^3R_1C_2}{A_1} = \frac{f_9^2}{f_6f_{18}} + \frac{f_3f_{18}^2}{f_6^2f_9}, \tag{25}$$

$$\frac{Q_1A_2 - q^2R_1B_2 + P_1C_2}{B_1} = \frac{f_9^2}{f_6f_{18}} + \frac{f_3f_{18}^2}{f_6^2f_9}, \tag{26}$$

and

$$\frac{R_1A_2 + P_1B_2 - qQ_1C_2}{C_1} = \frac{f_9^2}{f_6f_{18}} + \frac{f_3f_{18}^2}{f_6^2f_9}. \tag{27}$$

Proof. Setting $\{a=-q, b=-q^8, c=1, d=q^9\}$, $\{a=-q^2, b=-q^7, c=1, d=q^9\}$, and $\{a=-q^4, b=-q^5, c=1, d=q^9\}$ in Identity (6), respectively, and using basic properties of $f(a,b)$, we obtain

$$f(-q,-q^{17}) = \frac{f(-q,-q^8)f(q^9,q^{27})}{f(-q^8,-q^{10})}, \tag{28}$$

$$f(-q^7,-q^{11}) = \frac{f(-q^2,-q^7)f(q^9,q^{27})}{f(-q^2,-q^{16})}, \tag{29}$$

and

$$f(-q^5,-q^{13}) = \frac{f(-q^4,-q^5)f(q^9,q^{27})}{f(-q^4,-q^{14})}. \tag{30}$$

Multiplying Identities (28)-(30) together and using Lemma 1 and Identity (10), after some simplifications, we obtain Identity (21).

Setting $\{a=-q^4, b=-q^5, c=q^3, d=q^6\}$ in Identity (8), we obtain

$$\begin{aligned} &f(-q^4,-q^5)f(q^3,q^6) \\ &\quad = f(-q^7,-q^{11})f(-q^8,-q^{10}) + q^3 f(-q,-q^{17})f(-q^2,-q^{16}). \end{aligned} \tag{31}$$

Using Identity (11) with $q = q^3$ and Lemma 1 in Identity (31), we obtain Identity (22). The proofs of Identities (23) and (24) are similar to Identity (22), and hence, are omitted here.

Adding Identities (30) and (31) together and then by using Identity (11) with $q = q^2$ and Lemma 1, we obtain Identity(25). The proofs of Identities (26) and (27) are similar to Identity (25), and hence, are omitted here. □

4. Applications to the Theory of Partitions

In this section, we present partition theoretic interpretations of Identities (16) and (27). For simplicity, we define

$$(q^{\pm r};q^s)_\infty := (q^r, q^{s-r}; q^s)_\infty,$$

where r and s are positive integers and $r < s$.

Definition 1. A positive integer n has k *colors* if there are k copies of n available and all of them are viewed as distinct objects. Partitions of a positive integer into parts with colors are called *colored partitions.*

For example, if 2 is allowed to have two colors, say r (red), and g (green), then all colored partitions of 3 are 3, $2_r + 1$, $2_g + 1$, and $1 + 1 + 1$.

An important fact is that

$$\frac{1}{(q^u;\ q^v)_\infty^k}$$

is the generating function for the number of partitions of n, where all the parts are congruent to u modulo v and have k colors.

Let $P_1(n)$ denote the number of partitions of n into parts congruent to ± 7, ± 11, 18 modulo 36 with two colors, parts congruent to ± 1, ± 3, ± 4, ± 5, ± 8, ± 10, ± 13, ± 15, ± 16, ± 17 modulo 36 with three colors, parts congruent to ± 2, ± 6, ± 12, ± 14 modulo 36 with four colors, and parts congruent to ± 9 modulo 36 with five colors.

Let $P_2(n)$ denote the number of partitions of n into parts congruent to ± 5, ± 13, 18 modulo 36 with two colors, parts congruent to ± 1, ± 2, ± 3, ± 4, ± 7, ± 8, ± 11, ± 15, ± 16, ± 17 modulo 36 with three colors, parts congruent to ± 6, ± 10, ± 12, ± 14 modulo 36 with four colors, and parts congruent to ± 9 modulo 36 with five colors.

Let $P_3(n)$ denote the number of partitions of n into parts congruent to ± 1, ± 3, ± 17, 18 modulo 36 with two colors, parts congruent to ± 4, ± 5, ± 7, ± 8, ± 11, ± 13, ± 14, ± 15, ± 16 modulo 36 with three colors, parts congruent to ± 2, ± 6, ± 10, ± 12 modulo 36 with four colors, and parts congruent to ± 9 modulo 36 with five colors.

Let $P_4(n)$ denote the number of partitions of n into parts congruent to ± 1, ± 3, ± 4, ± 5, ± 7, ± 8, ± 9, ± 11, ± 13, ± 15, ± 16, ± 17 modulo 36 with three colors, and parts congruent to ± 2, ± 6, ± 10, ± 12, ± 14 modulo 36 with four colors.

Let $P_5(n)$ denote the number of partitions of n into parts congruent to ± 1, ± 2, ± 3, ± 4, ± 5, ± 6, ± 7, ± 8, ± 10, ± 11, ± 13, ± 14, ± 15, ± 16, ± 17 modulo 36, parts congruent to ± 12 modulo 36 with two colors, and parts congruent to ± 9 modulo 36 with three colors.

Let $P_6(n)$ denote the number of partitions of n into parts congruent to ± 3, ± 15, ± 17 modulo 36, parts congruent to ± 1, ± 4, ± 5, ± 6, ± 7, ± 8, ± 10, ± 11, ± 13, ± 16, ± 18 modulo 36 with two colors, parts congruent to ± 2, ± 12, ± 14 modulo 36 with three colors, and parts congruent to ± 9 modulo 36 with four colors.

Theorem 3. *For any positive integer $n \geq 3$, we have*

$$P_1(n) - P_2(n-3) + P_3(n-1) - \frac{1}{2}P_4(n+1) + \frac{1}{2}P_5(n+1) - P_6(n-2) = 0, \quad (32)$$

where $P_i(n)$, $1 \leq i \leq 6$, are defined above.

Proof. Using Identities (2)-(5) in Identity (16), and then rewriting all the products on both sides of the resulting identity subject to the common base q^{36}, we obtain

$$
\begin{aligned}
&\left\{\frac{1}{(q^{\pm7},q^{\pm11},q^{18};q^{36})_\infty^2(q^{\pm1},q^{\pm3},q^{\pm4},q^{\pm5},q^{\pm8},q^{\pm10},q^{\pm13},q^{\pm15},q^{\pm16},q^{\pm17};q^{36})_\infty^3}\right.\\
&\left.\times\frac{1}{(q^{\pm2},q^{\pm6},q^{\pm12},q^{\pm14};q^{36})_\infty^4\quad(q^{\pm9};q^{36})_\infty^5}\right\}\\
&-\left\{\frac{q^3}{(q^{\pm5},q^{\pm13},q^{18};q^{36})_\infty^2(q^{\pm1},q^{\pm2},q^{\pm3},q^{\pm4},q^{\pm7},q^{\pm8},q^{\pm11},q^{\pm15},q^{\pm16},q^{\pm17};q^{36})_\infty^3}\right.\\
&\left.\times\frac{1}{(q^{\pm6},q^{\pm10},q^{\pm12},q^{\pm14};q^{36})_\infty^4\quad(q^{\pm9};q^{36})_\infty^5}\right\}\\
&+\left\{\frac{q}{(q^{\pm1},q^{\pm3},q^{\pm17},q^{18};q^{36})_\infty^2(q^{\pm4},q^{\pm5},q^{\pm7},q^{\pm8},q^{\pm11},q^{\pm13},q^{\pm14},q^{\pm15},q^{\pm16};q^{36})_\infty^3}\right.\\
&\left.\times\frac{1}{(q^{\pm2},q^{\pm6},q^{\pm10},q^{\pm12};q^{36})_\infty^4\quad(q^{\pm9};q^{36})_\infty^5}\right\}
\end{aligned}
$$

$$
\begin{aligned}
&-\left\{\frac{q^{-1}}{2\ (q^{\pm1},q^{\pm3},q^{\pm4},q^{\pm5},q^{\pm7},q^{\pm8},q^{\pm9},q^{\pm11},q^{\pm13},q^{\pm15},q^{\pm16},q^{\pm17};q^{36})_\infty^3}\right.\\
&\left.\times\frac{1}{(q^{\pm2},q^{\pm6},q^{\pm10},q^{\pm12},q^{\pm14};q^{36})_\infty^4}\right\}\\
&+\left\{\frac{q^{-1}}{2\ (q^{\pm1},q^{\pm2},q^{\pm3},q^{\pm4},q^{\pm5},q^{\pm6},q^{\pm7},q^{\pm8},q^{\pm10},q^{\pm11},q^{\pm13},q^{\pm14},q^{\pm15};q^{36})_\infty}\right.\\
&\left.\times\frac{1}{(q^{\pm16},q^{\pm17};q^{36})_\infty(q^{\pm12};q^{36})_\infty^2\quad(q^{\pm9};q^{36})_\infty^3}\right\}\\
&-\left\{\frac{q^2}{(q^{\pm3},q^{\pm15},q^{\pm17};q^{36})_\infty\quad(q^{\pm1},q^{\pm4},q^{\pm5},q^{\pm6},q^{\pm7},q^{\pm8},q^{\pm10},q^{\pm11},q^{\pm13};q^{36})_\infty^2}\right.\\
&\left.\times\frac{1}{(q^{\pm16},q^{18};q^{36})_\infty^2(q^{\pm2},q^{\pm12},q^{\pm14};q^{36})_\infty^3\quad(q^{\pm9};q^{36})_\infty^4}\right\}=0.
\end{aligned}
\tag{33}
$$

Observe that the left-hand side of Equation (33) represents the generating functions for $P_1(n)$, $P_2(n)$, $P_3(n)$, $P_4(n)$, $P_5(n)$, and $P_6(n)$, respectively. Hence, the above equation is equivalent to

$$
\begin{aligned}
&\sum_{n=0}^{\infty}P_1(n)q^n-\sum_{n=0}^{\infty}P_2(n)q^{n+3}+\sum_{n=0}^{\infty}P_3(n)q^{n+1}-\frac{1}{2}\sum_{n=0}^{\infty}P_4(n)q^{n-1}\\
&\quad+\frac{1}{2}\sum_{n=0}^{\infty}P_5(n)q^{n-1}-\sum_{n=0}^{\infty}P_6(n)q^{n+2}=0.
\end{aligned}
$$

Equating the coefficients of q^n $(n\geq 3)$ on both sides of the above relation, we obtain Equation (32). □

Example 1. Table 2 illustrates the case $n = 4$ in Theorem 3.

$P_1(4) = 58$	$4_r,\ 4_g,\ 4_y,\ 3_r + 1_r,\ 3_r + 1_g,\ 3_r + 1_y,\ 3_g + 1_r,\ 3_g + 1_g,$ $3_g + 1_y,\ 3_y + 1_r,\ 3_y + 1_g,\ 3_y + 1_y,\ 2_r + 2_r,\ 2_r + 2_g,$ $2_g + 2_g,\ 2_r + 2_y,\ 2_y + 2_y,\ 2_r + 2_w,\ 2_w + 2_w,$ $2_g + 2_y,\ 2_g + 2_w,\ 2_y + 2_w,\ 2_r + 1_r + 1_r,$ $2_r + 1_r + 1_g,\ 2_r + 1_r + 1_y,\ 2_r + 1_r + 1_y,$ $2_r + 1_y + 1_y,\ 2_r + 1_g + 1_y,\ 2_g + 1_r + 1_r,\ 2_g + 1_r + 1_g,$ $2_g + 1_r + 1_y,\ 2_g + 1_r + 1_y, 2_g + 1_y + 1_y,\ 2_g + 1_g + 1_y,$ $2_y + 1_r + 1_r,\ 2_y + 1_r + 1_g,\ 2_y + 1_r + 1_y,\ 2_y + 1_r + 1_y,$ $2_y + 1_y + 1_y,\ 2_y + 1_g + 1_y,\ 2_w + 1_r + 1_r,\ 2_w + 1_r + 1_g,$ $2_w + 1_r + 1_y,\ 2_w + 1_r + 1_y, 2_w + 1_y + 1_y,\ 2_w + 1_g + 1_y,$ $1_r + 1_r + 1_r + 1_r,\ 1_r + 1_r + 1_r + 1_g,\ 1_r + 1_r + 1_g + 1_g,$ $1_r + 1_g + 1_g + 1_g,\ 1_g + 1_g + 1_g + 1_g,\ 1_r + 1_r + 1_g + 1_y,$ $1_r + 1_r + 1_y + 1_y,\ 1_r + 1_y + 1_y + 1_y,\ 1_y + 1_y + 1_y + 1_y,$ $1_g + 1_g + 1_g + 1_y,\ 1_g + 1_g + 1_y + 1_y,\ 1_r + 1_y + 1_y + 1_y$
$P_2(1) = 3$	$1_r,\ 1_g,\ 1_y$
$P_3(3) = 14$	$3_r,\ 3_g,\ 2_r + 1_r,\ 2_r + 1_g,\ 2_g + 1_r,\ 2_g + 1_g,\ 2_y + 1_r,$ $2_y + 1_g,\ 2_w + 1_r,\ 2_w + 1_g,\ 1_r + 1_r + 1_r,\ 1_r + 1_r + 1_g,$ $1_r + 1_g + 1_g,\ 1_g + 1_g + 1_g$
$P_4(5) = 133$	$5_r,\ 5_g,\ 5_y,\ 4_r + 1_r,\ 4_r + 1_g,\ 4_r + 1_y,$ $4_g + 1_r,\ 4_g + 1_g,\ 4_g + 1_y,\ 4_y + 1_r,\ 4_y + 1_g,\ 4_y + 1_y,$ $3_r + 2_r,\ 3_r + 2_g,\ 3_r + 2_y,\ 3_r + 2_w,\ 3_g + 2_r,\ 3_g + 2_g$ $3_g + 2_y,\ 3_g + 2_w,\ 3_y + 2_r,\ 3_y + 2_g,\ 3_y + 2_y,\ 3_y + 2_w$ $3_r + 1_r + 1_r,\ 3_r + 1_r + 1_g,\ 3_r + 1_g + 1_g,\ 3_r + 1_r + 1_y,$ $3_r + 1_y + 1_y,\ 3_r + 1_g + 1_y,\ 3_g + 1_r + 1_r,\ 3_g + 1_r + 1_g,$ $3_g + 1_g + 1_g,\ 3_g + 1_r + 1_y,\ 3_g + 1_y + 1_y,\ 3_g + 1_g + 1_y,$ $3_y + 1_r + 1_r,\ 3_y + 1_r + 1_g,\ 3_y + 1_g + 1_g,\ 3_y + 1_r + 1_y,$ $3_y + 1_y + 1_y,\ 3_y + 1_g + 1_y,\ 2_r + 2_r + 1_r,\ 2_r + 2_g + 1_r,$ $2_g + 2_g + 1_r,\ 2_r + 2_y + 1_r,\ 2_y + 2_y + 1_r,\ 2_r + 2_w + 1_r,$ $2_w + 2_w + 1_r,\ 2_g + 2_y + 1_r,\ 2_g + 2_w + 1_r,\ 2_y + 2_w + 1_r,$ $2_r + 2_r + 1_g,\ 2_r + 2_g + 1_g,\ 2_g + 2_g + 1_g,\ 2_r + 2_y + 1_g,$ $2_y + 2_y + 1_g,\ 2_r + 2_w + 1_g,\ 2_w + 2_w + 1_g,\ 2_g + 2_y + 1_g,$ $2_g + 2_w + 1_g,\ 2_y + 2_w + 1_g,\ 2_r + 2_r + 1_y,\ 2_r + 2_g + 1_y$ $2_g + 2_g + 1_y,\ 2_r + 2_y + 1_y,\ 2_y + 2_y + 1_y,\ 2_r + 2_w + 1_y,$ $2_w + 2_w + 1_y,\ 2_g + 2_y + 1_y, 2_g + 2_w + 1_y,\ 2_y + 2_w + 1_y$ $2_r + 1_r + 1_r + 1_r,\ 2_r + 1_r + 1_r + 1_g,\ 2_r + 1_r + 1_g + 1_g,$ $2_r + 1_g + 1_g + 1_g,\ 2_r + 1_r + 1_r + 1_y,\ 2_r + 1_r + 1_y + 1_y,$ $2_r + 1_y + 1_y + 1_y,\ 2_r + 1_g + 1_g + 1_y,\ 2_r + 1_g + 1_y + 1_y,$ $2_r + 1_r + 1_g + 1_y,\ 2_g + 1_r + 1_r + 1_r,\ 2_g + 1_r + 1_r + 1_g,$ $2_g + 1_r + 1_g + 1_g,\ 2_g + 1_g + 1_g + 1_g,\ 2_g + 1_r + 1_r + 1_y,$ $2_g + 1_r + 1_y + 1_y,\ 2_g + 1_y + 1_y + 1_y,\ 2_g + 1_g + 1_g + 1_y,$ $2_g + 1_g + 1_y + 1_y,\ 2_g + 1_r + 1_g + 1_y$

	$2_y+1_r+1_r+1_r,\ 2_y+1_r+1_r+1_g,\ 2_y+1_r+1_g+1_g,$ $2_y+1_g+1_g+1_g,\ 2_y+1_r+1_r+1_y,\ 2_y+1_r+1_y+1_y,$ $2_y+1_y+1_y+1_y,\ 2_y+1_g+1_g+1_y,\ 2_y+1_g+1_y+1_y,$ $2_y+1_r+1_g+1_y,\ 2_w+1_r+1_r+1_r,\ 2_w+1_r+1_r+1_g,$ $2_w+1_r+1_g+1_g,\ 2_w+1_g+1_g+1_g,\ 2_w+1_r+1_r+1_y,$ $2_w+1_r+1_y+1_y,\ 2_w+1_y+1_y+1_y,\ 2_w+1_g+1_g+1_y,$ $2_w+1_g+1_y+1_y,\ 2_w+1_r+1_g+1_y$ $1_r+1_r+1_r+1_r+1_r,\ 1_r+1_r+1_r+1_r+1_g,$ $1_r+1_r+1_r+1_g+1_g,\ 1_r+1_r+1_g+1_g+1_g,$ $1_g+1_g+1_g+1_g+1_y,\ 1_g+1_g+1_g+1_y+1_y,$ $1_g+1_g+1_y+1_y+1_y,\ 1_g+1_y+1_y+1_y+1_y,$ $1_r+1_r+1_r+1_r+1_y,\ 1_r+1_r+1_r+1_y+1_y,$ $1_r+1_r+1_y+1_y+1_y,\ 1_r+1_y+1_y+1_y+1_y,$ $1_y+1_y+1_y+1_y+1_y,\ 1_r+1_r+1_r+1_g+1_y,$ $1_g+1_g+1_g+1_r+1_y,\ 1_y+1_y+1_y+1_r+1_g,$ $1_r+1_r+1_g+1_g+1_y,\ 1_r+1_r+1_y+1_y+1_g,$ $1_g+1_g+1_y+1_y+1_r$
$P_5(5)=7$	$5,\ 4+1,\ 3+2, 3+1+1,\ 2+2+1,\ 2+1+1+1,$ $1+1+1+1+1$
$P_6(2)=6$	$2_r,\ 2_g,\ 2_y,\ 1_r+1_r,\ 1_g+1_g,\ 1_r+1_g$

Table 2: Partition function values and colored partitions

Let $Q_1(n)$ denote the number of partitions of n into parts congruent to ±2, ±5, ±6 modulo 18 and parts congruent to ±1, ±8, 9 modulo 18 with two colors. Let $Q_2(n)$ denote the number of partitions of n into parts congruent to ±4, ±6, ±7, ±8 modulo 18, parts congruent to ±1 modulo 18 with two colors, and parts congruent to 9 modulo 18 with four colors. Let $Q_3(n)$ denote the number of partitions of n into parts congruent to ±1, ±2, ±4, ±5, ±6, ±7, ±8 modulo 18 and parts congruent to 9 modulo 18 with two colors. Let $Q_4(n)$ denote the number of partitions of n into parts congruent to ±1, ±2, ±3, ±4, ±5, ±6, ±7, ±8 modulo 18. Let $Q_5(n)$ denote the number of partitions of n into parts congruent to ±1, ±2, ±4, ±5, ±6, ±7, ±8 modulo 18 and parts congruent to 9 modulo 18 with two colors.

Theorem 4. *For any positive integer $n \geq 1$, we have*

$$Q_1(n) - Q_2(n-1) + Q_3(n) - Q_4(n) - Q_5(n) = 0, \tag{34}$$

where $Q_i(n)$, $1 \leq i \leq 5$, are defined above.

Proof. Using Identities (2)-(5), in Identity (27), and then rewriting all the products on both sides of the resulting identity subject to the common base q^{18}, we obtain

$$\begin{aligned}
&\frac{1}{(q^{\pm 1},q^{\pm 8},q^{9};q^{18})^2_\infty (q^{\pm 2},q^{\pm 5},q^{\pm 6},q^{\pm 8};q^{18})_\infty}\\
&-\frac{q}{(q^{\pm 1};q^{18})^2_\infty (q^{\pm 4},q^{\pm 6},q^{\pm 7},q^{\pm 8};q^{18})_\infty (q^{9};q^{18})^4_\infty}\\
&+\frac{1}{(q^{\pm 1},q^{\pm 2},q^{\pm 4},q^{\pm 5},q^{\pm 6},q^{\pm 7},q^{\pm 8};q^{18})_\infty (q^{9};q^{18})^2_\infty}\\
&-\frac{1}{(q^{\pm 1},q^{\pm 2},q^{\pm 3},q^{\pm 4},q^{\pm 5},q^{\pm 6},q^{\pm 7},q^{\pm 8};q^{18})_\infty}\\
&-\frac{1}{(q^{\pm 1},q^{\pm 2},q^{\pm 4},q^{\pm 5},q^{\pm 6},q^{\pm 7},q^{\pm 8};q^{18})_\infty (q^{9},q^{18})^2} = 0. \qquad (35)
\end{aligned}$$

Observe that the left-hand side of Equation (35) represents the generating functions for $Q_1(n)$, $Q_2(n)$, $Q_3(n)$, $Q_4(n)$, and $Q_5(n)$, respectively. Hence it is equivalent to

$$\sum_{n=0}^{\infty} Q_1(n)q^n - q\sum_{n=0}^{\infty} Q_2(n)q^n + \sum_{n=0}^{\infty} Q_3(n)q^n - \sum_{n=0}^{\infty} Q_4(n)q^n - \sum_{n=0}^{\infty} Q_5(n)q^n = 0,$$

where we set $Q_1(0) = Q_2(0) = Q_3(0) = Q_4(0) = Q_5(0) = 1$. Equating coefficients of q^n $(n \geq 1)$ on both sides yields of the above relation, with which we obtain Equation (34). □

Example 2. Table 3 illustrates the case $n = 5$ in the above theorem.

$P_1(5) = 13$	$5,\ 2+2+1_r,\ 2+2+1_g,\ 2+1_r+1_r+1_r,\ 2+1_r+1_r+1_g,$ $2+1_r+1_g+1_g,\ 2+1_g+1_g+1_g,\ 1_r+1_r+1_r+1_r+1_r$ $1_r+1_r+1_r+1_r+1_g,\ 1_r+1_r+1_g+1_g+1_g,$ $1_r+1_r+1_r+1_g+1_g,\ 1_r+1_g+1_g+1_g+1_g,$ $1_g+1_g+1_g+1_g+1_g$
$P_2(4) = 6$	$4,\ 1_r+1_r+1_r+1_r,\ 1_r+1_r+1_r+1_g,$ $1_r+1_r+1_g+1_g,\ 1_r+1_g+1_g+1_g,\ 1_g+1_g+1_g+1_g$
$P_3(5) = 5$	$5,\ 4+1,\ 2+2+1, 2+1+1+1,\ 1+1+1+1+1$
$P_4(5) = 7$	$5,\ 4+1,\ 3+2,\ 3+1+1,$ $2+2+1,\ 2+1+1+1,\ 1+1+1+1+1$
$P_5(5) = 5$	$5,\ 4+1,\ 2+2+1,\ 2+1+1+1,\ 1+1+1+1+1$

Table 3: Partition function values and colored partitions

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