RAINBOW-FREE COLORINGS AND RAINBOW NUMBERS FOR $x - y = z^k$

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Abstract

An exact r-coloring of a set S is a surjective function $c : S \to \{1, 2, \ldots, r\}$. A rainbow solution to an equation over S is a solution such that all components are a different color. We prove that every 3-coloring of $\mathbb{N}$ with an upper density greater than $(4^s - 1)/(3 \cdot 4^s)$ contains a rainbow solution to $x - y = z^k$. The rainbow number for an equation in the set S is the smallest integer r such that every exact r-coloring has a rainbow solution. We compute the rainbow numbers of $\mathbb{Z}_p$ for the equation $x - y = z^k$, where p is prime and $k \geq 2$.

1. Introduction

Given a set S, a coloring is a function that assigns a color to each element of S. While Ramsey theory is the study of the existence of monochromatic subsets, anti-Ramsey theory is the study of rainbow subsets. A subset $X \subseteq S$ is a *rainbow* subset if each element in X is assigned a distinct color. For example, in the equation

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$x_1 + x_2 = x_3$, a *rainbow solution* is a solution $\{a, b, a + b\}$ in a set, for instance $\mathbb{Z}$ or $\mathbb{Z}_n$, such that each of a, b, $a + b$ are assigned a distinct color. A coloring is said to be rainbow-free for an equation if no rainbow solutions exist. Several papers have looked at the existence of rainbow-free 3-colorings for linear equations over $\mathbb{Z}$ and $\mathbb{Z}_n$ in [2], [8], [9], and [10]. In [14], Zhan studied the existence of rainbow-free colorings for the equation $x - y = z^2$ over $\mathbb{Z}$ with certain density conditions.

The rainbow number of S for eq, denoted $\mathrm{rb}(S, \mathrm{eq})$, is the smallest number of colors such that for every exact $\mathrm{rb}(S, \mathrm{eq})$-coloring of S, there exists a rainbow solution to eq. Several papers have looked at rainbow numbers over $\mathbb{Z}_n$. For instance, the authors in [5] looked at anti-van der Waerden numbers over both $\mathbb{Z}$ and $\mathbb{Z}_n$. The rainbow numbers of the equation $x_1 + x_2 = kx_3$ in $\mathbb{Z}_p$ were studied in [3], and rainbow numbers of linear equations $a_1x_1 + a_2x_2 + a_3x_3 = b$ over $\mathbb{Z}_n$ were computed in [1]. In [7], the authors consider rainbow numbers of $x_1 + x_2 = x_3$ over subsets $[m] \times [n]$ of $\mathbb{Z} \times \mathbb{Z}$.

In this paper, we generalize the results of Zhan in [14] to classify rainbow-free 3-colorings for the equation $x - y = z^k$ for $k \geq 2$. We also compute the rainbow number of $\mathbb{Z}_n$ for $x - y = z^k$.

In Section 2, we establish some preliminary notation and prove results on rainbow solutions to the equation $x - y = z^k$ in a 3-coloring of the natural numbers. The first result extends Theorem 1 of [14] to equations of the form $x - y = z^k$ for $k \geq 2$. In Section 3 we show the existence of rainbow solutions to the equation $x - y = z^k$ in three-colorings of the natural numbers that satisfy a density condition on the sizes of the color classes. In Section 4, we consider the modular case. We establish bounds on the color classes in rainbow-free colorings of $\mathbb{Z}_n$. We then establish a connection between rainbow-free colorings and the function digraph for the function $f(x) = x^k$ whose edges are of the form (x, x^k). Using these digraphs, we compute the rainbow numbers of $\mathbb{Z}_p$ for $x - y = z^k$ when k is prime.

2. Rainbow-Free 3-Colorings of $x - y = z^k$ in $\mathbb{N}$

In this section, we employ the same approach as Zhan to extend [14] Theorem 1 for the quadratic equation $x - y = z^2$ to equations of the form $x - y = z^k$, where $k \geq 2$.

An exact r-coloring of a set S is a surjective function $c : S \to [r]$. Throughout this section, let $c : \mathbb{N} \to \{R, B, G\}$ be an exact 3-coloring of the set of natural numbers. The coloring induces a partition into three disjoint color classes, which we call $\mathcal{R}$ $\mathcal{B}, \mathcal{G}$ color classes of red, blue, and green, respectively, where $\mathcal{R} = \{i \mid c(i) = R\}$. For any subset S of $\mathbb{N}$ and any $n \in \mathbb{N}$, define $S(n) = |[n] \cap S|$; hence $\mathcal{R}(n) = |[n] \cap \mathcal{R}|$. We define $\mathcal{B}(n)$ and $\mathcal{G}(n)$ in a similar way. A rainbow solution in $\mathbb{N}$ to the equation $x - y = z^k$ is a solution (a_1, a_2, a_3) so that $\{c(a_1), c(a_2), c(a_3)\} = \{R, G, B\}$; that is, each component of the solution has a different color. We say that c is *rainbow-free* for $x-y=z^k$ if there is no rainbow solution to the equation $x-y=z^k$ with respect to c.

A string of length ℓ at position i consists of numbers i, $i+1$, $i+2, \ldots, i+\ell-1$ where $i, \ell \in \mathbb{N}$. A string is *monochromatic* if it contains only one color. Similarly a string is *bichromatic* if it contains exactly two colors. Observe that a color is *dominant* if every bichromatic string contains that color. Note that if a dominant color exists for a 3-coloring it must be unique and nondominant colors cannot be adjacent. A string is an R-monochromatic string of length ℓ at position i if $c(i)=R$ for $i \le j \le i+\ell-1$ and $c(i-1), c(i+\ell) \ne R$. We similarly define B-monochromatic strings and G-monochromatic strings.

The following lemmas are direct extensions of [14] to the equation $x-y=z^k$. We include their proofs here for completeness.

Lemma 1. *Let $c:\mathbb{N} \to [r]$ be an exact rainbow-free coloring for $x-y=z^k$. Then $c(1)$ is dominant.*

Proof. Without loss of generality, assume $c(1)$ is red. It suffices to show that if $c(i) \ne c(i+1)$, then either $c(i)=R$ or $c(i+1)=R$. Since $(i+1, i, 1)$ is a solution to $x-y=z^k$, either $c(i)=R$ or $c(i+1)=R$. Thus, red is a dominant color, as desired. □

Throughout the rest of the paper we will assume that R is a dominant color. Here we establish that that B- or G-monochromatic strings remain monochromatic after moving j^k positions.

Lemma 2. *Let $c:\mathbb{N} \to \{R,G,B\}$ be rainbow-free for $x-y=z^k$ with dominant color R. If $c(j)=B$, then for any monochromatic string at position i of length ℓ of color G, the string of position $i \pm j^k$ of length ℓ is monochromatic of color either B or G.*

Proof. Suppose $c(j)=B$ and $c(i)=c(i+1)=\ldots=c(i+\ell-1)=G$. For all $0 \le h \le \ell-1$, $(i+h+j^k, i+h, j)$ is a solution to $x-y=z^k$, and so $c(i+h+j^k) \in \{B,G\}$. As B and G are nondominant colors, the string at position $i+j^k$ of length ℓ must be monochromatic of either blue or green. Since $(i+h, i+h-j^k, j)$ is a solution to $x-y=z^k$, a similar argument shows that the strings at position $i-j^k$ of length ℓ are monochromatic of color either B or G. □

Lemma 3 ([14])**.** *Suppose some set $S \subset \mathbb{N}$ satisfies $\limsup_{n\to\infty}(S(n)-\frac{n}{n_0})=\infty$ for some integer $n_0 \ge 2$. then there exists a $d \le n_0-1$ such that for any i, there exists $j>i$ such that j and $j+d$ are both elements of S.*

Lemma 4. *Let $c:\mathbb{N} \to \{R,G,B\}$ be rainbow-free for $x-y=z^k$ such that*

$$\limsup_{n\to\infty}\left(\min\{\mathcal{R}(n), \mathcal{B}(n), \mathcal{G}(n)\} - \frac{n}{n_0}\right) = \infty$$

for some integer $n_0 \geq 2$. *Then the length of every nondominant monochromatic string is bounded above.*

Proof. Since the upper density of every color class is finite, there cannot be any monochromatic strings of infinite length.

Suppose that R is the dominant color in c. Let $i_0 = \min\{i \in \mathbb{N} \,|\, c(i) = B\}$. Assume, for the sake of contradiction, that there exist G-monochromatic strings of arbitrary length. Suppose there exists a G-monochromatic string at position $j \geq i_0$ of length ℓ such that $\ell \geq i_0^k$. Without loss of generality, let j be the first green element in the string so that $c(j-1) = R$. Then $c(j + i_0^k - 1) = G$. It follows that $(j + i_0^k - 1, j - 1, i_0)$ is a rainbow solution to the equation $x - y = z^k$, which is a contradiction. Hence, the lengths of nondominant monochromatic strings in c are bounded above. □

An infinite arithmetic progression with initial term i and common difference d is monochromatic if $c(i) = c(i + hd)$ for all $h \in \mathbb{N}$. An element $j \in \mathbb{N}$ has the *A-property* if $c(j)$ is nondominant and there exists a monochromatic infinite arithmetic progression of the other nondominant color with common difference j^k.

Lemma 5. *Let* $c : \mathbb{N} \to \{R, G, B\}$ *be rainbow-free for* $x - y = z^k$ *with dominant color* R. *If* $c(j_1) = c(j_2) \neq R$ *and* $\gcd(j_1, j_2) = 1$, *then at most one of* j_1 *and* j_2 *has the A-property.*

Proof. Suppose j_1 and j_2 satisfy $\gcd(j_1, j_2) = 1$ and $c(j_1) = c(j_2) = G$. Furthermore, assume by way of contradiction that both j_1 and j_2 have the A-property. Then there exist two B-monochromatic infinite arithmetic progressions, one with initial term i_1 and common difference j_1^k and the other with initial term i_2 and common difference j_2^k.

Suppose the B-monochromatic string at position i_1 has length ℓ_0. By Lemma 2, there exist B-monochromatic strings of length ℓ_0 at all integers of the form $i_1 + mj_1^k$ and $i_2 + mj_1^k$ for all non-negative integers m. Since $\gcd(j_1^k, j_2^k) = 1$, there exist positive integers u_1, u_2 such that $u_1 j_1^k - u_2 j_2^k = i_2 - i_1$, and so $i_1 + u_1 j_1^k = i_2 + u_2 j_2^k$. This gives a common value in both arithmetic progressions; call this common value $i_3 = i_1 + u_1 j_1^k = i_2 + u_2 j_2^k$.

Consider the B-monochromatic string at position i_3 of length ℓ_1. Since i_3 is part of both B-monochromatic arithmetic progressions, by Lemma 2 there exist B-monochromatic strings of length ℓ_1 at positions $i_3 + mj_1^k$ and $i_3 + mj_2^k$ for all non-negative m. Since $\gcd(j_1, j_2) = 1$, there exist integers v_1 and v_2 such that $v_1 j_1^k - v_2 j_2^k = 1$. Thus, $(i_3 + v_1 j_1^k) - (i_3 + v_2 j_2^k) = 1$, so $i_3 + v_1 j_1^k = 1 + i_3 + v_2 j_2^k$, and $c(i_3 + v_1 j_1^k) = c(i_3 + v_2 j_2^k) = B$.

Applying Lemma 2 again, there exists a B-monochromatic string of length ℓ_1 at position $i_3 + v_2 j_2^k$. Since $i_3 + v_1 j_1^k = 1 + i_3 + v_2 j_2^k$, there exists a B-monochromatic string of length $\ell_1 + 1$ at $i_3 + v_1 j_1^k$. This process can be repeated to obtain arbitrarily long B-monochromatic strings, contradicting Lemma 4. Therefore, j_1 and j_2 cannot both have the A-property. □

Again following the approach of [14], we introduce some notation here that will be used in the following lemma. Define the magnitude function $M(u, v, w, i, D) = i + ud_1^k + vd_2^k + wd_3^k$, where $u, v, w, i \in \mathbb{Z}$ and $D = (d_1, d_2, d_3)$. Let m be the length of the lattice path $P = \{(a_{p,1}, b_{p,1}), (a_{p_2}, b_{p_2}), \ldots, (a_{p,m}, b_{p,m})$.

The following theorem gives that when each nondominant color class contains a pair of relatively prime elements, a rainbow solution must exist.

Theorem 1. *Let $c : \mathbb{N} \to \{R, G, B\}$ be a coloring with a dominant color R, satisfying the density condition of Lemma 4. Assume that there exist i, $i_1 \in \mathcal{B}$ contains a pair of integers i, i_1 such that $\gcd(i, i_1) = 1$ and there exist $j, j_1 \in \mathcal{G}$ such that $\gcd(j, j_1) = 1$. Then c contains a rainbow solution to $x - y = z^k$.*

Proof. Assume by contradiction that c is a rainbow-free coloring for $x - y = z^k$. Suppose that i, $i_1 \in \mathcal{B}$ with $\gcd(i, i_1) = 1$ and j, $j_1 \in \mathcal{G}$ with $\gcd(j, j_1) = 1$. According to Lemma 5, at most one of i and i_1 have the A-property, and at most one of j and j_1 have the A-property. Without loss of generality, assume that i and j do not have the A-property. Since $\gcd(i_1, i, j) = 1$, there exist integers u_0, v_0, w_0 such that $u_0 > 0$ and v_0, $w_0 < 0$ such that $u_0 i_1^k + v_0 i^k + w_0 j^k = -1$. Let $D = (i_1, i, j)$.

Choose i_2 such that $c(i_2) \neq R$ and $i_2 > \max\{-v_0 i^k, -w_0 j^k\} \geq i$. Let ℓ be the length of the nondominant monochromatic string at position i_2. Construct a lattice path as follows: let $(\alpha_0, \beta_0) = (0, 0)$ and recursively define (α_t, β_t) by

$$(\alpha_{t+1}, \beta_{t+1}) = \begin{cases} (\alpha_t + 1, \beta_t) & \text{if } c(M(\alpha_t, 0, \beta_t, i_2, D)) = G \\ (\alpha_t, \beta_t + 1) & \text{if } c(M(\alpha_t, 0, \beta_t, i_2, D)) = B. \end{cases}$$

By Lemma 2, at each $M(\alpha_t, 0, \beta_t, i_2, D)$ there exists a nondominant monochromatic string of length ℓ. Suppose that $\alpha_t < u_0$ for all t. Then for some t, the string of β_t must increase infinitely many times consecutively, so $M(\alpha_t, 0, \beta_t, i_2, D) = i_2 + \alpha_t i_1^k + \beta_t j^k$ are all blue for t sufficiently large. This gives an infinitely long blue monochromatic sequence with common difference of $M(\alpha_{t+1}, 0, \beta_{t+1}, i_2, D) - M(\alpha_t, 0, \beta_t, i_2, D) = j^k$. Since $c(j) = G$, this contradicts that j does not have has the A-property. Therefore, there exists some t_0 such that $\alpha_{t_0} = u_0$. Let $q_1 = \beta_{t_0}$ and so we consider the point $(\alpha_{t_0}, \beta_{t_0}) = (u_0, v_0)$ in the uw-plane. By Lemma 2, there exists a monochromatic nondominant string of length $\ell' \geq \ell$ at position $M(u_0, 0, q_1, i_2, D) = i_2 + u_0 i_1^k + q_1 j^k$.

Construct another lattice path in the wv-plane P_0 as follows. Let $(v_{P_0,1}, w_{P_0,1}) = (0, q_1)$. Recursively define $(v_{P_0,t}, w_{P_0,t})$ by

$$(v_{P_0,t+1}, w_{P_0,t+1}) = \begin{cases} (v_{P_0,t}+1, w_{P_0,t}) & \text{if } c(M(u_0, v_{P_0,t}, w_{P_0,t}, i_2, D)) = G \\ (v_{P_0,t}, w_{P_0,t}-1) & \text{if } c(M(u_0, v_{P_0,t}, w_{P_0,t}, i_2, D)) = B. \end{cases}$$

By construction of P_0 and Lemma 2, there exist monochromatic nondominant strings of length ℓ' at all $M(u_0, v_{P_0,t}, w_{P_0,t}, i_2, D)$. Since i does not have the A-property, there does not exist an infinite green arithmetic progression with common difference i^k. Therefore, there does not exist t' such that for all $t > t'$, $(v_{P_0,t+1}, w_{P_0,t+1}) = (v_{P_0,t}+1, w_{P_0,t})$. Thus, there must exist $q_1 - w_0$ integers t such that $(v_{P_0,t+1}, w_{P_0,t+1}) = (v_{P_0,t}, w_{P_0,t}-1)$. Therefore, there exists some point $(v_{P_0,m_0}, w_{P_0,m_0})$ on P_0 where $w_{P_0,m_0} = w_0$. We terminate P_0 at this point. Note that for all $1 \le t \le m_0$, we have $M(u_0, v_{P_0,t}, w_{P_0,t}, i_2, D) = i_2 + u_0 i_1^k + v_{P_0,t} i^k + w_{P_0,t} j^k > 0$, since $u_0, > 0$, $v_{P_0,t} \ge 0$ and $i_2 + w_{P_0,t} j^k \ge i_2 + w_0 j^k \ge 0$ by construction of P_0 and choice of i_2. Then by Lemma 2 and our construction of P_0, there are nondominant strings of length ℓ' at each position $M(u_0, v_{P_0,t}, w_{P_0,t}, i_2, D)$ for $1 \le t \le m_0$.

We construct another path P_0' as follows. Let $(v_{P_0',1}, w_{P_0',1}) = (0, q_1)$. Recursively define

$$(v_{P_0',t+1}, w_{P_0',t+1}) = \begin{cases} (v_{P_0',t}-1, w_{P_0',t}) & \text{if } c(M(u_0, v_{P_0',t}, w_{P_0',t}, i_2, D)) = G \\ (v_{P_0',t}, w_{P_0',t}+1) & \text{if } c(M(u_0, v_{P_0',t}, w_{P_0',t}, i_2, D)) = B. \end{cases}$$

We again use Lemma 2 to conclude that there exists a nondominant string of length at least ℓ' at all positions of the form $M(u_0, v_{P_0',t}, w_{P_0',t}, i_2, D)$ whenever $M(u_0, v_{P_0',t}, w_{P_0',t}, i_2, D) > 0$ which will be satisfied as long as $v_{P_0',t} > v_0$. Since j does not have the A-property, at some point m_0', we have $v_{P_0',m_0'} = v_0$. Terminate P_0' at the point $(v_{P_0',m_0'}, w_{P_0',m_0'})$.

Let P_1 be the union of P_0 and P_0'. The path P_1 is connected, since $(0, q_1)$ is on both paths and has length $m_0 + m_0' - 1$. Define $(v_{P_1,1}, w_{P_1,1}) = (v_{P_0',m_0'}, w_{P_0',m_0'})$ so that $(v_{P_1,m_0+m_0'-1}, w_{P_1,m_0+m_0'-1}) = (v_{P_0,m_0}, w_{P_0,m_0})$. Define P_1' to be the path in the vw-plane satisfying $(v_{P_1',t}, w_{P_1',t}) = (v_{P_1,t} - v_0, w_{P_1,t} - w_0)$. Note that $(v_{P_1',1}, w_{P_1',1}) = (0, w_{P_0',m_0'} - w_0)$ and $(v_{P_1',t}, w_{P_1',t}) = (v_{P_0,m_0} - v_0, 0)$.

Finally, construct a path P_2 defined as follows. Let $(v_{P_2,1}, w_{P_2,1}) = (0,0)$. Recursively define $(v_{P_0,t}, w_{P_0,t})$ by

$$(v_{P_2,t+1}, w_{P_2,t+1}) = \begin{cases} (v_{P_2,t}+1, w_{P_2,t}) & \text{if } c(M(0, v_{P_2,t}, w_{P_2,t}, i_2, D)) = G \\ (v_{P_2,t}, w_{P_2,t}+1) & \text{if } c(M(0, v_{P_2,t}, w_{P_2,t}, i_2, D)) = B. \end{cases}$$

Again by Lemma 2, there exists a monochromatic nondominant string of length at least ℓ' at all positions of the form $M(0, v_{P_2,t}, w_{P_2,t}, i_2, D)$.

As Figure 1 illustrates, by construction of P_1' and P_2, there must be a point of intersection of the two paths, say (v_0', w_0') with $v_0', w_0' > 0$. Consider the corresponding

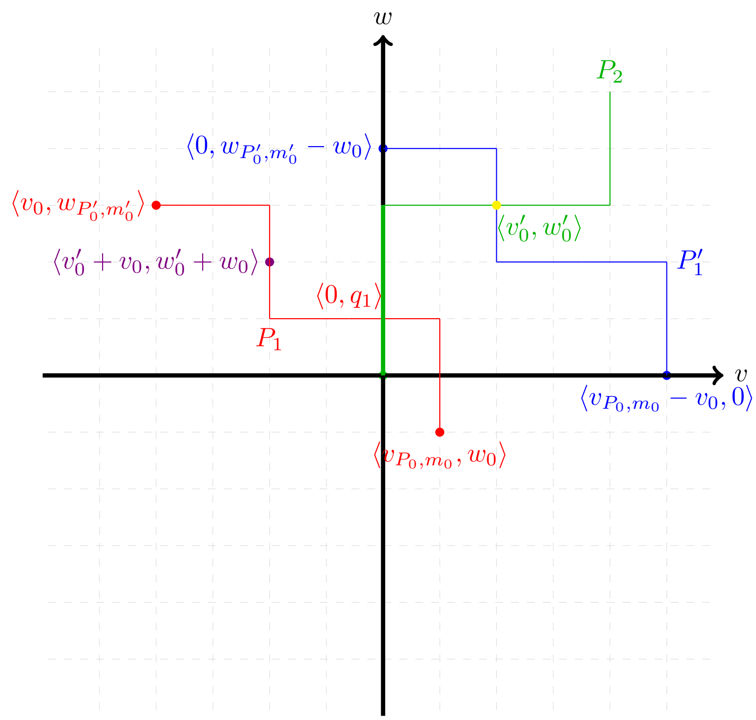

Figure 1: Paths P_1, P_1', and P_2

point $(v_0' + v_0, w_0' + w_0)$ on P_1 which corresponds to magnitude

$$M(u_0, v_0' + v_0, w_0' + w_0, i_2, D) = i_2 + u_0 i_1^k + v_0' i^k + v_0 i^k + w_0' j^k + w_0 j^k.$$

On P_2 the point (v_0', w_0') corresponds to magnitude $M(0, v_0', w_0', i_0, D) = i_2 + v_0' i^k + w_0' j^k$. Subtracting the two magnitudes gives $u_0 i_1^k + v_0 i^k + w_0 j^k = -1$ by choice of u_0, v_0, w_0. Therefore, $M(u_0, v_0' + v_0, w_0' + w_0, i_2, D)$ and $M(0, v_0', w_0', i_0, D)$ are adjacent, positive, and each has a nondominant string of length at least ℓ' in the nondominant color. Thus, a string of length at least $\ell' + 1$ exists at $M(u_0, v_0' + v_0, w_0' + w_0, i_2, D)$, which allows us to generate arbitrarily long nondominant monochromatic strings, contradicting Lemma 4. Thus, we conclude that c contains a rainbow solution to $x - y = z^k$. □

3. A Density Condition for Rainbow- Free Colorings over $\mathbb{N}$

Using Theorem 1, we show that 3-colorings of $\mathbb{N}$ satisfying a certain density condition contain rainbow solutions to $x - y = z^k$. When $k = 2$, the upper density is $\frac{1}{4}$

as in [14].

We use the following generalization of the Frobenius coin problem in the proof of Lemma 6.

Theorem 2. *Suppose two integers i and j satisfy $gcd(i,j) = k$. Then there exists an integer n_0 such that all numbers greater than n_0 divisible by k can be written in the form $ui + vj$ for non-negative integers u and v.*

In the following lemma, we use the stronger density condition to generalize [14, Lemma 7].

Lemma 6. *Let $c : \mathbb{N} \to \{R, G, B\}$ be rainbow-free for $x - y = z^k$ such that*

$$\limsup_{n\to\infty} \left(\min\{\mathcal{R}(n), \mathcal{B}(n), \mathcal{G}(n)\} - \frac{4^s - 1}{3 \cdot 4^s} \right) = \infty$$

where $s = \left\lfloor \frac{k}{2} \right\rfloor$ and R is the dominant color. Then both $\mathcal{B}$ and $\mathcal{G}$ must contain a pair of relatively prime integers.

Proof. Suppose $\mathcal{B}$ and $\mathcal{G}$ contain no pairs of consecutive integers. Since R is a dominant color, for all i, $c(i) = R$ or $c(i+1) = R$. Then for all n, $|\mathcal{R}(n)| \geq n/2$ and so

$$\liminf_{n\to\infty} (\mathcal{R}(n) - (4^s - 1)/(3 \cdot 4^s)) \geq \liminf_{n\to\infty} (n/2 - (4^s - 1)/(3 \cdot 4^s)) \geq 0.$$

Therefore,

$$\limsup_{n\to\infty} (\min\{\mathcal{B}(n), \mathcal{G}(n)\} - (4^s - 1)/(3 \cdot 4^s)) \leq 0,$$

a contradiction. Therefore, there exists an i such that i and $i + 1$ must be in $\mathcal{B}$ or $\mathcal{G}$. Without loss of generality, suppose i and $i+1$ are in B. Then B contains a pair of relatively prime integers.

Assume $\mathcal{G}$ does not have a pair of relatively prime integers. Let d be the minimum difference between any two elements in $\mathcal{G}$. Since $(4^s - 1)/(3 \cdot 4^s) \geq 1/4$, $\mathcal{G}$ satisfies the conditions of Lemma 3 with $n_0 = 4$, so we have that $d \leq 3$. Therefore, there exists a j such that j, $j + d \in \mathcal{G}$.

First consider $d = 2$. Since j and $j+2$ are not relatively prime, $\gcd(j, j+2) = 2$. There exists a B-monochromatic string at position i of length $\ell \geq 2$. By Theorem 2, there exists an integer n_0 such that all integers greater than n_0 that are divisible by $\gcd(j^k, (j+2)^k) = 2^k$ can be expressed in the form $j^k u + (j+2)^k v$ for some non-negative integers u and v. Hence, all integers greater than $i + n_0$ that are congruent to $i \mod 2^k$ can be expressed in the form $i + j^k u + (j+2)^k v$, and so there exist B-monochromatic strings at positions $i + j^k$ and $i + (j+2)^k$ of length at least 2 by Lemma 2. By induction, for any non-negative u and v, there exist B-monochromatic strings at $i + j^k u + (j+2)^k v$ of length at least 2. Thus, at some n_1, there exists a blue string at of length at least 2 at all integers of the form $n_1 + 2^k m$.

Consider a string of length 2^k at position $n_1 + 2^k m$. By our assumption $c(n_1 + 2^k m) = c(n_1 + 2^k m + 1) = B$. By Lemma 1, $c(n_1 + 2^k m + 2) \neq G$ and $c(n_1 + 2^k(m) + 2^k - 1) \neq G$. Since $d = 2$, every green element is followed by a red element, so

$$|\mathcal{G} \cap [n_1 + 2^k m, n_1 + 2^k(m+1) - 1]\}| \leq |\mathcal{R} \cap [n_1 + 2^k m, n_1 + 2^k(m+1) - 1]\}| - 1.$$

When k is even, one has that $\frac{4^s-1}{3\cdot 4^s} = \frac{2^k-1}{3\cdot 2^k}$. Thus, by the density condition for m sufficiently large, we have that

$$\begin{aligned}|\mathcal{G} \cap [n_1 + 2^k m, n_1 + 2^k(m+1) - 1]\}| &> \frac{2^k-1}{3}, \\ |\mathcal{R} \cap [n_1 + 2^k m, n_1 + 2^k(m+1) - 1]\}| &> \frac{2^k-1}{3} + 1, \\ |\mathcal{B} \cap [n_1 + 2^k m, n_1 + 2^k(m+1) - 1]\}| &> \frac{2^k-1}{3},\end{aligned}$$

a contradiction. When k is odd, $\frac{4^s-1}{3\cdot 4^s} = \frac{2^k-2}{3\cdot 2^k}$. By a similar argument we get a contradiction here.

Now suppose $d = 3$. Since j and $j+3$ are not relatively prime, $\gcd(j, j+3) = 3$. There is a monochromatic blue string of length 2 at position i. By Corollary 2, there exists an integer n_2 such that all integers greater than n_2 that are divisible by $\gcd(j^k, (j+3)^k) = 3^k$ can be expressed in the form $j^k u + (j+3)^k v$. As above, there is an integer n_3 such that there exists a blue string of length at least 2 at all numbers of the form $n_3 + 3^k m$. Since every green element is followed by at least two red elements since $d = 3$, the density condition on each color class cannot hold, a contradiction.

Therefore, there exists a relatively prime pair of integers colored green. □

Theorem 3. *Let $s = \lfloor k/2 \rfloor$. Every exact 3-coloring of the set of natural numbers with the upper density of each color class greater than $(4^s - 1)/(3 \cdot 4^s)$ contains a rainbow solution to $x - y = z^k$.*

Proof. Suppose that there is a rainbow-free 3-coloring c of $\mathbb{N}$ for the equation $x - y = z^k$ satisfying the density condition above. By Lemma 1, there exists a dominant color, say red. Since red is dominant, by Lemma 6, $\mathcal{B}$ and $\mathcal{G}$ each contain a pair of relatively prime integers. By Theorem 1, c contains a rainbow-solution, a contradiction. □

4. Rainbow Numbers of $\mathbb{Z}_n$ for $x - y = z^k$

Using the results in the previous sections on rainbow colorings over $\mathbb{Z}$, we compute rainbow numbers for $x - y = z^k$ over $\mathbb{Z}_p$.

Note rainbow-free 3-coloring of $\mathbb{Z}_n$ yield rainbow-free 3-coloring of $\mathbb{N}$.

Lemma 7. *If $\overline{c} : \mathbb{Z}_n \to \{R, G, B\}$ is rainbow-free for $x - y = z^k$, then the coloring $c : \mathbb{N} \to \{R, G, B\}$ given by $c(i) = \overline{c}(i \mod n)$ where $i \equiv j \mod n$ is rainbow-free for $x - y = z^k$.*

The following lemma is used to find pairs of relatively prime pairs in $\mathbb{N}$ in non-dominant colors. The proof in [14] does not depend on the equation.

Lemma 8 ([14])**.** *Let $\overline{c} : \mathbb{Z}_n \to \{R, G, B\}$ be an exact 3-coloring of $\mathbb{Z}_n$ and let $c : \mathbb{N} \to \{R, G, B\}$ be defined by $c(i) = \overline{c}(i \mod n)$. If two integers i_1 and i_2 in $\mathbb{N}$ satisfy $\gcd(|i_1 - i_2|, n) = 1$, then there exists a pair of relatively prime integers j_1 and j_2 where $c(i_1) = c(j_1)$, $c(i_2) = c(j_2)$, and $|i_1 - i_2| = |j_1 - j_2|$.*

The following theorem generalizes [14, Theorem 14].

Theorem 4. *Let n be odd and let r_1 be the smallest prime factor of n. Let $\overline{c} : \mathbb{Z}_n \to \{R, G, B\}$ be an exact 3-coloring of $\mathbb{Z}_n$ with corresponding color classes $\mathcal{R}, \mathcal{B}, \mathcal{G}$. If $\overline{c}$ is rainbow-free for $x - y = z^k$, then $\min\{|\mathcal{R}|, |\mathcal{B}|, |\mathcal{G}|\} \le \frac{n}{r_1}$.*

Proof. Define $c : \mathbb{N} \to \{R, G, B\}$ by $c(i) = \overline{c}(i \mod n)$. By Lemma 7, c is rainbow-free for $x - y = z^k$. Denote the corresponding color classes of c as $\mathcal{R}'$, $\mathcal{G}'$, and $\mathcal{B}'$. By Lemma 1, there exists a dominant color, say R. Suppose by contradiction that $\min\{|\mathcal{R}|, |\mathcal{B}|, |\mathcal{G}|\} > \frac{n}{r_1}$. Since $\limsup_{n' \to \infty} \left(\mathcal{B}'(n') - \frac{n'}{r_1}\right) = \infty$, there exists an i_1 and $k_1 \le r_1 - 1$ such that i_1, $i_1 + k_1 \in \mathcal{B}'$ by Lemma 3. By Lemma 8, there exists a pair of relatively prime integers j_1, j_2 where $c(j_1) = c(i_1) = B$ and $c(j_2) = c(i_1 + k_1) = B$. Similarly there exists a pair of relatively prime integers in $\mathcal{G}$. By Theorem 1, c is not rainbow-free for $x - y = z^k$, a contradiction. □

For primes, we immediately get the following corollary.

Corollary 1. *Let p be prime. Let $\overline{c} : \mathbb{Z}_n \to \{R, G, B\}$ be an exact 3-coloring of $\mathbb{Z}_p$ with corresponding color classes $\mathcal{R}, \mathcal{B}, \mathcal{G}$. If $\overline{c}$ is rainbow-free for $x - y = z^k$, then $\min\{|\mathcal{R}|, |\mathcal{B}|, |\mathcal{G}|\} = 1$.*

For the remainder of the section, we determine the structure of 3-colorings of $\mathbb{Z}_p$ that are rainbow-free for $x - y = z^k$ for p an odd prime.

When p is prime and $a \ne 0$, the set $0, a^k, 2a^k, \ldots, (p-1)a^k$ forms a complete residue system for $\mathbb{Z}_p$. We generalize the notion of a dominant color to this complete residue system. We say that a a^k string of length ℓ at position ia^k consists of numbers $ia^k, (i+1)a^k, \ldots, (i+\ell-1)a^k$, where $i, \ell \in \mathbb{Z}_p$. An a^k-string is *bichromatic* if it contains exactly two colors. A color is *a^k-dominant* if every bichromatic string contains that color. As with dominant colors, if an a^k-dominant color exists for a 3-coloring it must be unique for the complete residue system $0, a^k, 2a^k, \ldots, (p-1)a^k$. Here we generalize Lemma 1 to a^k-dominant colors.

Lemma 9. *If* $c : \mathbb{Z}_p \to \{R, G, B\}$ *is an exact rainbow-free 3-coloring for* $x - y = z^k$, *then* $c(a)$ *is* a^k*-dominant.*

Proof. Without loss of generality, assume $c(a) = R$. It suffices to show that if $c(ia^k) \neq c\left((i+1)a^k\right)$, then either $c(ia^k) = R$ or $c\left((i+1)a^k\right) = R$. Since c is rainbow-free for $x - y = z^k$ and $\left((i+1)a^k, ia^k, a\right)$ is a solution to $x - y = z^k$, we get the desired conclusion. □

Lemma 10. *If* $c : \mathbb{Z}_p \to \{R, G, B\}$ *is an exact rainbow-free 3-coloring for* $x - y = z^k$ *then* $c(a) = c(-a)$.

Proof. Let $a \neq 0$. If k is even, $a^k = (-a)^k$. By Lemma 9, $c(a)$ is a^k- dominant and $c(-a)$ is a^k-dominant. Since a^k-dominant colors are unique, $c(a) = c(-a)$. Now consider k odd. Note that if R is an a^k-dominant color for $0, a^k, 2a^k, \ldots, (p-1)a^k$, it is also $(-a)^k$-dominant for $0, (-a)^k, \ldots, (p-1)a^k$, since the latter is the former in reverse. Since dominant colors are unique, $c(a) = c(-a)$. □

The following corollary follows immediately from Corollary 1 and Lemma 10.

Corollary 2. *Let* $c : \mathbb{Z}_p \to \{R, G, B\}$ *be an exact rainbow-free coloring for* $x - y = z^k$. *If* $c(0) = B$, *then* $\mathcal{B} = \{0\}$. *That is,* 0 *is the only element in its color class.*

To finalize our classification of rainbow-free 3-colorings for $x - y = z^k$ over $\mathbb{Z}_p$, we consider the associated digraph from powers modulo p. For any function $f : \mathbb{Z}_m \to \mathbb{Z}_m$, we construct a digraph that has the elements of $\mathbb{Z}_m$ as vertices and a directed edge (a, b) if and only if $f(a) \equiv b \mod m$.

In some cases, the digraph associated to a function $f(x)$ gives additional structure on rainbow-free colorings for the equation $x - y = f(x)$. If c is a coloring of $\mathbb{Z}_n$ and D a component of G, let $c(D) = \{c(a) | a \in D\}$. A component D is *monochromatic* if $|c(D)| = 1$.

Lemma 11. *Let* G *be a digraph associated to a function* $f(x)$ *on* $\mathbb{Z}_n$ *and let* D *be a component of* G. *Let* $c : \mathbb{Z}_n \to [t]$ *be a rainbow-free exact* t*-coloring of* $\mathbb{Z}_n$ *for the equation* $x - y = f(x)$. *Suppose that* $c(0) \notin c(D)$. *Then* D *is monochromatic.*

Proof. Suppose that D is not monochromatic. Then there exists two adjacent vertices a and $f(a)$ in D such that $c(a) \neq c((f(a))$. Since $c(0) \notin c(D)$, $(f(a), 0, a)$ is a rainbow solution to $x - y = f(x)$. □

Throughout the rest of the section let G_p^k be the digraph associated to the function $f(x) = x^k \mod p$. The structure of such digraphs has been well-studied in [4], [6], [11], [12], and [13]. For example, when $k = 2$ and $p = 10$, we have the digraph as shown in Figure 2.

We use digraphs to classify exact 3-colorings of $\mathbb{Z}_p$ that are rainbow-free for $x - y = z^k$.

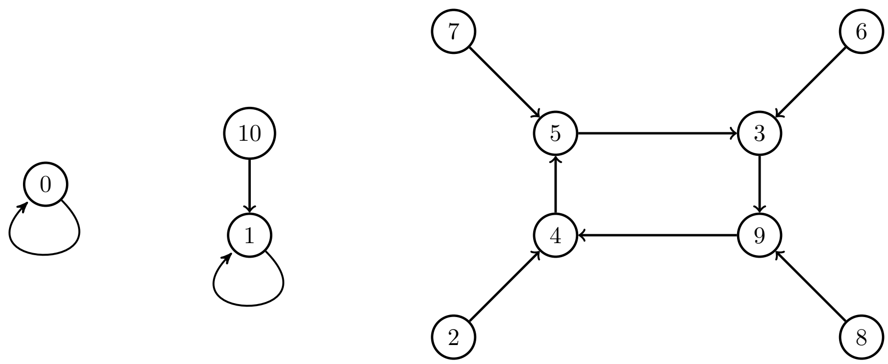

Figure 2: Function digraph for $f(x) = x^2$ over $\mathbb{Z}_{11}$

Theorem 5. *Let $c : \mathbb{Z}_p \to \{R, G, B\}$ be an exact 3-coloring. Then c is rainbow-free for $x - y = z^k$ if and only if the following hold:*

1. *0 is the only element in its color class*
2. *every component of G_p^k is monochromatic*
3. *$c(a) = c(-a)$ for all $a \in \mathbb{Z}_p$.*

Proof. Suppose c is rainbow-free for $x - y = z^k$. Then by Corollary 2, 0 is in its own color class. By Lemma 11 and since 0 is not in any other component, every component of G_p^k is monochromatic. By Lemma 10, $c(a) = c(-a)$ for all $a \in \mathbb{Z}_p$.

Now suppose that 0 is the only element in its color class, every component of G_p^k is monochromatic, and $c(a) = c(-a)$ for all $a \in \mathbb{Z}_p$. We show that c is rainbow-free. Let (a_1, a_2, a_3) be a rainbow solution to $x-y = z^k$. Then one of a_1, a_2, a_3 is 0, since 0 is the only element in its color class. If $a_3 = 0$ then $a_1 = a_2$, contradicting that a_1 and a_2 are distinct colors. If $a_2 = 0$, $a_1 = a_3^k$, so there is a directed edge (a_3, a_1) in the digraph G_p^k, a contradiction, since the components of G_p^k are monochromatic. Finally, suppose that $a_1 = 0$. Then $-a_2 = a_3^k$. There is a directed edge $(a_3, -a_2)$, so $c(a_3) = c(-a_2) = c(a_2)$, a contradiction. Thus, the coloring c is rainbow-free. □

As repeated iteration of $f(x) = x^k$ leads to cycles, G_p^k has the following property.

Lemma 12 ([11])**.** *Let G_p^k be the digraph associated to the function $f(x) = x^k mod p$, where p is prime. Every component of G_p^k contains exactly one cycle.*

The following theorem determines the number of components in the digraph G_p^k. In [11], Lucheta, Miller, and Reiter consider digraphs whose vertices include only nonzero residues. We restate the theorems here for digraphs whose vertices are the elements of $\mathbb{Z}_p$.

Theorem 6 ([11]). *Let p be an odd prime. Let $p-1 = wt$, where t is the largest factor of $p-1$ relatively prime to k. Let $c \neq 0$ be a nonzero vertex of G_p^k. The vertex a is a cycle vertex of G_p^k if and only if $\operatorname{ord}_p a | t$.*

It follows as in [11, Corollary 16] that there are precisely $t+1$ vertices in cycles.

Let p and t be as in Theorem 6. Then G_p^k has exactly 2 components if and only if $t = 1$. Using the proposition, the prime factorizations of k and $p-1$ determine the number of components of G_p^k

Proposition 1. *Let k be even. If $k = 2^{\alpha_0} q_1^{\alpha_1} q_2^{\alpha_2} \dots q_\ell^{\alpha_\ell}$, q_i prime for $1 \leq i \leq \ell$, $\alpha_i \geq 1$, then the digraph G_p^k has two components if and only if $p-1 = 2^{\beta_0} q_1^{\beta_1} q_2^{\beta_2} \dots q_\ell^{\beta_\ell}$ where $\beta_i \geq 0$.*

Proof. As a result of [11, Corollary 16], the number of cycle vertices in G_p^k is the t as in the statement of Theorem 6. It follows from that theorem that $t = 1$ if and only if $p-1 = 2^{\beta_0} q_1^{\beta_1} q_2^{\beta_2} \dots q_\ell^{\beta_\ell}$.

Suppose $t > 1$. The digraph G_p^k has at least 3 cycle vertices. Since $0^k = 0$ and $1^k = 1$, there are at least 2 cycles of length 1, so there must be at least one vertex on a different cycle. Thus, G_p^k has more than 2 components.

If $t = 1$, the only cycles are the length 1 cycles formed by 0 and 1 so G_p^k has two components. □

Proposition 2. *Let $k \geq 3$ be odd and let $k = q_1^{\alpha_1} q_2^{\alpha_2} \dots q_\ell^{\alpha_\ell}$, $q_i > 2$ prime for $1 \leq i \leq \ell$, $\alpha_i \geq 1$. The digraph G_p^k has exactly three components if and only if $p-1 = 2q_1^{\beta_1} q_2^{\beta_2} \dots q_\ell^{\beta_\ell}$ where $\beta_i \geq 0$.*

Proof. It follows from [11, Corollary 16] that the number of cycle vertices in G_p^k is t, as in the statement of Theorem 6. We see that $t = 2$ if and only if $p-1 = 2q_1^{\beta_1} q_2^{\beta_2} \dots q_\ell^{\beta_\ell}$ where $\beta_i \geq 0$.

Suppose $t > 2$. By Theorem 6, G_p^k has at least 4 cycle vertices. Since $0^k = 0$, $(-1)^k = -1$, and $1^k = 1$, there are at least 3 cycles of length 1, so there must be at least one vertex on a different cycle. Thus, G_p^k has more than 3 components.

If $t = 2$, the only cycles are the length 1 cycles formed by 0, 1, and -1 so G_p^k has three components. □

When k is odd, $-a$ may not be in the same component as a, but the components are symmetric.

When an even digraph has at least three components in $\mathbb{Z}_p$, we can give a rainbow-free 3-coloring of $\mathbb{Z}_p$ by coloring the component with 0 using one color, the component with 1 a second color, and coloring everything else a third color. For instance, in the digraph in Figure 2, the coloring $c(0) = R$, $c(1) = c(10) = B$, and $c(2) = \ldots = c(9) = G$ gives a rainbow-free coloring of $\mathbb{Z}_{11}$ for $x - y = z^2$.

We now compute the rainbow number when k is even.

Theorem 7. *Suppose $k = 2^{\alpha_0} q_1^{\alpha_1} q_2^{\alpha_2} \dots q_\ell^{\alpha_\ell}$, q_i prime for $1 \le i \le \ell$, $\alpha_i \ge 1$. Then we have*

$$\mathrm{rb}(\mathbb{Z}_p, x - y = z^k) = \begin{cases} 3 & \text{if } p - 1 = 2^{\beta_0} q_1^{\beta_1} q_2^{\beta_2} \dots q_\ell^{\beta_\ell} \text{ where } \beta_i \ge 0, \\ 4 & \text{otherwise.} \end{cases}$$

Proof. Let $k = 2^{\alpha_0} q_1^{\alpha_1} q_2^{\alpha_2} \dots q_\ell^{\alpha_\ell}$, q_i prime for $1 \le i \le \ell$, $\alpha_i \ge 1$. Suppose $p - 1 = 2^{\beta_0} q_1^{\beta_1} q_2^{\beta_2} \dots q_\ell^{\beta_\ell}$. By Corollary 1, the digraph G_p^k has exactly two components. Let $c : \mathbb{Z}_p \to \{R, G, B\}$ be an exact 3-coloring. Since the components of G_p^k are not monochromatic, by Theorem 5, c contains a rainbow solution to $x - y = z^k$. Thus, $\mathrm{rb}(\mathbb{Z}_p, x - y = z^k) = 3$.

Now suppose $p - 1 \ne 2^{\beta_0} q_1^{\beta_1} q_2^{\beta_2} \dots q_\ell^{\beta_\ell}$. By Corollary 1, the digraph G_p^k has at least 3 components. Define a 3-coloring $c : \mathbb{Z}_p \to \{R, G, B\}$ as follows:

$$c(a) = \begin{cases} R & \text{if } a = 0, \\ B & \text{if } a \text{ is in the same component as } 1, \\ G & \text{otherwise.} \end{cases}$$

Since k is even, (a, a^k) and $(-a, a^k)$ are edges in G_p^k. Thus, a and $-a$ are in the same component for all a. Since each component is monochromatic, 0 is in its own color class, and $c(a) = c(-a)$ for all a, it follows by Theorem 5 that c does not contain a rainbow-solution to $x - y = z^k$. Thus, $\mathrm{rb}(\mathbb{Z}_p, x - y = z^k) \ge 4$.

Suppose $c : \mathbb{Z}_p \to \{R, B, G, Y\}$ is an exact 4-coloring. Suppose that 0 is red. Define an exact 3-coloring $\overline{c} : \mathbb{Z}_p \to \{B, G, Y\}$ by combining the color class that contains 0 with another color class. Since 0 is not in its own color class in $\overline{c}$, by Theorem 5, $\overline{c}$ contains a rainbow solution to $x - y = z^k$. By construction, c also contains a rainbow solution and so every exact 4-coloring contains a rainbow solution. Thus, $\mathrm{rb}(\mathbb{Z}_p, x - y = z^k) \le 4$. □

It follows that the rainbow number of $\mathbb{Z}_p$ for $x - y = z^2$ is 3 if and only if p is a Fermat prime.

When an odd digraph has at least three components in $\mathbb{Z}_p$, we can give a rainbow-free 3-coloring of $\mathbb{Z}_p$ by coloring the component with 0 using one color, the components containing 1 and $p - 1$ with a second color, and coloring everything else a third color.

Theorem 8. *Suppose $k = q_1^{\alpha_1} q_2^{\alpha_2} \dots q_\ell^{\alpha_\ell}$, $q_i \ge 3$ prime for $1 \le i \le \ell$, $\alpha_i \ge 1$, with $k \ge 3$. Then*

$$\mathrm{rb}(\mathbb{Z}_p, x - y = z^k) = \begin{cases} 3 & \text{if } p - 1 = 2 q_1^{\beta_1} q_2^{\beta_2} \dots q_\ell^{\beta_\ell} \text{ where } \beta_i \ge 0, \\ 4 & \text{otherwise.} \end{cases}$$

Proof. Let $k = q_1^{\alpha_1} q_2^{\alpha_2} \dots q_\ell^{\alpha_\ell}$, q_i prime for $1 \le i \le \ell$ and $\alpha_i \ge 1$.

Suppose $p - 1 = 2q_1^{\beta_1} q_2^{\beta_2} \dots q_\ell^{\beta_\ell}$ where $\beta_i \ge 0$. By Corollary 2, the digraph G_p^k has 3 components. Since 1 and -1 are both cycle vertices in G_p^k, 1 and -1 are in distinct components. Let $c : \mathbb{Z}_p \to \{R, G, B\}$ be an exact 3-coloring. If the components are monochromatic, each component must be a distinct color. In particular, $c(1) \ne c(-1)$, so Theorem 5 shows that there exists a rainbow solution to $x - y = z^k$ in c. Otherwise, the components are not chromatic, and again there is a rainbow solution to $x - y = z^k$. Thus, $\mathrm{rb}(\mathbb{Z}_p, x - y = z^k) = 3$.

Now suppose $p - 1 \ne 2q_1^{\beta_1} q_2^{\beta_2} \dots q_\ell^{\beta_\ell}$. By Corollary 2, the digraph G_p^k has at least 4 components and 1 and -1 are in distinct components. Define a 3-coloring $c : \mathbb{Z}_p \to \{R, G, B\}$ as follows:

$$c(a) = \begin{cases} R & \text{if } a = 0, \\ B & \text{if } a \text{ is in the same component as 1 or -1}, \\ G & \text{otherwise.} \end{cases}$$

Suppose that (a, a^k) is an edge in the component containing 1. Then $(-a, -a^k)$ is an edge in the component containing -1. Thus, $c(a) = c(-a)$ for all $a \in \mathbb{Z}_p$. Furthermore, each component is monochromatic, and 0 is in its own color class. It follows by Theorem 5 that c does not contain a rainbow-solution to $x - y = z^k$. Thus, $\mathrm{rb}(\mathbb{Z}_p, x - y = z^k) \ge 4$.

Let $c : \mathbb{Z}_p \to \{R, B, G, Y\}$ be an exact 4-coloring. Suppose that $c(0) = R$. Define an exact 3-coloring $\bar{c} : \mathbb{Z}_p \to \{R, B, G\}$ by combining the red and yellow color classes. That is, $\bar{c}(i) = c(i)$ if $c(i) \in \{R, B, G\}$, and $\bar{c}(i) = R$ if $c(i) = Y$. Since 0 is not in its own color class in $\bar{c}$, according to Theorem 5, $\bar{c}$ contains a rainbow solution to $x - y = z^k$.

By construction, c also contains a rainbow solution. Therefore, we conclude that every exact 4-coloring contains a rainbow solution, and thus, $\mathrm{rb}(\mathbb{Z}_p, x - y = z^k) \le 4$. Hence, we have shown that $\mathrm{rb}(\mathbb{Z}_p, x - y = z^k) = 4$. □

5. Conclusion

The technique of using graphs to study rainbow numbers could be applied to other families of equations when we know that 0 is the only element in its color class in every rainbow-free 3-coloring of $\mathbb{Z}_p$.

For example, consider the equation $a_1 x_1 + a_2 x_2 + a_3 x_3 = 0$. If we take the union of the three digraphs obtained by setting each $x_i = 0$, we get digraphs $f(x_i) = -a_j / a_i x_j$. The associated graph has exactly 2 components precisely when $|\langle d_1, d_2, \dots, d_6 \rangle| = p - 1$, where $d_1 = -a_3 a_1^{-1}$, $d_2 = -a_2 a_1^{-1}$, $d_3 = -a_1 a_2^{-1}$, $d_4 = -a_3 a_2^{-1}$, $d_5 = -a_1 a_3^{-1}$, and $d_6 = -a_2 a_3^{-1}$, recovering the result of [8, Corollary 8].

We can extend these concepts to equations of the form $x - y = mz^k$. Many of the results in Section 2 can be generalized to this equation. By investigating the function digraphs for $f(x) = mx^k$, we can determine the rainbow numbers for this equation over $\mathbb{Z}_p$. Through a similar approach, we conjecture that the rainbow numbers $\mathrm{rb}(\mathbb{Z}_p, x - y = z^k)$ and $\mathrm{rb}(\mathbb{Z}_p, x + y = z^k)$ are equal.

Lastly, when n is composite, it would be intriguing to compute the rainbow numbers for $x - y = z^k$ in $\mathbb{Z}_n$. Considerable knowledge exists about function digraphs G_n^k. The presence of nonlinearity complicates the computation of the rainbow number for $\mathbb{Z}_{pq}$ when p and q are prime compared to the linear equations case explored in [1].

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