THE ODD/EVEN DICHOTOMY FOR THE SET OF NUMBERS THAT ARE BOTH k-FULL AND ℓ-FREE

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Abstract

A $[k,\ell]$-*integer* is a natural number that is both k-full and ℓ-free. Using an elementary method, an asymptotic ratio of the set of odd $[k,\ell]$-integers to that of even $[k,\ell]$-integer is derived.

1. Introduction and Results

Let k and ℓ be two fixed integers such that $2 \le k < \ell$. A natural number n is called ℓ-*free* if for all primes p dividing n, we have that p^ℓ does not divide n. A natural number n is called k-*full* if for all primes p dividing n, we have $p^k \mid n$. A natural number $n_{k,\ell}$ is called $[k,\ell]$-*integer* if for all primes p dividing n, we have $p^k \mid n$ but $p^\ell \nmid n$. Let $N_{k,\ell}$ denote the set of all $[k,\ell]$-integers, and let $N_{k,\ell}(x)$ denote the number of integers that are in $N_{k,\ell}$ and that do not exceed x. In 1995, Krätzel [5] and Seibold [9] studied the asymptotic behavior of $N_{k,\ell}(x)$ and showed that

$$N_{k,\ell}(x) = \sum_{n=k}^{\min\{2k,\ell\}-1} c_{k,\ell}^{(n)} x^{1/n} + \Delta(x), \tag{1}$$

where

$$c_{k,\ell}^{(n)} = \operatorname*{Res}_{s=1/n} \frac{F_{k,\ell}(s)}{s}, \quad F_{k,\ell}(s) = \prod_p \Big(1 + \sum_{v=k}^{\ell-1} p^{-vs}\Big),$$

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and $\Delta(x)$ is the error term. The distribution of these numbers $n_{k,\ell}$ attracted the attention of many authors; see [5, 6, 9, 11, 12, 13].

In 2008 Scott [8] conjectured that one third of the square-free numbers are even and this was proven by Jameson in [3]. He proved that

$$\lim_{x\to\infty} \frac{S_o(x)}{S_e(x)} = 2, \tag{2}$$

where $S_o(x)$ and $S_e(x)$ denote the number of all odd and even square-free numbers that do not exceed x, respectively. In 2020 Srichan [10] used an elementary method to prove that

$$\lim_{x\to\infty} \frac{N_o(x)}{N_e(x)} = 2 - \sqrt{2}, \tag{3}$$

where $N_o(x)$ and $N_e(x)$ denote the number of all odd and even square-full numbers that do not exceed x, respectively. One year later, Puttasontiphot and Srichan [7] considered this for the case of cube-full numbers and proved that,

$$\lim_{x\to\infty} \frac{C_o(x)}{C_e(x)} = 2 - 2^{2/3}, \tag{4}$$

where $C_o(x)$ and $C_e(x)$ denote the number of all odd and even cube-full numbers that do not exceed x, respectively. In 2021, Jameson [4] reproved his result in [3] by using the same method as in [10]. Thus, it is interesting to use the elementary method in [10] to generalize this to the case of $[k,\ell]$-integers.

We will use the following notation. For a given set $\mathcal{A}$ and $x > 1$, $\mathcal{A}(x)$ denotes the number of elements in $\mathcal{A}$ that do not exceed x. Here, for two fixed integers k and ℓ such that $2 \le k < \ell < \infty$, we denote by $O_{k,\ell}$ and $E_{k,\ell}$ the set of all odd and even $[k,\ell]$-integers, respectively. The symbol $f(x) \sim g(x)$ means $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$, and we say that $f(x)$ is asymptotic to $g(x)$ as $x \to \infty$. The notation $\lfloor x \rfloor$ denotes the greatest integer not exceeding real x.

Here we prove the following results.

Theorem 1. *As $x \to \infty$, we have*

$$\frac{O_{k,\ell}(x)}{E_{k,\ell}(x)} \sim \frac{2 - 2^{1-(1/k)}}{1 - 2^{1-(\ell/k)}}.$$

Should the preceding definition and notation be allowed for $k = 1$, then 1-full numbers are merely positive integers and generalized ℓ-free integers are just $[1,\ell]$-integers. The following corollary generalizes (2).

Corollary 1. *As $x \to \infty$, we have*

$$\frac{O_{1,\ell}(x)}{E_{1,\ell}(x)} \sim \frac{2^\ell}{2^\ell - 2}.$$

Moreover, we note that, for x large enough and $2 \leq k < x$, all $[k, \lfloor x \rfloor]$-integers not exceeding x are k-full integers. Thus, Theorem 1 recovers (3) and (4). Namely, we obtain the following corollary.

Corollary 2. *As $x \to \infty$, we have*

$$\frac{O_{k,\lfloor x \rfloor}(x)}{E_{k,\lfloor x \rfloor}(x)} = \frac{K_o(x)}{K_e(x)} \sim 2 - 2^{1-(1/k)},$$

where $K_o(x)$ and $K_e(x)$ denote the number of all odd and even k-full numbers that do not exceed x, respectively.

Proof. This follows immediately from the fact that

$$\lim_{x\to\infty} \frac{2 - 2^{1-(1/k)}}{1 - 2^{1-(\lfloor x \rfloor/k)}} = 2 - 2^{1-(1/k)}.$$

□

2. Proof of Theorem 1

We now prove Theorem 1.

Proof of Theorem 1. Let k and ℓ be two fixed integers such that $2 \leq k < \ell$. Recall that $N_{k,\ell}$ denotes the set of all $[k, \ell]$-integers and $O_{k,\ell}$ and $E_{k,\ell}$ denote the set of all odd and even $[k, \ell]$-integers, respectively. Assume that, as $x \to \infty$,

$$O_{k,\ell}(x) \sim ax^{1/k} \qquad \text{and} \qquad E_{k,\ell}(x) \sim bx^{1/k}, \qquad \text{for some } a, b \in \mathbb{R}^+. \tag{5}$$

We will show that, as $x \to \infty$,

$$\frac{a}{b} = \frac{2 - 2^{1-(1/k)}}{1 - 2^{1-(\ell/k)}}. \tag{6}$$

For $0 \leq i \leq \ell - k - 1$, we define $\mathcal{A}_i = \{n \in E_{k,\ell} : n = 2^{k+i}m,\ m \in O_{k,\ell}\}$. By this definition, we have $\mathcal{A}_i(x) = O_{k,\ell}\Big(\frac{x}{2^{k+i}}\Big)$, for $0 \leq i \leq \ell - k - 1$. Thus, we have

$$E_{k,\ell}(x) = \sum_{i=0}^{\ell-k-1} O_{k,l}\Big(\frac{x}{2^{k+i}}\Big). \tag{7}$$

In view of (5) and (7), we have

$$bx^{1/k} = \sum_{i=0}^{\ell-k-1} a \left(\frac{x}{2^{k+i}}\right)^{1/k}.$$

This proves (6).

Now it remains to prove the existence of a and b. In view of (7), we write

$$N_{k,\ell}(x) = O_{k,\ell}(x) + E_{k,\ell}(x) = O_{k,\ell}(x) + \sum_{i_1=0}^{\ell-k-1} O_{k,\ell}\left(\frac{x}{2^{k+i_1}}\right). \tag{8}$$

For $0 \le i_2 \le \ell - k - 1$, we replace x in (8) by $\frac{x}{2^{k+i_2}}$ and then sum that equation over all i_2. Thus,

$$\sum_{i_2=0}^{\ell-k-1} N_{k,\ell}\left(\frac{x}{2^{k+i_2}}\right) = \sum_{i_2=0}^{\ell-k-1} O_{k,\ell}\left(\frac{x}{2^{k+i_2}}\right) + \sum_{i_2=0}^{\ell-k-1}\sum_{i_1=0}^{\ell-k-1} O_{k,\ell}\left(\frac{x}{2^{2k+i_1+i_2}}\right). \tag{9}$$

In view of (8) and (9), we have

$$N_{k,\ell}(x) - \sum_{i_2=0}^{\ell-k-1} N_{k,\ell}\left(\frac{x}{2^{k+i_2}}\right) = O_{k,\ell}(x) - \sum_{i_2=0}^{\ell-k-1}\sum_{i_1=0}^{\ell-k-1} O_{k,\ell}\left(\frac{x}{2^{2k+i_1+i_2}}\right). \tag{10}$$

We redo this process and get

$$\begin{aligned}\sum_{i_2=0}^{\ell-k-1}\sum_{i_1=0}^{\ell-k-1} N_{k,\ell}\left(\frac{x}{2^{2k+i_1+i_2}}\right) &= \sum_{i_2=0}^{\ell-k-1}\sum_{i_1=0}^{\ell-k-1} O_{k,\ell}\left(\frac{x}{2^{2k+i_1+i_2}}\right)\\ &\quad + \sum_{i_3=0}^{\ell-k-1}\sum_{i_2=0}^{\ell-k-1}\sum_{i_1=0}^{\ell-k-1} O_{k,\ell}\left(\frac{x}{2^{3k+i_1+i_2+i_3}}\right). \end{aligned}\tag{11}$$

In view of (10) and (11), we have

$$\begin{aligned}N_{k,\ell}(x) - \sum_{i_2=0}^{\ell-k-1} N_{k,\ell}\left(\frac{x}{2^{k+i_2}}\right) + \sum_{i_2=0}^{\ell-k-1}\sum_{i_1=0}^{\ell-k-1} N_{k,\ell}\left(\frac{x}{2^{2k+i_1+i_2}}\right)\\ = O_{k,\ell}(x) + \sum_{i_3=0}^{\ell-k-1}\sum_{i_2=0}^{\ell-k-1}\sum_{i_1=0}^{\ell-k-1} O_{k,\ell}\left(\frac{x}{2^{3k+i_1+i_2+i_3}}\right).\end{aligned}$$

Repeating this, we have, for $v \ge 2$,

$$\begin{aligned}N_{k,\ell}(x) - \sum_{i_1=0}^{\ell-k-1} N_{k,\ell}\left(\frac{x}{2^{k+i_1}}\right) + \sum_{i_1=0}^{\ell-k-1}\sum_{i_2=0}^{\ell-k-1} N_{k,\ell}\left(\frac{x}{2^{2k+i_1+i_2}}\right) + \dots\\ + \cdots + (-1)^v \sum_{i_1=0}^{\ell-k-1}\sum_{i_2=0}^{\ell-k-1}\cdots\sum_{i_v=0}^{\ell-k-1} N_{k,\ell}\left(\frac{x}{2^{vk+i_1+i_2+\cdots+i_v}}\right)\\ =O_{k,\ell}(x) + (-1)^v \sum_{i_1=0}^{\ell-k-1}\sum_{i_2=0}^{\ell-k-1}\cdots\sum_{i_{v+1}=0}^{\ell-k-1} O_{k,\ell}\left(\frac{x}{2^{(v+1)k+i_1+i_2+\cdots+i_{v+1}}}\right).\end{aligned}\tag{12}$$

In view of (1), we have $N_{k,\ell}(x) \sim cx^{1/k}$ for some $c \in \mathbb{R}^+$. For $\epsilon > 0$, we take x_0 such that $(c-\epsilon)x^{1/k} \le N_{k,\ell}(x) \le (c+\epsilon)x^{1/k}$, for all $x > x_0$. For $x > x_0$, we choose an odd positive integer v such that $x_0^k > \frac{x}{2^{v(k-1)}}$ and $x_0 < \frac{x}{2^{(\ell-1)v}}$. From (12), we have, for $x > x_0 2^{(\ell-1)v}$,

$$\begin{aligned}O_{k,\ell}(x) \ge& (c-\epsilon)x^{1/k} - (c+\epsilon)x^{1/k}\sum_{i_1=0}^{\ell-k-1}\left(\frac{1}{2^{k+i_1}}\right)^{1/k}\\ &+ (c-\epsilon)x^{1/k}\sum_{i_1=0}^{\ell-k-1}\sum_{i_2=0}^{\ell-k-1}\left(\frac{1}{2^{2k+i_1+i_2}}\right)^{1/k} + \dots\\ &- (c+\epsilon)x^{1/k}\sum_{i_1=0}^{\ell-k-1}\cdots\sum_{i_v=0}^{\ell-k-1}\left(\frac{1}{2^{vk+i_1+i_2+\cdots+i_v}}\right)^{1/k}\\ &+ \sum_{i_1=0}^{\ell-k-1}\cdots\sum_{i_{v+1}=0}^{\ell-k-1} O_{k,\ell}\left(\frac{x}{2^{(v+1)k+i_1+i_2+\cdots+i_{v+1}}}\right).\end{aligned}$$

From the fact that $O_{k,\ell}(x) \ge 0$ for all $x > 0$, it follows that

$$\begin{aligned}O_{k,\ell}(x) \ge& (c-\epsilon)x^{1/k} - (c+\epsilon)x^{1/k}\sum_{i_1=0}^{\ell-k-1}\left(\frac{1}{2^{k+i_1}}\right)^{1/k}\\ &+ (c-\epsilon)x^{1/k}\sum_{i_1=0}^{\ell-k-1}\sum_{i_2=0}^{\ell-k-1}\left(\frac{1}{2^{2k+i_1+i_2}}\right)^{1/k} + \dots\\ &- (c+\epsilon)x^{1/k}\sum_{i_1=0}^{\ell-k-1}\cdots\sum_{i_v=0}^{\ell-k-1}\left(\frac{1}{2^{vk+i_1+i_2+\cdots+i_v}}\right)^{1/k}\\ =& \left(c\left(\frac{1+\left(\frac{A}{2}\right)^{v+1}}{1+\frac{A}{2}}\right) - \epsilon\left(\frac{1-\left(\frac{A}{2}\right)^{v+1}}{1-\frac{A}{2}}\right)\right)x^{1/k},\end{aligned} \tag{13}$$

where $A = \sum_{i=0}^{\ell-k-1} 2^{-i/k}$. Now we consider the upper bound of $O_{k,\ell}(x)$,

$$\begin{aligned}O_{k,\ell}(x) \le& (c+\epsilon)x^{1/k} - (c-\epsilon)x^{1/k}\sum_{i_1=0}^{\ell-k-1}\left(\frac{1}{2^{k+i_1}}\right)^{1/k}\\ &+ (c+\epsilon)x^{1/k}\sum_{i_1=0}^{\ell-k-1}\sum_{i_2=0}^{\ell-k-1}\left(\frac{1}{2^{2k+i_1+i_2}}\right)^{1/k} + \dots\\ &- (c-\epsilon)x^{1/k}\sum_{i_1=0}^{\ell-k-1}\cdots\sum_{i_v=0}^{\ell-k-1}\left(\frac{1}{2^{vk+i_1+i_2+\cdots+i_v}}\right)^{1/k}\\ &+ \sum_{i_1=0}^{\ell-k-1}\cdots\sum_{i_{v+1}=0}^{\ell-k-1} O_{k,\ell}\left(\frac{x}{2^{(v+1)k+i_1+i_2+\cdots+i_{v+1}}}\right).\end{aligned}$$

Thus,

$$O_{k,\ell}(x) \leq \left(c\left(\frac{1+\left(\frac{A}{2}\right)^{v+1}}{1+\frac{A}{2}}\right) + \epsilon\left(\frac{1-\left(\frac{A}{2}\right)^{v+1}}{1-\frac{A}{2}}\right)\right)x^{1/k} + \sum_{i_1=0}^{\ell-k-1} \cdots \sum_{i_{v+1}=0}^{\ell-k-1} O_{k,\ell}\left(\frac{x}{2^{(v+1)k+i_1+i_2+\cdots+i_{v+1}}}\right). \tag{14}$$

From (7) and choosing v such that $x_0^k > \frac{x}{2^{v(k-1)}}$, we have

$$\begin{aligned}\sum_{i_1=0}^{\ell-k-1} \cdots \sum_{i_{v+1}=0}^{\ell-k-1} O_{k,\ell}\left(\frac{x}{2^{vk+k+i_1+\cdots+i_{v+1}}}\right) &= \sum_{i_1=0}^{\ell-k-1} \cdots \sum_{i_v=0}^{\ell-k-1} E_{k,\ell}\left(\frac{x}{2^{(vk+i_1+\cdots+i_v}}\right) \\ &< \sum_{i_1=0}^{\ell-k-1} \cdots \sum_{i_v=0}^{\ell-k-1} \frac{x}{2^{vk+i_1+\cdots+i_v}} \\ &< \frac{x_0^k}{2^v} \sum_{i_1=0}^{\ell-k-1} \cdots \sum_{i_v=0}^{\ell-k-1} \frac{1}{2^{i_1+\cdots+i_v}} \\ &= \frac{x_0^k}{2^v}\left(\sum_{i=0}^{\ell-k-1} \frac{1}{2^i}\right)^v \\ &< x_0^k \\ &< x_0^k\left(\frac{1-\left(\frac{A}{2}\right)^{v+1}}{1-\frac{A}{2}}\right)\Bigg). \end{aligned} \tag{15}$$

In view of (14) and (15), we have, for $x > \frac{x_0^{k^2}}{\epsilon^k}$,

$$O_{k,\ell}(x) \leq \left(c\left(\frac{1+\left(\frac{A}{2}\right)^{v+1}}{1+\frac{A}{2}}\right) + 2\epsilon\left(\frac{1-\left(\frac{A}{2}\right)^{v+1}}{1-\frac{A}{2}}\right)\right)x^{1/k}. \tag{16}$$

From (13) and (16), we have

$$\left(c\left(\frac{1+\left(\frac{A}{2}\right)^{v+1}}{1+\frac{A}{2}}\right) - \epsilon\left(\frac{1-\left(\frac{A}{2}\right)^{v+1}}{1-\frac{A}{2}}\right)\right)x^{1/k} \leq O_{k,\ell}(x) \leq \left(c\left(\frac{1+\left(\frac{A}{2}\right)^{v+1}}{1+\frac{A}{2}}\right) + 2\epsilon\left(\frac{1-\left(\frac{A}{2}\right)^{v+1}}{1-\frac{A}{2}}\right)\right)x^{1/k}.$$

This inequality proves the existence of a and from (8) the existence of b follows from the existence a. □

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