INTERSECTIONS IN PAIRS OF WORDS

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Abstract

Given an alphabet $[k]$, we obtain generating functions for pairs of non-intersecting words of any length and in particular for pairs of words of the same length. We then consider pairs of words which may contain the same letters and track the total number of letters in the second word that were also used in the first word. These generating functions are also specialized to account for pairs of weakly decreasing words.

1. Introduction

Historically, pairs of partitions and compositions have been considered in the sense of tracking certain properties of the pair by means of generating functions. See for example [8], where Wilf produced a generating function tracking the number of pairs of non-intersecting partitions of positive integers. Also see [4] where one of the current authors and others tracked pairs of compositions of positive integers having the same number of parts. The latter was generalized and extended in [2].

In this context, we consider pairs of words over the alphabet $[k]$ and discuss firstly (in Section 3) when these have no parts in common and secondly (in Section 4) when they do have parts in common. For the latter, we provide a generating function for the number of parts in the second word which have already appeared in the first of the pair.

In Sections 5 and 6, we switch attention to pairs of words which are weakly decreasing and adjust the methods of Section 3 and 4 to study non-overlapping

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pairs and then overlapping pairs of the decreasing type.

There is an interesting difference between the overlapping cases of different types of pairs. Both Theorem 2 and Theorem 4 use the Temperley method, also known colloquially as "adding a slice", in its proof. Usually, this slice is added either at the beginning or the end of the model leading to the generating function. In the case of Theorem 2, it is added at the end. But as described in the proof of Theorem 4, the multi-part slice is added at various uniquely defined positions in the model which may be anywhere in the slice. This allows the case modeled to grow by multiple parts. The current authors and others have previously used a variant of a non-conventional type of adaptation in one example (see [1]), although this has been rare. There are countless examples of more conventional uses of the method in [5]. And as far as we are aware, adding multiple parts as in this case constitutes a new type of approach, which can only work in specific problems such as this one.

2. Generating Function for Words over the Alphabet $[k]$ Using All of j Letters

We require the following standard results.

The generating function for words over the alphabet $[j]$ using all of j letters, whose number of parts is tracked by x, is given in [7], by

$$P_0(x,j) := \frac{j!x^j}{\prod_{i=1}^{j}(1-ix)} = \sum_{n=0}^{\infty} j!\mathcal{S}_{n,j}x^n, \tag{1}$$

where $\mathcal{S}_{n,j}$ denotes the Stirling numbers of the second kind, i.e., the number of set partitions of n into j blocks.

On the other hand, the generating function for non-empty words over the alphabet $[j]$, made up of some or all of these letters, and whose length (number of parts) is tracked by x, is given by

$$\frac{jx}{1-jx}.$$

3. Pairs of Non-Intersecting Words

We let the number of parts in the first word be tracked by x and the number of parts in the second word be tracked by y. Suppose that the first word uses a total of j distinct letters, some of which may occur a multiple number of times. We define a bijection between the set of words using these j distinct letters and words over the alphabet $[j]$ as follows. First put these letters in numerical order and map the ith letter to the letter i in the alphabet. By so mapping each of the constituent letters, we extend this mapping from a word of length n onto a word of length n over the alphabet $[j]$. This bijection preserves the number of words of any particular length from its domain to its range.

Using the bijection already described, if the second word in the pair has no parts in common to the first, then the number of such words with length tracked by y is counted by

$$\frac{(k-j)y}{1-(k-j)y}.$$

We note that the number of choices for the j distinct letters in the first word is given by $\binom{k}{j}$. Hence, we have proved the following theorem.

Theorem 1. *The generating function tracking all pairs of non-intersecting words over the alphabet k is given by*

$$\begin{aligned} f_k(x,y) &= \sum_{j=1}^{k-1}\binom{k}{j}j!\frac{x^j}{\prod_{k=1}^{j}(1-kx)}\frac{(k-j)y}{1-(k-j)y} \\ &= \sum_{n=1}^{\infty}\sum_{j=1}^{k-1}\binom{k}{j}j!\mathcal{S}_{n,j}x^n\frac{(k-j)y}{1-(k-j)y}, \end{aligned} \tag{2}$$

where x and y track the number of parts in the first and second word of the pair, respectively.

Example 1. When $k=4$, the series given by Equation (2) begins

$$\begin{aligned} f_4(x,y) = x&\left(12y+36y^2+108y^3+324y^4+972y^5\right) \\ &+x^2\left(36y+84y^2+204y^3+516y^4+1356y^5\right) \\ &+\mathbf{x^3}\left(108y+204y^2+\mathbf{420y^3}+924y^4+2148y^5\right) \\ &+x^4\left(324y+516y^2+924y^3+1812y^4+3804y^5\right) \\ &+x^5\left(972y+1356y^2+2148y^3+3804y^4+7332y^5\right)+\cdots. \end{aligned} \tag{3}$$

We illustrate the term $\mathbf{420x^3y^3}$ above. Firstly, if word one uses a single letter three times, then there are $\binom{4}{1}1^3=4$ choices for word one and 3^3 choices for word two, giving a total of 108 words. Secondly, if word one uses two distinct letters, then there are $\binom{4}{2}6=36$ choices for word one and 2^3 choices for word two, giving a total of 288 words. Finally, if word one uses three distinct letters, then there are $\binom{4}{3}3!=24$ choices for word one and 1^3 choices for word two, giving a total of 24 words. Altogether there are 108+288+24=420 non-intersecting word pairs of length 3.

Remark 1. Note the symmetry, $f_k(x,y)=f_k(y,x)$. This follows because the bijection mapping any pair of words to the same pair in reverse order (also reversing the pair order) ensures that these objects are equinumerous.

The coefficent of y^n in the generating function $\frac{(k-j)y}{1-(k-j)y}$ is $(k-j)^n$. By extracting the coefficient $[x^n y^n]$ from Equation (2), we obtain the following corollary.

Corollary 1. *The generating function tracking all pairs of non-intersecting words with the same number of parts n over the alphabet k is given by*

$$g_k(x) = \sum_{n=1}^{\infty} \sum_{j=1}^{k-1} \binom{k}{j} j! \mathcal{S}_{n,j} (k-j)^n x^n, \tag{4}$$

where x now tracks the number of parts in each of the first and second words of the pair.

Next, we extract $[x^n]$ from Equation (4) and obtain our next corollary.

Corollary 2. *The number of pairs of non-intersecting words with the same number of parts n over the alphabet k is given by*

$$h_k(n) = \sum_{j=1}^{k-1} \binom{k}{j} j! \mathcal{S}_{n,j} (k-j)^n. \tag{5}$$

Combinatorially, we might obtain the above equation as follows. Firstly, we let j be the number of different letters used in the first of the pair. There is a choice of $\binom{k}{j}$ such letters out of the k available in the alphabet. Since the second word is non-intersecting, there are a total of $(k-j)^n$ such words. Once the j letters have been chosen, we must ensure that all these letters have in fact been used. This can be done in $j!\mathcal{S}_{n,j}$ ways.

Example 2. In the case of Equation (4) where $k = 4$, the series obtained begins

$$g_4(x) = 12x + 84x^2 + 420x^3 + 1812x^4 + 7332x^5 + 28884x^6 + 112740x^7 + 439572x^8,$$

which concurs with the diagonal terms in Equation (3) as expected. Using Equation (5) we find, for example, that $h_4(5) = 7332$.

4. Pairs of Words Where the Number of Overlaps of the Second Word with the First Is Tracked

As in the previous section, we let $[k]$ be the alphabet for both words, track the number of parts in the first by x, and let the number of distinct letters used in the first word be given by j. We use the same bijection as that given in the previous section and therefore the generating function for the number of such first words is given by Equation (1). Now let $P_r(x,q,j)$ be the generating function for all pairs of words using j distinct letters in the first word, whose length is tracked by x, and having r letters in the second word where the total number of these r letters that have already been used in the first is tracked by q. By adding a column to the right of any such word, we obtain the following recursion for $r \geq 1$:

$$P_r(x,q,j) = P_{r-1}(x,q,j)(qj + (k-j))$$

where $P_0(x,q,j) := P_0(x,j)$ is given by Equation (1). Repeatedly iterating this equation, we obtain

$$P_r(x,q,j) = (qj + (k-j))^r \sum_{n=1}^{\infty} j!\mathcal{S}_{n,j} x^n. \tag{6}$$

Next we define

$$P(x,y,q,j) := \sum_{r=1}^{\infty} P_r(x,q,j) y^r,$$

multiply each $P_r(x,q,j)$ in Equation (6) by y^r and sum over r to obtain

$$\begin{aligned} P(x,y,q,j) &= \sum_{r=1}^{\infty} (qj + (k-j))^r y^r \sum_{n=1}^{\infty} j!\mathcal{S}_{n,j} x^n \\ &= \frac{(qj+k-j)y}{1-(qj+k-j)y} \sum_{n=1}^{\infty} j!\mathcal{S}_{n,j} x^n. \end{aligned}$$

Finally, for each choice of j letters making up the first word, there are $\binom{k}{j}$ possibilities. Hence the generating function $P(x,y,q;k)$ for all such word pairs over the alphabet $[k]$ is given by the following theorem.

Theorem 2. *The generating function for the number of repeats (tracked by q) of letters in the second of a pair of words over the alphabet $[k]$ is given by*

$$P(x,y,q;k) := \sum_{j=1}^{k} \binom{k}{j} P(x,y,q,j) = \sum_{j=1}^{k} \binom{k}{j} \frac{(qj+k-j)y}{1-(qj+k-j)y} \sum_{n=1}^{\infty} j!\mathcal{S}_{n,j} x^n, \tag{7}$$

where x and y track the number of parts in the first and second of the pair of words, respectively. The index j is the number of distinct letters used in the first word.

Remark 2. By putting $q = 0$ above, we recover Theorem 1.

Example 3. The series expansion when $k = 7$ for Equation (7) begins

$$\begin{aligned} &P(x,y,q;7) = x\left(7(6+q)y + 7(6+q)^2y^2 + 7(6+q)^3y^3\right) \\ &\quad + x^2\left(7(36+13q)y + 7\left(186+132q+25q^2\right)y^2 + 49\left(138+144q+54q^2+7q^3\right)y^3\right) \\ &\quad + x^3\left(7(216+127q)y + 49\left(138+156q+49q^2\right)y^2 + \cdots\right) + \cdots . \end{aligned}$$

We illustrate $[q^2x^3y^2]P(x,y,q;7) = 49^2 = 2401$: Firstly, the case where word one uses one letter has 7 choices for the letter, and one case for word two to match that letter, leads to $7q^2$ cases. Secondly, word one uses two letters. If word two uses both those letters, this leads to $\binom{7}{2} \times 6 \times 2q^2 = 252q^2$ cases. On the other hand, if word two uses one of the letters twice, this leads again to $\binom{7}{2} \times 6 \times 2q^2 = 252q^2$ cases. Finally, word one uses three letters and word two uses two out of these three letters, which leads to $\binom{7}{3} \times 3! \times \binom{3}{2} \times 2q^2 = 1260q^2$ cases. Instead, word two may use one of the three letters used in word one, twice; then we have an additional $\binom{7}{3} \times 3! \times \binom{3}{1} q^2 = 630q^2$ cases. Summing all these cases together yields $2401q^2$.

Let us consider pairs of words of equal length. We therefore amend Equation (7) by replacing the generating function for the second word with an arbitrary number of parts by that for the second word with n parts, and thereafter replace the tracker y by x. By doing this, we have proved the following corollary.

Corollary 3. *The generating function $P(x,q;k)$ for the number of repeats (tracked by q) of letters in the second word in a pair of words of equal length, over the alphabet $[k]$, is given by*

$$P(x,q;k) = \sum_{j=1}^{k} \binom{k}{j} \sum_{n=1}^{\infty} j!\mathcal{S}_{n,j}(qj+k-j)^n x^{2n},$$

where x tracks the number of parts in each of the pair of words. The index j is the number of distinct letters used in the first word.

Once again, we provide a combinatorial explanation for the above corollary: $\binom{k}{j}$ is again the number of choices for the j letters in the first word. Now we allow intersections in the second word. The letters may be chosen from the original j, in which case each use is tracked with a q, or the letters come from the remaining non-intersecting letters of which there are $k-j$. So the generating function for all n letters in the second word is $(qj+k-j)^n$. To ensure that we use all j letters in the first word, we again have the factor $j!\mathcal{S}_{n,j}$.

Remark 3. By differentiating Equation (3) with respect to q and setting $q=1$, we obtain the generating function for the total number of repeats of parts in the second of a pair of words of equal length relative to those appearing in the first. This is

$$\sum_{j=1}^{k} \sum_{n=1}^{\infty} \binom{k}{j} jj!\mathcal{S}_{n,j} jk^{n-1} nx^{2n}.$$

5. Pairs of Weakly Decreasing Non-Intersecting Words

Here we consider pairs of weakly decreasing words. As already explained in Section 3, we let the number of parts in the first word be tracked by x and suppose that the first word uses a total of j distinct letters, some of which may occur a multiple number of times. As before, we let the number of parts in the second word be tracked by y. And we again use the bijection explained in Section 3, which means we use the alphabet $[j]$ as the set for the parts in the first word and use this to obtain a generating function accounting for the number of all such decreasing first words, while using the set $[k-j]$ to obtain a generating function accounting (bijectively) for all such second words whose parts come from $[k-j]$. We note that the second word does not necessarily use all the elements of $[k-j]$.

According to the analysis in [3], choosing (allowing replacement of) p objects from n different objects without regard to the order of choice is called a *combination with repetition*. The number of possible such combinations is a basic and elementary statistic. Indeed, choosing a combination of j objects with repetition from a set of n objects can be done in $\binom{n+j-1}{j}$ ways. By definition, such a combination is of the form $\{c_1, c_2, \ldots, c_j\}$ where $1 \leq c_1 \leq c_2 \leq \cdots \leq c_j \leq n$. In other words, a combination is by definition (in reverse order) a weakly decreasing word of arbitrary length. We let $card(\pi)$ denote the cardinality of the combination π of $[n]$, let C_n be the set of all such combinations of arbitrary length, and let $P_n(p)$ be the generating function for the number of combinations π of $[n]$ with repetition according to the statistics *card* tracked by p:

$$P_n(p) := \sum_{\pi \in C_n} p^{card(\pi)}.$$

Since each combination of $[n]$ either contains n or not, we have

$$P_n(p) = P_{n-1}(p) + P_n(p|n), \tag{8}$$

where $P_n(p|i)$ is defined to be the generating function for such combinations ending in i. We consider the two cases of cardinality 1 or larger (where the last two columns are jn) and therefore obtain:

$$P_n(p|n) = p + p\sum_{j=1}^{n} P_j(p|j), \tag{9}$$

which implies

$$P_{n-1}(p|n-1) = p + p\sum_{j=1}^{n-1} P_j(p|j) \tag{10}$$

with $P_1(p|1) = P_1(p) = \frac{p}{1-p}$. Subtracting Equation (10) from (9), we obtain

$$P_n(p|n) = \frac{P_{n-1}(p|n-1)}{1-p},$$

which we iterate $n-1$ times to obtain

$$P_n(p|n) = \frac{p}{(1-p)^n}.$$

So from Equation (8),

$$P_n(p) = P_{n-1}(p) + \frac{p}{(1-p)^n}.$$

Again, we iterate the latter equation to obtain

$$P_n(p) = p\sum_{j=1}^{n} \frac{1}{(1-p)^j} = (1-p)^{-n} - 1.$$

So, in the case of word two, where y tracks the number of parts, we have the generating function $P_{2,n}(y)$ given by

$$P_{2,n}(y) = (1-y)^{-n} - 1.$$

In the case of word one, every letter from $[j]$ must appear at least once in decreasing order. Again in decreasing order, all the cases of the repeats of these letters slot into the first occurrences of the first elements from $[j]$, each in a unique position. Thus, in the case of word one, where x tracks the number of parts, the generating function is given by

$$P_{1,j}(x) = x^j(1-x)^{-j}.$$

In the case of $P_{2,n}(y)$, we put $n = k-j$ representing the fact that the parts of word two belong to the alphabet $[k]$, but have no intersection with the parts of word one. Hence, we have proved the following theorem.

Theorem 3. *The generating function tracking all pairs of non-intersecting weakly decreasing words over the alphabet $[k]$ is given by*

$$f_{w,k}(x,y) = \sum_{j=1}^{k-1} \binom{k}{j} x^j(1-x)^{-j}\left((1-y)^{-(k-j)} - 1\right), \tag{11}$$

where x and y track the number of parts in the first and second word of the pair, respectively, and the index j tracks the number of distinct letters in the first word.

Example 4. The series expansion when $k = 4$ for Equation (11) begins

$$\begin{aligned} f_{w,4}(x,y) = &x\left(12y + 24y^2 + 40y^3 + 60y^4 + 84y^5\right) \\ &+ x^2\left(24y + 42y^2 + 64y^3 + 90y^4 + 120y^5 +\right) \\ &+ x^3\left(40y + 64y^2 + 92y^3 + 124y^4 + 160y^5\right) \\ &+ x^4\left(60y + 90y^2 + 124y^3 + \mathbf{162}y^4 + 204y^5\right) \\ &+ x^5\left(84y + 120y^2 + 160y^3 + 204y^4 + 252y^5\right) + \cdots, \end{aligned} \tag{12}$$

from the extension of which we extract the diagonal series beginning

$$12xy + 42(xy)^2 + 92(xy)^3 + \mathbf{162}(xy)^4 + 252(xy)^5 + 362(xy)^6 + 492(xy)^7.$$

This is sequence A005901 in [6]. However it is a new interpretation of this sequence.

We illustrate the bold coefficient from Equation (12) above . Firstly, the case where word one uses one letter has 4 choices for the letter and one case for word two to have one letter (3 choices) which leads to 12 cases. Or, word two uses two letters, which leads to $4\binom{3}{2} \times 3$ cases. The last possibility for this case is where word two uses three letters leading to 4×3 cases. Secondly, word one uses two letters, which leads to $\binom{4}{2} \times 3$ possibilities for this word. If word two uses one letter, this leads to $\binom{4}{2} \times 3 \times \binom{2}{1}$ choices. If word two uses two letters, this leads to $\binom{4}{2} \times 3 \times 3$ cases. Finally, word one using three letters leads to $\binom{4}{3} \times 3$ possibilities for this word. Word two then has one possibility. Summing all these cases yields $162x^4y^4$.

By using the binomial theorem, the above theorem may be expressed in a nicer form given in the next corollary.

Corollary 4. *The generating function tracking all pairs of non-intersecting weakly decreasing words over the alphabet* $[k]$ *is given by*

$$f_{w,k}(x,y) = \left(\frac{x}{1-x} + \frac{1}{1-y}\right)^k - \left(\frac{1}{1-x}\right)^k - \left(\frac{1}{1-y}\right)^k + 1,$$

where x *and* y *track the number of parts in the first and second word of the pair, respectively.*

Below, we extract the diagonal coefficients from Equation (11). We see that

$$[(xy)^n]f_{w,k}(x,y) = \sum_{j=1}^{k-1}\binom{k}{j}\binom{n-1}{j-1}\binom{n+k-j-1}{n}.$$

So we have proved the following corollary.

Corollary 5. *The generating function counting pairs of non-intersecting weakly decreasing words of equal length over the alphabet* $[k]$ *is given by*

$$\sum_{n=1}^{\infty}\sum_{j=1}^{k-1}\binom{k}{j}\binom{n-1}{j-1}\binom{n+k-j-1}{n}z^n$$

where z tracks the length of each word in the pair.

For the final time we give a combinatorial explanation of the above corollary. Firstly, choose the j distinct letters in the first word in $\binom{k}{j}$ ways. Since this is a weakly decreasing word, there are $\binom{n-1}{j-1}$ choices of position where there is a left-to-right strict decrease after the largest letter goes into the first position. These are all placed in decreasing order. All the positions other than these $j-1$ are where the letter to its left are repeated (i.e., a weak decrease). The second word (weakly decreasing, non-intersecting) is constituted from a subset of the remaining $k-j$ letters and is counted by $\binom{n+k-j-1}{n}$ which is the number of combinations with repetition.

Remark 4. The k values 4 and 54 in Corollary 5 correspond to Sequences A005901 and A0035879, respectively, in [6]. Other k values between 4 and 54 correspond to other sequence numbers between the above two extremes. The non-intersecting word interpretation are all new instances of these sequences and the original sequences from [6] all describe a different but inter-related problem, which we invite the interested reader to explore.

Remark 5. As explained in Remark 1, we have the symmetry, $f_{w,k}(x,y) = f_{w,k}(y,x)$.

6. Pairs of Decreasing Words where the Number of Overlaps of the Second Word with the First Is Tracked

Here, we will use a variant on the method of adding a slice. The variation is as follows. Firstly, instead of adding one part at a time, we add r parts simultaneously which is enabled because each of the r parts fits into the existing second of the pair of decreasing words in a uniquely defined position. Moreover, each of these parts has an identical effect on the existing generating function for the non-intersecting parts. Secondly, the more conventional treatment adds the new slice either at the beginning or end of the existing word, but this particular problem enables the r new slices to fit in at some other uniquely defined positions.

Our starting point will be to use Theorem 3, which gives the generating function tracking pairs of non-intersecting weakly decreasing words over the alphabet $[k]$. For each $r \geq 1$, we will additionally track r extra parts in the second of the pair, all of which have already occurred in the first of the pair. As stated in the second paragraph of Section 5, for each r, there are $\binom{r+j-1}{r}$ choices (in weakly decreasing order) for the r repeated parts from the j distinct letters in the first word. We track the occurrence of each overlap with variable q and, as before, each (additional) part by y. Hence, the generating function given by Equation (11) needs to be modified to allow the part of it tracking the non-overlapping parts of the second word in the pair (i.e., $(1-y)^{-(k-j)} - 1$) to also track the overlapping additional r parts in the second of the pair, where each of these overlapping parts in the second word is placed exactly in the unique rightmost possible position to ensure that the second word remains weakly decreasing. Note that the generating function $\left((1-y)^{-(k-j)}-1\right)$ excludes the empty case but the additional r parts are allowed to be appended to the empty case for which the generating function is $(1-y)^{-(k-j)}\sum_{r=1}^{\infty}\binom{r+j-1}{r}q^r y^r$. Altogether the term $(1-y)^{-(k-j)}-1$ in Equation (11) needs to be replaced by

$$(1-y)^{-(k-j)}-1+(1-y)^{-(k-j)}\sum_{r=1}^{\infty}\binom{r+j-1}{r}q^r y^r.$$

Thus we have proved the following theorem.

Theorem 4. *The generating function $g_{w,k}(x,y,q)$ tracking all pairs of weakly decreasing words over the alphabet $[k]$ is given by*

$$\sum_{j=1}^{k-1}\binom{k}{j}x^j(1-x)^{-j}\left((1-y)^{-(k-j)}-1+(1-y)^{-(k-j)}\sum_{r=1}^{\infty}\binom{r+j-1}{r}q^r y^r\right)$$
$$=\sum_{j=1}^{k-1}\binom{k}{j}x^j(1-x)^{-j}\left(-1+(1-y)^{-(k-j)}(1-qy)^{-j}\right), \tag{13}$$

where x and y track the number of parts in the first and second word of the pair, respectively, the index j tracks the number of distinct letters in the first word and q tracks the number of overlaps in the second word.

Remark 6. By putting $q=0$ above, we recover Theorem 3.

Example 5. The series expansion when $k=7$ for Equation (13) begins

$$\begin{aligned}&x\left(7(6+q)y+7(21+q(6+q))y^2+7\left(56+q\left(21+6q+q^2\right)\right)y^3\right)\\&+x^2\left(49(3+q)y+14\left(33+18q+5q^2\right)y^2+7\left(161+111q+51q^2+13q^3\right)y^3\right)\\&+x^3\left(196(2+q)y+49\left(23+18q+7q^2\right)y^2+21\left(122+117q+72q^2+25q^3\right)y^3\right)\\&+\cdots.\end{aligned}$$

We illustrate $21\times 25x^3y^3q^3$ from the above series:

If word 1 uses one letter there are seven choices, and word 2 must use that letter three times, for a total $7q^3$ possibilities.

If word 1 uses two letters, there are $\binom{7}{2}\times 2$ choices for word 1. Then word 2 can use one of them three times, with two choices, or both of them, so two choices, for a total of $\binom{7}{2}\times 2\times 4q^3$ possibilities.

Finally, if word 1 has three letters, there are $\binom{7}{3}$ choices for word 1. Next if word 2 uses one out of the three, there are three choices, or two out of the three gives 3×2 choices, or three out of three so one choice. This gives a total of $\binom{7}{3}(3+6+1)q^3$ possibilities.

Summing all possibilities together yields $525x^3y^3q^3$ as indicated in the series.

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