ON THE PARALLELOGRAM UNIQUENESS OF SQUARE NUMBERS FOR MULTIPLICATIVE FUNCTIONS

Poo-Sung Park[1]
Department of Mathematics Education, Kyungnam University, Changwon, Republic of Korea
pspark@kyungnam.ac.kr

Received: 7/7/24, Revised: 8/20/24, Accepted: 11/26/24, Published: 12/9/24

Abstract

Pak and Kang showed that if a multiplicative function f satisfies $f(p+q)+f(p-q)=2f(p)+2f(q)$ for all primes p and q, then f is uniquely determined assuming Goldbach's conjecture. In this paper, we prove that if a multiplicative function f satisfies $f(m^2+n^2)+f(m^2-n^2)=2f(m^2)+2f(n^2)$ for all positive integers $m>n$, then there are three solutions for f. If $f(4)\neq 0$, then f is uniquely determined to be $f(n)=n^2$.

1. Introduction

A function $f:\mathbb{N}\to\mathbb{C}$ is called *multiplicative* if $f(1)=1$ and $f(mn)=f(m)f(n)$ for all positive integers m and n with $\gcd(m,n)=1$. Let S be a set of multiplicative functions and E be a set of positive integers. If $f\in S$ is uniquely determined under the condition $f(m+n)=f(m)+f(n)$ for all $m,n\in E$, then we call E an *additive uniqueness set* for S.

Spiro [8] coined the notion of additive uniqueness and showed that the set of primes is an additive uniqueness set for multiplicative functions f provided that there exists a prime p_0 such that $f(p_0)\neq 0$. Since her paper many mathematicians have studied various additive uniqueness sets. For example, Dubickas and Šarka [3] extended Spiro's result to apply to more than two primes; that is,

$$f(p_1+p_2+\cdots+p_k)=f(p_1)+f(p_2)+\cdots+f(p_k)$$

for all primes p_i.

Chung [2] considered the condition $f(a^2+b^2)=f(a^2)+f(b^2)$ for all $a,b\in\mathbb{N}$. However, the multiplicative function f is not uniquely determined. Later, the author

DOI: 10.5281/zenodo.14339987
[1]This work was supported by the National Research Foundation of Korea (NRF) grant funded by the Korean government (MSIT) (NRF-2021R1A2C1092930).

[6] proved that if $k \geq 3$ and a multiplicative function f satisfies

$$f(a_1^2 + a_2^2 + \cdots + a_k^2) = f(a_1^2) + f(a_2^2) + \cdots + f(a_k^2)$$

for all positive integers a_i, then f is the identity function.

As a variation, Bašić [1] studied multiplicative functions f satisfying $f(a^2+b^2) = f(a)^2 + f(b)^2$ for all $a, b \in \mathbb{N}$. However, this is also not uniquely determined. The author [5] also showed that f is uniquely determined if $k \geq 3$ and

$$f(a_1^2 + a_2^2 + \cdots + a_k^2) = f(a_1)^2 + f(a_2)^2 + \cdots + f(a_k)^2$$

holds for all positive integers a_i.

Recently, Pak and Kang [4] studied the different functional equation

$$f(p+q) + f(p-q) = 2f(p) + 2f(q)$$

for all primes $p > q$. They showed that the multiplicative function is uniquely determined to be $f(n) = n^2$ if Goldbach's conjecture is true. They call this property *parallelogram uniqueness*.

In this paper, we study the parallelogram uniqueness of square numbers for multiplicative functions. There are three solutions for the functional equation: one is $f(n) = n^2$ (Theorem 1) and two other functions are possible (Theorem 2).

Theorem 1. *If a multiplicative function f satisfies*

$$f(m^2 + n^2) + f(m^2 - n^2) = 2f(m^2) + 2f(n^2)$$

for all positive integers $m > n$ and $f(4) \neq 0$, then f is uniquely determined to be $f(n) = n^2$.

Theorem 2. *If a multiplicative function f satisfies*

$$f(m^2 + n^2) + f(m^2 - n^2) = 2f(m^2) + 2f(n^2)$$

for all positive integers $m > n$ and $f(4) = 0$, then f is one of the following:

1. $f(n) = \begin{cases} 0, & \text{if } n \equiv 0 \pmod 4, \\ 4, & \text{if } n \equiv 2 \pmod 4, \\ 1, & \text{otherwise}, \end{cases}$

2. $f(2) = 4$, $f(2^r) = 0$ *for all* $r \geq 2$, *and*

$$f(p^r) = \left(\frac{-1}{p}\right)^r p^r$$

for all $r \geq 1$ and odd primes p. Here, $\left(\frac{\cdot}{\cdot}\right)$ is the Jacobi symbol.

2. Proofs of Theorems 1 and 2

In this section we will provide the proofs of Theorems 1 and 2. The proofs rely on several preliminary results (Theorem 3-6 and Lemmas 1-5), that we present first.

If a multiplicative function f satisfies

$$f(m^2+n^2)+f(m^2-n^2)=2f(m^2)+2f(n^2)$$

for all positive integers $m>n$, then we can evaluate some $f(n)$ by solving a system of equations. For example, it contains

$$f(2^2+1^2)+f(2^2-1^2)=2f(2^2)+2f(1^2)$$

or

$$f(5)+f(3)=2f(4)+2f(1).$$

We use the following `Mathematica` code by setting `fn` $= f(n)$.

```
Solve[f1 == 1 && f5 + f3 == 2*f4 + 2*f1
  && f4 f17 + f4 f3 f5 == 2*f64 + 2*f4
  && f5 f13 + f9 f7 == 2*f64 + 2*f1
  && f25 + f7 == 2*f16 + 2*f9 && f4 f25 + f4 f7 == 2*f64 + 2*f4 f9
  && f5 f17 + f7 f11 == 2*f81 + 2*f4 && f2 f5 + f8 == 2*f9 + 2*f1
  && f2 f13 + f8 f3 == 2*f25 + 2*f1 && f13 + f5 == 2*f9 + 2*f4
  && f9 f13 + f9 f5 == 2*f81 + 2*f4 f9
  && f5 f17 + f13 == 2*f49 + 2*f4 f9
  && f4 f13 + f4 f5 == 2*f4 f9 + 2*f16
  && f2 f17 + f16 == 2*f25 + 2*f9
  && f41 + f9 == 2*f25 + 2*f16 && f5 f13 + f3 f11 == 2*f49 + 2*f16
  && f2 f25 + f16 f3 == 2*f49 + 2*f1
  && f2 f41 + f16 f5 == 2*f81 + 2*f1,
 {f1, f2, f3, f4, f5, f7, f8, f9, f11, f13, f16, f17, f25, f41, f49,
 f64, f81}]
```

The above system of equations has three sets of solutions as in Table 1.

We can determine $f(2^r)$ according to the value of $f(4)$.

Theorem 3. *If $f(4)=0$, then $f(2^r)=0$ for $r\geq 2$. Otherwise, $f(2^r)=2^{2r}$ for $r\geq 1$.*

Proof. From Table 1, if $f(4)=0$, then $f(2^3)=f(2^4)=0$. We use induction on $f(2^r)$, assuming that $f(2^t)=0$ for all $2\leq t<r$.

n	2	3	4	5	7	8	9	11	13	16	17	25	41	49	64	81
$f(n)$	4	1	0	1	1	0	1	1	1	0	1	1	1	1	0	1
$f(n)$	4	-3	0	5	-7	0	9	-11	13	0	17	25	41	49	0	81
$f(n)$	2^2	3^2	4^2	5^2	7^2	8^2	9^2	11^2	13^2	16^2	17^2	25^2	41^2	49^2	64^2	81^2

Table 1: $f(n)$ for several n's

Consider $r = 2s+1 \geq 5$. To evaluate $f(2^{2s+1})$ by using the equation

$$f(A^2+B^2) + f(A^2-B^2) = 2f(A^2) + 2f(B^2),$$

we set $A = 3 \cdot 2^{s-1}$ and $B = 2^{s-1}$. Then, the above equation becomes

$$f(5)f\big(2^{2s-1}\big) + f\big(2^{2s+1}\big) = 2f(9)f\big(2^{2s-2}\big) + 2f\big(2^{2s-2}\big).$$

Rearranging the equation, we obtain $f(2^{2s+1}) = 0$.

If $r = 2s+2 \geq 4$, then we set $A = 3 \cdot 2^s$ and $B = 2^{s+1}$. Since

$$f(13)f\big(2^{2s}\big) + f(5)f\big(2^{2s}\big) = 2f(9)f\big(2^{2s}\big) + 2f\big(2^{2s+2}\big),$$

we obtained $f(2^{2s+2}) = 0$.

If $f(4) \neq 0$, then we already computed $f(4) = 4^2$, $f(8) = 8^2$, and $f(16) = 16^2$. In this case we obtain $f(2^r) = 2^{2r}$ by induction by using the same process as the above two cases. $\square$

Now we start the proof of Theorem 1. We prove that $f(p^r) = p^{2r}$ through induction on odd primes p and $r \in \mathbb{N}$. In our proof that $f(p) = p^2$, we assume that $f(q^s) = q^{2s}$ for all primes $q < p$ and all $s \in \mathbb{N}$. Using the fact that $f(p) = p^2$, we then prove that $f(p^r) = p^{2r}$ for all r.

The difficult part is to show that $f(p) = p^2$ for $p \equiv 3 \pmod 4$. If $p \equiv 1 \pmod 4$, then p is the sum of two squares and we can use induction.

First, we show that $f(p^r) = p^{2r}$ with $r \geq 2$ under the assumption that $f(p) = p^2$. To this end we need a lemma.

Lemma 1. *Assume that $f(q^s) = q^{2s}$ for all primes $q < p$ and $s \in \mathbb{N}$. If $f(p) = p^2$, then $f(p^2+1) = (p^2+1)^2$.*

Proof. Note that all prime divisors of $p+1$ and $p-1$ are smaller that p. Then, since

$$f\left(\frac{p^2+1}{2}\right) + f(p) = 2f\left(\left(\frac{p+1}{2}\right)^2\right) + 2f\left(\left(\frac{p-1}{2}\right)^2\right),$$

we obtain $f(p^2+1) = (p^2+1)^2$. $\square$

Theorem 4. *Assume that* $f(q^s) = q^{2s}$ *for all primes* $q < p$ *and* $s \in \mathbb{N}$*. If* $f(p) = p^2$*, then* $f(p^r) = p^{2r}$ *for* $r \geq 2$*.*

Proof. Suppose r is even. We write $r = 2k$. If $f(p^s) = p^{2s}$ for all $s < 2k$, then $f(p^{2k}) = p^{4k}$ by induction since

$$f(p^{2k-2})\, f(p^2+1) + f(p^{2k-2})\, f((p+1)(p-1)) = 2f(p^{2k}) + 2f(p^{2k-2}).$$

When r is odd, we write $r = 2k+1$. If $f(p^s) = p^{2s}$ for all $s < 2k+1$, then the equation

$$f(p^{2k})\, f\left(\frac{p^2+1}{2}\right) + f(p^{2k+1}) = 2f(p^{2k})\, f\left(\left(\frac{p+1}{2}\right)^2\right) + 2f(p^{2k})\, f\left(\left(\frac{p-1}{2}\right)^2\right)$$

gives $f(p^{2k+1}) = p^{2(2k+1)}$. □

We already have $f(p) = p^2$ for $p = 3, 5, 7, 11, 13$, and 17. So, we only need to show that $f(p) = p^2$ for $p \geq 19$ under the induction hypothesis.

Theorem 5. *Assume that* $f(q^s) = q^{2s}$ *for all primes* $q < p$ *and* $s \in \mathbb{N}$*. If* $p \equiv 1 \pmod 4$*, then* $f(p) = p^2$*.*

Proof. Since $p \equiv 1 \pmod 4$, $p = a^2 + b^2$ for some positive integers $a > b$. Then, all prime divisors of $a^2 - b^2$, a^2, and b^2 are smaller than p. Thus, since

$$f(p) + f(a^2 - b^2) = 2f(a^2) + 2f(b^2),$$

we obtain $f(p) = p^2$. □

Now, we prove that $f(p) = p^2$ when $p \equiv 3 \pmod 4$. To use induction, we assume that $f(q^r) = q^{2r}$ for all primes $q < p$.

Lemma 2. *Assume that* $f(q^s) = q^{2s}$ *for all primes* $q < p$ *and* $s \in \mathbb{N}$*. If* $p \equiv 3 \pmod 4$*, then* $f(p+2) = (p+2)^2$ *and* $f\big((p+2)^2\big) = (p+2)^4$*.*

Proof. If $p+2$ is composite, then all prime divisors of $p+2$ are smaller than p. So, $f(p+2) = (p+2)^2$ and $f\big((p+2)^2\big) = (p+2)^4$.

If $p+2$ is prime, then $p+2 \equiv 1 \pmod 4$ and thus, $p+2 = a^2 + b^2$ with some integers $a > b$. Thus, $f(p+2) = (p+2)^2$ by the previous method in the proof of Lemma 5.

Similarly, since

$$(p+2)^2 = (a^2 - b^2)^2 + (2ab)^2$$

and $a^2 - b^2 < p$ and $2ab < a^2 + b^2 = p$, we have that $f\big((p+2)^2\big) = (p+2)^4$. □

Lemma 3. *Assume that* $f(q^s) = q^{2s}$ *for all primes* $q < p$ *and* $s \in \mathbb{N}$*. If* $p = 4k-1$*, then* $f(k^2+1) = (k^2+1)^2$ *and* $f(4k^2+1) = (4k^2+1)^2$*.*

Proof. Note that

$$f(k^2+1)+f((k+1)(k-1))=2f(k^2)+2f(1)$$

and

$$f(4k^2+1)+f((2k+1)(2k-1))=2f(4k^2)+2f(1).$$

Since prime divisors of each factor are less than p, we obtain $f(k^2+1)=(k^2+1)^2$ and $f(4k^2+1)=(4k^2+1)^2$. □

Lemma 4. *Assume that $f(q^s)=q^{2s}$ for all primes $q<p$ and $s\in\mathbb{N}$. If $p=4k-1\geq 19$ and k is not a multiple of 4, then $f(5k+3)=(5k+3)^2$.*

Proof. Suppose $5k+3$ is prime. Then, $5k+3=a^2+b^2$ for some $a>b\geq 1$. Note that $a<\sqrt{5k+3}<4k-1=p$. Thus, $f(5k+3)=(5k+3)^2$ similarly as in the proof of Theorem 5.

If $5k+3$ is a composite number, then it has a divisor $d\geq 2$ and

$$\frac{5k+3}{d}\leq\frac{5k+3}{2}<4k-1=p.$$

Thus, all prime divisors of $5k+3$ are smaller than p and $f(5k+3)=(5k+3)^2$ by the induction hypothesis. □

Similarly, we have the following lemma.

Lemma 5. *Assume that $f(q^s)=q^{2s}$ for all primes $q<p$ and $s\in\mathbb{N}$. If $p=4k-1\geq 19$ and $k\not\equiv 3 \pmod 4$, then $f(6k+1)=(6k+1)^2$.*

We have now finished the setup for the proof.

Theorem 6. *Assume that $f(q^s)=q^{2s}$ for all primes $q<p$ and $s\in\mathbb{N}$. If $p=4k-1\geq 19$, then $f(p)=p^2$.*

Proof. The previous lemmas established that for $p=4k-1$, we have $f(5k+3)=(5k+3)^2$ or $f(6k+1)=(6k+1)^2$.

Suppose that $4\nmid k$. Then, $f(5k+3)=(5k+3)^2$ by Lemma 4. Also, $f(k^2+1)=(k^2+1)^2$ by Lemma 3. If $17^s\|(k^2+1)$ with $s\geq 0$, then

$$f(k^2+1)=f(17^s)\,f\left(\frac{k^2+1}{17^s}\right)=17^{2s}\,f\left(\frac{k^2+1}{17^s}\right)=(k^2+1)^2$$

and thus,

$$f\big(17(k^2+1)\big)=f(17^{s+1})\,f\left(\frac{k^2+1}{17^s}\right)=17^2(k^2+1)^2.$$

Note that

$$f\big(17(k^2+1)\big) + f((5k+3)(3k-5)) = 2f(p^2) + 2f\big((k+4)^2\big)$$

and $\gcd(5k+3, 3k-5)$ is 1, 2, 17, or 34. In every case, we have

$$f((5k+3)(3k-5)) = (5k+3)^2(3k-5)^2$$

since $3k-5 < p$.

Then, we have that $f(16k^2+1) = (16k^2+1)^2$, since

$$f(2)\, f(16k^2+1) + f(16k) = 2f\big((p+2)^2\big) + 2f(p^2)$$

and $f\big((p+2)^2\big) = (p+2)^4$ by Lemma 2.

Thus, we can deduce that $f(p) = p^2$ since

$$f(16k^2+1) + f(p+2)\, f(p) = 2f(16k^2) + 2f(1)$$

and $f(p+2) = (p+2)^2$ by Lemma 2.

Now, suppose that $4 \mid k$. Because $5k+3 \equiv 3 \pmod 4$, we cannot use $f(5k+3)$. Instead, we use $6k+1 \not\equiv 3 \pmod 4$. We have $f(6k+1) = (6k+1)^2$ by Lemma 5 and $f(4k^2+1) = (4k^2+1)^2$ by Lemma 3. Then, $f(p^2) = p^4$ since

$$f\big(5(4k^2+1)\big) + f((6k+1)(2k-3)) = 2f(p^2) + 2f\big(4(k+1)^2\big).$$

Therefore, $f(16k^2+1) = (16k^2+1)^2$ and $f(p) = p^2$ by the previous method. □

Now, we prove Theorems 1 and 2.

Proof of Theorem 1. We have that $f(2^r) = 2^{2r}$ by Theorem 3. Let p be an odd prime and assume that $f(q^r) = q^{2r}$ for all primes $q < p$. Then, $f(p) = p^2$ by Theorem 5 and Theorem 6. If $f(p^s) = p^{2s}$ for all $s < r$, then $f(p^r) = p^{2r}$ by Theorem 4. □

Proof of Theorem 2. We can prove Theorem 2 in the same way. For example, $f(2^r) = 0$ with $r \geq 2$ was already shown in Theorem 3. It is obvious that $f(n) = 1$ for all odd n satisfies the condition. So, suppose that $f(3) \neq 1$. Then, Lemma 1 for this case should be stated as follows: If $f(p) = \left(\frac{-1}{p}\right)p$, then $f\left(\frac{p^2+1}{2}\right) = \frac{p^2+1}{2}$. This can be verified since

$$f\left(\left(\frac{p+1}{2}\right)^2\right) = \begin{cases} \left(\frac{p+1}{2}\right)^2 & \text{if } p \equiv 1 \pmod 4, \\ 0 & \text{if } p \equiv 3 \pmod 4, \end{cases}$$

$$f\left(\left(\frac{p-1}{2}\right)^2\right) = \begin{cases} 0 & \text{if } p \equiv 1 \pmod 4, \\ \left(\frac{p-1}{2}\right)^2 & \text{if } p \equiv 3 \pmod 4. \end{cases}$$

By a similar process we obtain $f(p^r) = \left(\frac{-1}{p}\right)^r p^r$ for all odd primes p. □

Acknowledgements. The author would like to thank the referees and editors for their valuable comments and advice.

References

[1] B. Bašić, Characterization of arithmetic functions that preserve the sum-of-squares operation, *Acta Math. Sin. (Engl. Ser.)* **30** (4) (2014), 689-695.

[2] P. V. Chung, Multiplicative functions satisfying the equation $f(m^2 + n^2) = f(m^2) + f(n^2)$, *Math. Slovaca* **46** (1996), 165-171.

[3] A. Dubickas and P. Šarka, On multiplicative functions which are additive on sums of primes, *Aequationes Math.* **86** (1-2) (2013), 81-89.

[4] H. C. Pak and D. Kang, Prime numbers as a uniqueness set of the parallelogram equation via the Goldbach's conjecture, preprint, arXiv:2302.04037v3 (2023).

[5] P.-S. Park, Multiplicative functions commutable with sums of squares, *Int. J. Number Theory* **14** (2) (2018), 469-478.

[6] P.-S. Park, On k-additive uniqueness of the set of squares for multiplicative functions, *Aequationes Math.* **92** (3) (2018), 487-495.

[7] P.-S. Park, On multiplicative functions which are additive on positive cubes, *Aequationes Math.*, to appear.

[8] C. A. Spiro, Additive uniqueness sets for arithmetic functions, *J. Number Theory* **42** (2) (1992), 232-246.