#A109 INTEGERS 24 (2024)

THE 2-COMPLEXITY OF EVEN POSITIVE INTEGERS

Pengcheng Zhang
Max Planck Institute for Mathematics, Bonn, Germany
pzhang@mpim-bonn.mpg.de

Received: 6/11/24, Revised: 8/19/24, Accepted: 11/27/24, Published: 12/9/24

Abstract

The question of integer complexity asks about the minimal number of 1's that are needed to express a positive integer using only addition and multiplication (and parentheses). In this paper, we propose the notion of l-complexity of multiples of l, which specializes to integer complexity when $l = 1$, prove several elementary results on 2-complexity of even positive integers, and raise some interesting questions on 2-complexity and in general l-complexity.

1. Introduction

Given a positive integer n, the *integer complexity* of n, denoted as $\|n\|$, is defined as the minimal number of 1's that are needed to express n in terms of 1 using only addition and multiplication (and parentheses). Alternatively, one could also define the integer complexity recursively via

$$\|1\| := 1 \quad \text{and} \quad \|n\| := \min_{\substack{a,b\in\mathbb{Z}^+ \\ a+b=n \text{ or } ab=n}} \big(\|a\| + \|b\|\big).$$

The notion of integer complexity was first raised by Mahler–Popken [6], and it has been a longstanding problem to determine the integer complexity of some given n. Integer complexity grows logarithmically:

$$3\log_3 n \;\leq\; \|n\| \;\leq\; 3\log_2 n.$$

According to Guy [5], Selfridge proved that the lower bound can be attained when $n = 3^m$ and raised the question of whether there exists $a \in \mathbb{Z}^+$ such that $\|2^a\| < 2a$. This question is nowadays usually incorporated into the following conjecture.

Conjecture 1. For $a \geq 1$ and $b \geq 0$, $\|2^a \cdot 3^b\| = 2a + 3b$.

DOI: 10.5281/zenodo.14340011

There has been much progress on this topic, specifically on improving the upper bound for 'generic' positive integers, algorithms for computing integer complexity, and partial results on Conjecture 1. Here we refer to the papers by Steinerberger [7], Cordwell et al. [4], Altman–Zelinsky [2], and Altman [1] for discussions of previous results and recent advances on various aspects. We would also like to mention the paper by Arias de Reyna [3], which raised many other conjectures on integer complexity, though most of them have been settled.

Now, let l be a positive integer. Given a positive integer $n \in l\mathbb{Z}^+$, define the *l-complexity* of n, denoted as $\|n\|_l$, as the minimal number of l's that are needed to express n in terms of l using only addition and multiplication (and parentheses). As before, one could also define the l-complexity recursively via

$$\|l\|_l := 1 \quad \text{and} \quad \|n\|_l := \min_{\substack{a,b\in l\mathbb{Z}^+ \\ a+b=n \text{ or } ab=n}} \big(\|a\|_l + \|b\|_l\big).$$

Indeed, l-complexity specializes to integer complexity when $l = 1$, and in this article, we will mostly focus on $l = 2$. Our main result is a complete classification of $n \in 2\mathbb{Z}^+$ with $\|n\|_2 = m + 1$ and $\|n\|_2 = m + 2$, where $m = \lceil \log_2 n \rceil - 1$.

Theorem 1. *Let $n \in 2\mathbb{Z}^+$ and $m = \lceil \log_2 n \rceil - 1$, i.e., $2^m < n \leq 2^{m+1}$.*

(1) $\|n\|_2 = m + 1$ *if and only if* $n = 2^{m+1}$ *or* $n = 2^m + 2^{m'}$ *for* $1 \leq m' < m$.

(2) $\|n\|_2 = m + 2$ *if and only if n is of one of the following forms*

  (a) $2^{m_1} + 2^{m_2} + 2^{m_3}$ *for* $m = m_1 > m_2 > m_3 \geq 1$;

  (b) $2^{m_1} + 2^{m_2} + 2^{m_3} + 2^{m_4}$ *for* $m = m_1 > m_2 > m_3 > m_4 \geq 2$ *with* $m_1 + m_4 = m_2 + m_3$;

  (c) $2^{m_1} + 2^{m_1-3} + 2^{m_2} + 2^{m_2-1}$ *for* $m = m_1 \geq m_2 + 3 \geq 6$;

  (d) $2^m + 2^{m-5} + 2^{m-6} + 2^{m-7}$ *for* $m \geq 10$.

Using this classification, we are able to prove the following result on the 2-complexity of even positive integers of certain particular forms.

Theorem 2. *For $m \geq 0$, we have*

(1) $\|2^m \cdot 6^r\|_2 = m + 3r$ *for* $1 \leq r \leq 7$, *and* $m + 3r - 1 \leq \|2^m \cdot 6^r\|_2 \leq m + 3r$ *for* $8 \leq r \leq 9$;

(2) $\|2^m \cdot 10^r\|_2 = m + 4r$ *for* $1 \leq r \leq 4$, *and* $m + 19 \leq \|2^m \cdot 10^5\|_2 \leq m + 20$.

We would also like to propose the following conjecture cautiously, which may well be entirely wrong for large r. One should compare this conjecture with Conjecture 1 in the realm of integer complexity. In Section 4, we also propose a list of other interesting questions to investigate.

Conjecture 2. For $m \geq 0$ and $r \geq 1$, $\|2^m \cdot 6^r\|_2 = m+3r$ and $\|2^m \cdot 10^r\|_2 = m+4r$.

2. Preparatory Lemmas

Throughout this section, let $l > 1$ be a fixed positive integer. The main result that we will prove in this section is that l-complexity also grows logarithmically. For convenience, we also adopt the notational convention that $\|1\|_l = 0$ for $l > 1$. Note that this notation is compatible with that $\|ab\|_l \leq \|a\|_l + \|b\|_l$ when a or b is 1.

Proposition 1. *Let $l > 1$ and $n \in l\mathbb{Z}^+$. Then, $\log_l n \leq \|n\|_l \leq l \log_l n - 1$.*

The lower bound follows from Lemma 1, which may seem trivially true but does depend on the assumption that $l > 1$; the upper bound follows from Lemma 2.

Lemma 1. *Let $l > 1$ and $m \in \mathbb{Z}^+$. Then, l^m is the largest number that one can obtain using addition and multiplication with m copies of l.*

Proof. Indeed, for $l > 1$, we always have $a + l \leq a \cdot l$ for all $a \in \mathbb{Z}^+$, so the lemma follows. □

Lemma 2. *Let $l > 1$. Then,*

$$\|a_1 l^{m_1} + a_2 l^{m_2} + \cdots + a_k l^{m_k}\|_l \ \leq\ m_1 + a_1 + a_2 + \cdots + a_k - 1$$

for $m_1 > m_2 > \cdots > m_k \geq 1$ and $0 \leq a_i \leq l - 1$ with $a_1 \neq 0$.

Proof. We prove this by induction on k. If $k = 1$, then

$$\|a_1 l^{m_1}\|_l \ \leq\ \|l^{m_1 - 1}\|_l + \|a_1 l\|_l \ \leq\ m_1 + a_1 - 1,$$

so the result holds. Now, suppose that $k > 1$ and that the result holds for $k - 1$. Then,

$$\begin{aligned}
\|a_1 l^{m_1} + \cdots + a_k l^{m_k}\|_l \ &\leq\ \|l^{m_k - 1}\|_l + \|a_1 l^{m_1 - m_k + 1} + a_2 l^{m_2 - m_k + 1} + \cdots + a_k l\|_l \\
&\leq\ \|l^{m_k - 1}\|_l + \|a_1 l^{m_1 - m_k + 1} + \cdots + a_{k-1} l^{m_{k-1} - m_k + 1}\|_l + \|a_k l\|_l \\
&\leq\ (m_k - 1) + \big((m_1 - m_k + 1) + (a_1 + \cdots + a_{k-1}) - 1\big) + a_k \\
&=\ m_1 + a_1 + a_2 + \cdots + a_k - 1,
\end{aligned}$$

where in the third line we use the induction hypothesis for $k - 1$. The result then follows from induction. □

Proof of Proposition 1. The lower bound follows from the fact that $n \leq l^{\|n\|_l}$ by Lemma 1. For the upper bound, write $n = a_1 l^{m_1} + a_2 l^{m_2} + \cdots + a_k l^{m_k}$ as in Lemma 2. Then, $m_1 \leq \log_l n$ and $k \leq m_1 \leq \log_l n$. Hence, by Lemma 2,

$$\begin{aligned}
\|n\|_l \ &\leq\ m_1 + a_1 + a_2 + \cdots + a_k - 1 \\
&\leq\ m_1 + (l - 1)k - 1 \\
&\leq\ \log_l n + (l - 1) \log_l n - 1 \ =\ l \log_l n - 1.
\end{aligned}$$

□

Corollary 1. *Let $l > 1$. Then, $\|l^m + l^{m'}\|_l = m + 1$ for $m \geq m' \geq 1$.*

Proof. It follows from Lemma 2 that $\|l^m + l^{m'}\|_l \leq m + 1$. On the other hand, it follows from Proposition 1 that $\|l^m + l^{m'}\|_l \geq \log_l(l^m + l^{m'}) > m$. The result then follows. □

Our last proposition also demonstrates the difference between $l = 1$ and $l > 1$. This proposition is essentially saying that given $n = b + al$ for $b \in l^2\mathbb{Z}^+$ and $0 \leq a \leq l - 1$, the most efficient way to write n using l is to first write b using l in the most efficient way, and then write al as $l + \cdots + l$ with a l's.

Proposition 2. *Let $l > 1$. For $n \in l\mathbb{Z}^+$, write $n = b + al$ uniquely for $b \in l^2\mathbb{Z}^+$ and $0 \leq a \leq l - 1$. Then, $\|n\|_l = \|b\|_l + a$.*

Proof. We will prove this by induction on n. If $b = 0$, then $\|al\|_l = a$ simply because we can only use addition in this case as $al < l \cdot l$ (or use the recursive definition inductively). Hence, suppose that $b > 0$, i.e., $n \geq l^2$. If $a = 0$, then indeed this holds, so suppose that $a > 0$. Now, suppose that the proposition holds for all $n' < n$. By the recursive definition, there exist $x, y \in l\mathbb{Z}^+$ with $x + y = b + al$ or $xy = b + al$ such that $\|x\|_l + \|y\|_l = \|b + al\|_l$. Since $l^2 \nmid b + al$, this can only happen when $x + y = b + al$.

Let $x = b_x + a_x l$ and $y = b_y + a_y l$ with $b_x, b_y \in l^2\mathbb{Z}^+$ and $0 \leq a_x, a_y \leq l - 1$. By the induction hypothesis for $x, y < n$,

$$\|x\|_l \; = \; \|b_x\|_l + a_x \quad \text{and} \quad \|y\|_l \; = \; \|b_y\|_l + a_y.$$

Note that we must have $a_x + a_y = a$ or $l + a$. Suppose by contradiction that $a_x + a_y = l + a$. On one hand,

$$l + 1 \; \leq \; l + a \; = \; a_x + a_y \; \leq \; 2l - 2$$

so $l \geq 3$. On the other hand,

$$b + al \; = \; x + y \; = \; b_x + b_y + l^2 + al$$

so

$$\begin{aligned} \|b_x\|_l + \|b_y\|_l + a_x + a_y \; = \; \|x\|_l + \|y\|_l \; = \; \|b + al\|_l \; &= \; \|b_x + b_y + l^2 + al\|_l \\ &\leq \; \|b_x\|_l + \|b_y\|_l + 2 + a. \end{aligned}$$

This implies that $l + a = a_x + a_y \leq 2 + a$, so $l \leq 2$, which leads to contradiction. Hence, $a_x + a_y = a$ and $b_x + b_y = b$, so

$$\|b_x\|_l + \|b_y\|_l + a \; = \; \|x\|_l + \|y\|_l \; = \; \|b + al\|_l \; \leq \; \|b\|_l + a \; \leq \; \|b_x\|_l + \|b_y\|_l + a.$$

The result then follows. □

3. 2-Complexity

In this section, we will specialize everything to the case when $l = 2$. This section is dedicated to proving Theorem 1 and Theorem 2. We first summarize the results that follow from Section 2.

Lemma 3. *The following are true:*

(1) $\|2^{m_1}\|_2 = m_1$ *and* $\|2^{m_1} + 2^{m_2}\|_2 = m_1 + 1$ *for* $m_1 > m_2 \geq 1$*;*

(2) $\|2^{m_1} + 2^{m_2} + \cdots + 2^{m_k}\|_2 \leq m_1 + k - 1$ *for* $m_1 > m_2 > \cdots > m_k \geq 1$*;*

(3) $\log_2 n \leq \|n\|_2 \leq 2\log_2 n - 1$.

Note that the first part of Lemma 3 essentially says that the inequality in the second part is an equality when $k = 1, 2$. In fact, the equality also holds when $k = 3$. To show this, we will first prove the following proposition, which is also the first part of Theorem 1.

Proposition 3. *Let* $m \geq 1$ *and* $n \in 2\mathbb{Z}^+$ *with* $2^m < n \leq 2^{m+1}$*. Then,* $\|n\|_2 = m+1$ *if and only if* n *is of the form* $2^m + 2^t$ *for some* $1 \leq t \leq m$*.*

Proof. Indeed, for $1 \leq t \leq m$, $\|2^m + 2^t\|_2 = m + 1$ by Corollary 1. Also, the result is trivially true when $m = 1$ and easy to verify when $m = 2$. We will now prove this by induction on m.

Let $m > 2$ and suppose that the result holds for all $m' < m$. Let $n \in 2\mathbb{Z}^+$ with $\|n\|_2 = m+1$ and $2^m < n \leq 2^{m+1}$. Then, there exist $a, b \in 2\mathbb{Z}^+$ such that $n = a+b$ or $n = ab$ with $\|a\|_2 + \|b\|_2 = \|n\|_2 = m + 1$. Let $m_a = \|a\|_2$ and $m_b = \|b\|_2$. If $n = a + b$, then $2^m < n \leq 2^{m_a} + 2^{m_b} \leq 2^m + 2$, so $n = 2^m + 2$ and hence is of the required form.

Now, suppose $n = ab$. Note that if $a \leq 2^{m_a-1}$, then $n = ab \leq 2^{m_a-1} \cdot 2^{m_b} \leq 2^m$, contradicting the assumption. Hence, $2^{m_a-1} < a \leq 2^{m_a}$ and similarly $2^{m_b-1} < b \leq 2^{m_b}$. If $m_a = 1$, then $a = 2, n = 2b$, and $\|b\|_2 = m$ with $2^{m-1} < b \leq 2^m$. As $m > m - 1 \geq 1$, $b = 2^{m-1} + 2^t$ for some $1 \leq t \leq m - 1$ by the induction hypothesis for $m - 1$, so $n = 2b$ is of the required form.

Hence, suppose that $m_a, m_b > 1$. For convenience, let $c \in \{a, b\}$. As $m > m_c - 1 \geq 1$ and $2^{m_c-1} < c \leq 2^{m_c}$, we have $c = 2^{m_c-1} + 2^{t_c}$ for some $1 \leq t_c \leq m_c - 1$ by the induction hypothesis for $m_c - 1$. Then,

$$\begin{aligned} 2^{m_a+m_b-1} \leq n = ab &= (2^{m_a-1} + 2^{t_a})(2^{m_b-1} + 2^{t_b}) \\ &= 2^{m_a+m_b-2} + 2^{m_a+t_b-1} + 2^{m_b+t_a-1} + 2^{t_a+t_b}. \end{aligned}$$

It is easy to see that one must have $t_c \geq m_c - 2$ for some $c \in \{a, b\}$ in order for the right hand side to be $\geq 2^{m_a+m_b-1}$. Without loss of generality, suppose that

$t_a \geq m_a - 2$. If $t_a = m_a - 1$, then $n = 2^{m_a} \cdot b$ so it is of the required form. Hence, suppose that $t_a = m_a - 2$. Then,

$$2^{m_a+m_b-1} \;\leq\; n \;=\; 2^{m_a+m_b-2} + 2^{m_a+m_b-3} + 2^{m_a+t_b-1} + 2^{m_a+t_b-2},$$

implying that $t_b \geq m_b - 2$. Now, if $t_b = m_b - 1$, then $n = 2^{m_a+m_b} = 2^m + 2^m$ is of the required form; if $t_b = m_b - 2$, then

$$n \;=\; (2^{m_a-1} + 2^{m_a-2})(2^{m_b-1} + 2^{m_b-2}) \;=\; 2^{m_a+m_b-1} + 2^{m_a+m_b-4}$$

so n is also of the required form (note that $m_a, m_b > 2$ in this case). The result then follows. □

An immediate corollary of this lemma is a slightly better lower bound for $\|n\|_2$ when n is not expressible as a sum of two powers of 2, as stated below.

Corollary 2. *Let $n \in 2\mathbb{Z}^+$. If n is not expressible as a sum of (one or) two powers of 2, then*

$$\log_2 n + 1 \;\leq\; \|n\|_2.$$

Proof. Let $m = \lceil \log_2 n \rceil - 1$ so that $2^m < n \leq 2^{m+1}$. Note that $\|n\|_2 \geq \lceil \log_2 n \rceil = m+1$ by Corollary 1, and that $\|n\|_2 \neq m+1$ by the assumption and Proposition 3, so it follows that $\|n\|_2 \geq m + 2 = \lceil \log_2 n \rceil + 1$. □

Corollary 3. *Let $m \geq 3$ and $n \in 2\mathbb{Z}^+$ such that $2^m < n \leq 2^{m+1}$. Then, $\|n\|_2 = m + 2$ if n is of one of the following forms:*

(a) $2^{m_1} + 2^{m_2} + 2^{m_3}$ *for* $m = m_1 > m_2 > m_3 \geq 1$*;*

(b) $2^{m_1} + 2^{m_2} + 2^{m_3} + 2^{m_4}$ *for* $m = m_1 > m_2 > m_3 > m_4 \geq 2$ *with* $m_1 + m_4 = m_2 + m_3$*;*

(c) $2^{m_1} + 2^{m_1-3} + 2^{m_2} + 2^{m_2-1}$ *for* $m = m_1 \geq m_2 + 3 \geq 6$*.*

(d) $2^m + 2^{m-5} + 2^{m-6} + 2^{m-7}$ *for* $m \geq 10$*.*

Proof. First, note that in each case n is not expressible as a sum of two powers of 2, so $m + 2 = \lceil \log_2 n \rceil + 1 \leq \|n\|_2$ by Corollary 2. It then suffices to show that $\|n\|_2 \leq m + 2$ in each case, and we will show this case by case.

(a): It follows from Lemma 3 that $\|2^{m_1} + 2^{m_2} + 2^{m_3}\|_2 \leq m_1 + 3 - 1 = m + 2$.

(b): As $m_1 + m_4 = m_2 + m_3$, we have

$$2^{m_1} + 2^{m_2} + 2^{m_3} + 2^{m_4} \;=\; 2^{m_4-2} \cdot (2^{m_2-m_4+1} + 2) \cdot (2^{m_3-m_4+1} + 2).$$

It then follows from Lemma 3 that

$$\begin{aligned}\|2^{m_1} + 2^{m_2} + 2^{m_3} + 2^{m_4}\|_2 \;&\leq\; m_4 - 2 + (m_2 - m_4 + 2) + (m_3 - m_4 + 2)\\ &=\; m_1 + 2 \;=\; m + 2.\end{aligned}$$

(c): Note that

$$2^{m_1} + 2^{m_1-3} + 2^{m_2} + 2^{m_2-1} \ = \ (2^{m_2-1} + 2^{m_2-2}) \cdot (2^{m_1-m_2} + 2^{m_1-m_2-1} + 2).$$

It then follows from Lemma 3 that

$$\|2^{m_1} + 2^{m_1-3} + 2^{m_2} + 2^{m_2-1}\|_2 \ \leq \ m_2 + (m_1 - m_2 + 2) \ = \ m + 2.$$

(d): Note that

$$2^m + 2^{m-5} + 2^{m-6} + 2^{m-7} \ = \ (2^{m-6} + 2^{m-9}) \cdot (2^5 + 2^4 + 2^3 + 2^2).$$

It then follows from Lemma 3 and (b) that

$$\|2^m + 2^{m-5} + 2^{m-6} + 2^{m-7}\|_2 \ \leq \ (m-5) + 7 \ = \ m + 2.$$

In conclusion, we always have $\|n\|_2 \leq m+2$ in each case. The result then follows. □

As promised before, it follows from this lemma that the inequality in the second part of Lemma 3 is an equality when $k = 3$ but is no longer an equality when $k \geq 4$. In fact, the four cases described in Corollary 3 are the only cases with $2^m < n \leq 2^{m+1}$ and $\|n\|_2 = m + 2$, as shown in the following proposition, which is also the second part of Theorem 1.

Proposition 4. *Let $m \geq 3$ and $n \in 2\mathbb{Z}^+$ with $2^m < n \leq 2^{m+1}$. Then, $\|n\|_2 = m+2$ if and only if n is of one of the forms described in Corollary 3.*

Proof. Throughout the proof, we also adopt the notation $c \in \{a, b\}$ for convenience as before. We will prove this by induction on m. The result is trivially true when $m = 3, 4$ so suppose that $m > 4$ and that the result holds for all $m' < m$.

First, let $a, b \in 2\mathbb{Z}^+$ such that $n = a + b$ with $\|a\|_2 + \|b\|_2 = \|n\|_2 = m + 2$. If $a, b > 2$, then $n = a + b \leq 2^{\|a\|_2} + 2^{\|b\|_2} \leq 2^m + 2^2$, implying that $n = 2^m + 2$ or $2^m + 2^2$, both of which contradict that $\|n\|_2 = m + 2$ by Lemma 3. Thus, one of a, b must be 2. Without loss of generality, let $a = 2$. Then, $\|b\|_2 = m + 1$ and $2^m - 2 < b \leq 2^{m+1} - 2$. This implies that $2^m < b < 2^{m+1}$ so $b = 2^m + 2^t$ for some $m > t \geq 1$ by Proposition 3. We have $t > 1$ as otherwise $n = 2^m + 2^2$ contradicting that $\|n\|_2 = m + 2$. Thus, $n = 2^m + 2^t + 2$ is of the form (a).

Now, let $a, b \in 2\mathbb{Z}^+$ such that $n = ab$ with $\|a\|_2 + \|b\|_2 = \|n\|_2 = m + 2$. Let $m_c = \|c\|_2$ for $c \in \{a, b\}$. If $a = 2^{m_a}$, then $2^{m-m_a} < b \leq 2^{m-m_a+1}$ and $\|b\|_2 = m - m_a + 2$, so b is of one of the forms by the induction hypothesis for $m - m_a$, and hence so is $n = 2^{m_a} \cdot b$. Thus, suppose that $c < 2^{m_c}$ for both $c \in \{a, b\}$, and note that this implies that $m_c > 2$ for both $c \in \{a, b\}$. If $a \leq 2^{m_a-2}$, then $n = ab \leq 2^{m_a-2} \cdot 2^{m_b} = 2^m$, contradicting that $n > 2^m$. Hence, $c > 2^{m_c-2}$ for both $c \in \{a, b\}$.

To summarize, we are now in the situation where $m_c > 2$ and $2^{m_c-2} < c < 2^{m_c}$ for both $c \in \{a, b\}$. If $c < 2^{m_c-1}$ for both $c \in \{a, b\}$, then $n = ab < 2^m$, which leads to contradiction. Suppose for now that $c > 2^{m_c-1}$ for both $c \in \{a, b\}$. Then, $2^{m_c-1} < c < 2^{m_c}$ and $\|c\|_2 = m_c$, so $c = 2^{m_c-1} + 2^{t_c}$ for some $1 \leq t_c < m_c - 1$ by Proposition 3. Hence,

$$n \;=\; ab \;=\; 2^{m_a+m_b-2} + 2^{m_a+t_b-1} + 2^{m_b+t_a-1} + 2^{t_a+t_b}$$

with $m_a + m_b - 2 > m_a + t_b - 1, m_b + t_a - 1 > t_a + t_b$, so n is of either the form (a) or (b). Thus, suppose without loss of generality that $a > 2^{m_a-1}$ and $b < 2^{m_b-1}$. As before, we still have $a = 2^{m_a-1} + 2^{t_a}$ for some $1 \leq t_a < m_a - 1$.

To summarize again, we are now in the situation where $m_a, m_b > 2$, $a = 2^{m_a-1} + 2^{t_a}$ with $1 \leq t_a < m_a - 1$, and $2^{m_b-2} < b < 2^{m_b-1}$ with $\|b\|_2 = m_b$. Note that the condition on b readily implies that $m_b \geq 5$. By the induction hypothesis for $m_b - 2$, b is of one of the forms, so $b \leq 2^{m_b-2} + 2^{m_b-3} + 2^{m_b-4} + 2^{m_b-5} = 15 \cdot 2^{m_b-5}$. Hence,

$$2^{m_a-1} + 2^{t_a} \;=\; \frac{n}{b} \;>\; \frac{2^m}{b} \;\geq\; \frac{2^m}{15 \cdot 2^{m_b-5}} \;=\; \frac{2^{m_a+3}}{15}$$

so $2^{t_a} > \frac{2^{m_a-1}}{15}$, implying that $t_a \geq m_a - 4$.

`Case i.` $t_a = m_a - 4$. Then,

$$b \;>\; \frac{2^m}{2^{m_a-1} + 2^{m_a-4}} \;=\; \frac{2^{m_b+2}}{9} \;=\; \frac{7 \cdot 2^{m_b+2}}{63} \;>\; 7 \cdot 2^{m_b-4} \;=\; 2^{m_b-2} + 2^{m_b-3} + 2^{m_b-4}.$$

Since b is of one of the forms, we must have $b = 2^{m_b-2} + 2^{m_b-3} + 2^{m_b-4} + 2^{m_b-5}$, so n is of the form (d).

`Case ii.` $t_a = m_a - 3$. Then,

$$b > \frac{2^m}{2^{m_a-1} + 2^{m_a-3}} = \frac{2^{m_b+1}}{5} = \frac{51 \cdot 2^{m_b+1}}{255} > 51 \cdot 2^{m_b-7} = 2^{m_b-2} + 2^{m_b-3} + 2^{m_b-6} + 2^{m_b-7}.$$

Since b is of one of the forms, we must have

(1) $b = 2^{m_b-2}+2^{m_b-3}+2^{m_b-5}$ so $n = 2^m+2^{m-6}$, contradicting that $\|n\|_2 = m+1$;
(2) $b = 2^{m_b-2} + 2^{m_b-3} + 2^{m_b-5} + 2^{m_b-6}$ so n is of the form (d);
(3) $b = 2^{m_b-2} + 2^{m_b-3} + 2^{m_b-4}$ so n is of the form (a);
(4) $b = 2^{m_b-2} + 2^{m_b-3} + 2^{m_b-4} + 2^{m_b-5}$ so n is also of the form (c).

`Case iii.` $t_a = m_a - 2$. Then,

$$b \;>\; \frac{2^m}{2^{m_a-1} + 2^{m_a-2}} \;=\; \frac{2^{m_b}}{3} \;=\; \frac{85 \cdot 2^{m_b}}{255} \;>\; 85 \cdot 2^{m_b-8} \;=\; 2^{m_b-2} + 2^{m_b-4} + 2^{m_b-6} + 2^{m_b-8}.$$

Since b is of one of the forms, we must have

(1) $b = 2^{m_b-2}+2^{m_b-4}+2^{m_b-5}$ so $n = 2^m+2^{m-5}$, contradicting that $\|n\|_2 = m+2$;
(2) $b = 2^{m_b-2} + 2^{m_b-4} + 2^{m_b-5} + 2^{m_b-7}$ so n is of the form (d);
(3) $b = 2^{m_b-2} + 2^{m_b-3} + 2^{t_b}$ with $1 \le t_b < m_b - 3$ so n is of the form (c);
(4) $b = 2^{m_b-2} + 2^{m_b-3} + 2^{t_b} + 2^{t_b-1}$ with $t_b < m_b - 3$ so n is of the form (b).

In conclusion, n must be of one of the forms described in Corollary 3. □

As before, this proposition also yields a slightly better lower bound for us.

Corollary 4. *Let $n \in 2\mathbb{Z}^+$. If n is not expressible as a sum of two powers of 2 or one of the forms described in Corollary 3, then $\log_2 n + 2 \le \|n\|_2$. In particular, if n is expressible as a sum of at least five distinct powers of 2, then $\log_2 n+2 \le \|n\|_2$.*

Having this corollary, we are now ready to prove Theorem 2.

Proposition 5. *For $m \ge 0$, we have*

$$\|2^m \cdot 6^r\|_2 \ = \ m + 3r \quad \textit{for } 1 \le r \le 7$$

and

$$m + 3r - 1 \ \le \ \|2^m \cdot 6^r\|_2 \ \le \ m + 3r \quad \textit{for } 8 \le r \le 9.$$

Proof. By Lemma 3, if $r = 1$, then $\|2^m \cdot 6\|_2 = \|2^{m+2} + 2^{m+1}\|_2 = m + 3$, and if $r = 2$, then $\|2^m \cdot 6^2\|_2 = \|2^{m+5} + 2^{m+2}\|_2 = m + 6$. Now, assume that $3 \le r \le 9$ so that $n = 2^m \cdot (2^2 + 2)^r$ is not a sum of two powers of 2. We then have $\log_2 n + 1 \le \|n\|_2$ by Corollary 2, so $m + (\log_2 6)r + 1 \le \|n\|_2$. On the other hand, $\|n\|_2 \le m + r\|6\|_2 = m + 3r$. Thus, $\lceil m + (\log_2 6)r + 1\rceil \le \|n\|_2 \le m + 3r$. When $r = 3, 4$, computation shows that $\lceil(\log_2 6)r + 1\rceil = 3r$, so the result follows.

Now, assume that $5 \le r \le 9$. Computation shows that 6^r is a sum of at least five distinct powers of 2 in this case, and so is $n = 2^m \cdot 6^r$. We then have $\log_2 n+2 \le \|n\|_2$ by Corollary 4, so $m+(\log_2 6)r+2 \le \|n\|_2$. Thus, $\lceil m+(\log_2 6)r+2\rceil \le \|n\|_2 \le m+3r$ for $5 \le r \le 9$. Now, computation shows that $\lceil(\log_2 6)r + 2\rceil = 3r$ when $5 \le r \le 7$, and that $\lceil(\log_2 6)r + 2\rceil = 3r - 1$ when $8 \le r \le 9$, so the result follows. □

Proposition 6. *For $m \ge 0$, we have*

$$\|2^m \cdot 10^r\|_2 \ = \ m + 4r \quad \textit{for } 1 \le r \le 4$$

and for $r = 5$,

$$m + 19 \ \le \ \|2^m \cdot 10^5\|_2 \ \le \ m + 20.$$

Proof. If $r = 1$, then $\|2^m \cdot 10\|_2 = \|2^{m+3} + 2^{m+1}\|_2 = m + 4$ by Lemma 3; if $r = 2$, then $\|2^m \cdot 10^2\|_2 = \|2^{m+6} + 2^{m+5} + 2^{m+2}\|_2 = m + 8$ by Corollary 3. Now, assume that $3 \le r \le 5$. Computation shows that 10^r is a sum of at least five distinct powers of 2 in this case, and so is $n = 2^m \cdot 10^r$. We then have $\log_2 n + 2 \le \|n\|_2$ by Corollary 4, so $m + (\log_2 10)r + 2 \le \|n\|_2$. On the other hand, $\|n\|_2 \le m + r\|10\|_2 = m + 4r$. Thus, $\lceil m + (\log_2 10)r + 2\rceil \le \|n\|_2 \le m + 4r$. Now, computation shows that $\lceil(\log_2 10)r + 2\rceil = 4r$ when $3 \le r \le 4$, and that $\lceil(\log_2 10)r + 2\rceil = 4r - 1$ when $r = 5$, so the result follows. □

4. Further Questions

In the final section, we will propose a list of questions on 2-complexity, and in general l-complexity, that seem interesting to investigate further.

Question 1. For $l > 1$, can one improve the bounds for $\|n\|_l$ in Proposition 1 for all $n \in l\mathbb{Z}^+$ or for a density 1 subset of $l\mathbb{Z}^+$?

Question 2. Can one determine the l-complexity of multiples of l that are of certain particular forms? For example, are the following true?

(1) For l not divisible by 2, $\|(2l)^r\|_l = 2r$ for all $r \ge 1$, and if $l > 1$, then $\|l^m \cdot (2l)^r\|_l = m + 2r$ for all $m \ge 0$ and $r \ge 1$. (Note that $l = 1$ is a special case of Conjecture 1.)

(2) For l not divisible by 3, $\|(3l)^r\|_l = 3r$ for all $r \ge 1$, and if $l > 1$, then $\|l^m \cdot (3l)^r\|_l = m + 3r$ for all $m \ge 0$ and $r \ge 1$. (Note that $l = 1$ follows from Selfridge's result and $l = 2$ is one of the cases of Conjecture 2.)

The next question should be compared with the result on integer complexity that for every $n \in \mathbb{Z}^+$, there exists $m_0 \ge 0$ such that for all $m \ge m_0$, $\|3^m \cdot n\|_1 = 3(m - m_0) + \|3^{m_0} \cdot n\|_1$ (see [2, Theorem 1.5]).

Question 3. Let $l > 1$ and define

$$A_l := \left\{ n \in l\mathbb{Z}^+ \,\middle|\, \|l^m \cdot n\|_l = m + \|n\|_l \text{ for all } m \ge 0 \right\}.$$

Can one say anything about the set A_l? For example, is it true that for every $n \in l\mathbb{Z}^+$, there exists $m_0 \ge 0$ such that $l^{m_0} \cdot n \in A_l$? (Note that e.g. $54 \notin A_2$ since $\|54\|_2 = 8$ and $\|2^2 \cdot 54\|_2 = \|6^3\|_2 = 9$, but $2^2 \cdot 54 \in A_2$ by Theorem 2.)

Question 4. For $l = 2$, should one expect Conjecture 2 to hold in a more general form, i.e., $\|2^m \cdot (2^u + 2)^r\|_2 = m + (u + 1)r$ for all $m \ge 0$, $u \ge 2$, and $r \ge 1$?

The last question is an interesting interplay between integer complexity and 2-complexity. Before stating the question, recall that if $\mathcal{R}$ is a most efficient representation of $n \in \mathbb{Z}^+$ in terms of 1's, then $\mathcal{R}$ does not contain $1 + \cdots + 1$ with more than five 1's. In particular, one can always replace $1+1+1+1$ with $(1+1)(1+1)$ and $1+1+1+1+1$ with $(1+1)(1+1)+1$ to assume that $\mathcal{R}$ does not contain $1 + \cdots + 1$ with more than three 1's.

Question 5. Let $n \in \mathbb{Z}^+$ and let $\mathcal{R}$ be a most efficient representation of n in terms of 1's that does not contain $1 + \cdots + 1$ with more than three 1's. Let $a(n, \mathcal{R})$ be the even number obtained by replacing all the 1's in $\mathcal{R}$ with 2's. Then, is it true that $\|a(n, \mathcal{R})\|_2 = \|n\|_1$? If not, can one say anything about pairs $(n, \mathcal{R})$ satisfying or violating this?

Acknowledgement. The author would like to thank the anonymous referee(s) for their detailed and useful suggestions.

References

[1] Harry Altman, Integer complexity: the integer defect, *Mosc. J. Comb. Number Theory* **8** (3) (2019), 193–217.

[2] Harry Altman and Joshua Zelinsky, Numbers with integer complexity close to the lower bound, *Integers* **12** (6) (2012), 1093–1125.

[3] Juan Arias de Reyna, Complexity of natural numbers, *Integers* **24** (2024), #A19.

[4] Katherine Cordwell, Alyssa Epstein, Anand Hemmady, Steven J. Miller, Eyvindur Palsson, Aaditya Sharma, Stefan Steinerberger, and Yen Nhi Truong Vu, On algorithms to calculate integer complexity, *Integers* **19** (2019), #A12.

[5] Richard K. Guy, Unsolved Problems: Some Suspiciously Simple Sequences, *Amer. Math. Monthly* **93** (3) (1986), 186–190.

[6] K. Mahler and J. Popken, On a maximum problem in arithmetic, *Nieuw Arch. Wisk. (3)* **1** (1953), 1–15.

[7] Stefan Steinerberger, A short note on integer complexity, *Contrib. Discrete Math.* **9** (1) (2014), 63–69.