ON THE DISTRIBUTION OF THE RESTRICTED SEQUENCE OF INTEGERS $2^{\omega(n)}$

Kampamolla Venkatasubbareddy
School of Mathematics and Statistics, University of Hyderabad, Hyderabad, Telangana, India
20mmpp02@uohyd.ac.in

Ayyadurai Sankaranarayanan
School of Mathematics and Statistics, University of Hyderabad, Hyderabad, Telangana, India
sank@uohyd.ac.in

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Abstract

This paper studies the distribution of the sequence $2^{\omega(n)}$ over squareful integers n (we call an integer $n > 1$ squareful if for any prime p with $p|n$ implies $p^2|n$). We provide an unconditional asymptotic formula for the discrete mean sum of $2^{\omega(n)}$ over squareful integers. Additionally, assuming the Strong Riemann hypothesis, we extract a few more main terms, thereby improving the error term bound.

1. Introduction

Define $\omega(1) = 0$, and for $n > 1$, let $\omega(n)$ denote the number of distinct prime factors of n. We call an integer $n > 1$ squareful if for any prime p with $p|n$ implies $p^2|n$. Now, we define the arithmetical function $f(n)$ as $f(1) = 1$, and for $n > 1$,

$$f(n) = \begin{cases} 2^{\omega(n)} & \text{if } n \text{ is squareful,} \\ 0 & \text{otherwise.} \end{cases}$$

Note that $f(n)$ is multiplicative. The L-function attached to the coefficients $f(n)$, namely

$$F(s) := \sum_{n=1}^{\infty} \frac{f(n)}{n^s}, \tag{1}$$

has the following Euler product

$$F(s) = \prod_p (1 + 2p^{-2s} + 2p^{-3s} + 2p^{-4s} + \cdots) \tag{2}$$

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for $\Re(s) > \frac{1}{2}$, and $F(s)$ converges absolutely and uniformly in $\Re(s) > \frac{1}{2}$.

The Riemann zeta function is defined by

$$\zeta(s) = \sum_{n\geq 1} \frac{1}{n^s}$$

for $\Re(s) > 1$, which has a meromorphic continuation to the entire complex plane $\mathbb{C}$ by the functional equation

$$\zeta(s) = \chi(s)\zeta(1-s), \tag{3}$$

where the conversion factor $\chi(s)$ behaves, in absolute value, as

$$\mid \chi(s) \mid \sim |t|^{\frac{1}{2}-\sigma} \tag{4}$$

for $s = \sigma + it$, $|t| \geq t_0$ (see [4]).

The *Riemann hypothesis* is a conjecture that states that all the nontrivial zeros of the Riemann zeta function $\zeta(s)$ lie on the critical line $\Re(s) = \frac{1}{2}$. The *stronger version of the Riemann hypothesis* states that all the nontrivial zeros of $\zeta(s)$ lie on the critical line, and each such zero is simple. A region Ω is said to be a *zero-free region* for $\zeta(s)$ if $\zeta(s) \neq 0$ for all $s \in \Omega$. For the best-known zero-free region we refer to Theorem 6.1 of [1].

Throughout the paper, ϵ is any small positive constant. The aim here is to establish the following theorems.

Theorem 1. *Let $x > 0$ be large and $\epsilon > 0$. Then we have unconditionally,*

$$\sum_{n\leq x} f(n) = \mathcal{C}_1 x^{\frac{1}{2}} \log x + \mathcal{C}_2 x^{\frac{1}{2}} + \mathcal{C}_3 x^{\frac{1}{3}} \log x + \mathcal{C}_4 x^{\frac{1}{3}} + O\left(x^{\frac{1}{4}} \exp\left\{c\epsilon\left(\frac{\log x}{\log\log x}\right)^{\frac{1}{3}}\right\}\right),$$

for some $c > 0$ and for real constants $\mathcal{C}_1,\ \mathcal{C}_2,\ \mathcal{C}_3$ and $\mathcal{C}_4$, that can be evaluated explicitly.

Theorem 2. *Let $x > 0$ be large and $\epsilon > 0$. Assuming the strong Riemann hypothesis namely, all the nontrivial zeros of $\zeta(ls)$ lie on the respective critical lines for $l = 2,\ 3,\ \cdots,\ 14$, and each such zero is simple, we have*

$$\begin{aligned}\sum_{n\leq x} f(n) = {} & \mathcal{C}_1 x^{\frac{1}{2}} \log x + \mathcal{C}_2 x^{\frac{1}{2}} + \mathcal{C}_3 x^{\frac{1}{3}} \log x + \mathcal{C}_4 x^{\frac{1}{3}} + \\ & + \sum_{\substack{\zeta(4\rho)=0,\ \Re(\rho)=\frac{1}{8},\\ 0<\left|\Im(4\rho)=\gamma_{\frac{1}{8}}\right|<x^{\frac{3}{11}}}} \mathcal{D}_{1,\ \gamma_{\frac{1}{8}}} x^{\frac{1}{8}+i\gamma_{\frac{1}{8}}} \\ & + \sum_{\substack{\zeta(5\rho)=0,\ \Re(\rho)=\frac{1}{10},\\ 0<\left|\Im(5\rho)=\gamma_{\frac{1}{10}}\right|<x^{\frac{3}{11}}}} \left(\mathcal{D}_{2,\ \gamma_{\frac{1}{10}}} x^{\frac{1}{10}+i\gamma_{\frac{1}{10}}} \log x + \mathcal{D}_{3,\ \gamma_{\frac{1}{10}}} x^{\frac{1}{10}+i\gamma_{\frac{1}{10}}}\right)\end{aligned}$$

$$+ \sum_{\substack{\zeta(6\rho)=0,\ \Re(\rho)=\frac{1}{12},\\ 0<\left|\Im(6\rho)=\gamma_{\frac{1}{12}}\right|<x^{\frac{3}{11}}}} \Big(\mathcal{D}_{4,\ \gamma_{\frac{1}{12}}} x^{\frac{1}{12}+i\gamma_{\frac{1}{12}}} (\log x)^2$$

$$+ \mathcal{D}_{5,\ \gamma_{\frac{1}{12}}} x^{\frac{1}{12}+i\gamma_{\frac{1}{12}}} \log x + \mathcal{D}_{6,\ \gamma_{\frac{1}{12}}} x^{\frac{1}{12}+i\gamma_{\frac{1}{12}}}\Big) + O\left(x^{\frac{5}{22}+\epsilon}\right),$$

where $\mathcal{C}_i$'s are certain real constants, and $\mathcal{D}_i$'s are certain complex constants that can be evaluated explicitly.

Remark 1. We observe that the main term

$$S_1 := \sum_{\substack{\zeta(4\rho)=0,\ \Re(\rho)=\frac{1}{8},\\ 0<\left|\Im(4\rho)=\gamma_{\frac{1}{8}}\right|<x^{\frac{3}{11}}}} \mathcal{D}_{1,\ \gamma_{\frac{1}{8}}} x^{\frac{1}{8}+i\gamma_{\frac{1}{8}}}$$

in Theorem 1, satisfies

$$|S_1| \leq x^{\frac{1}{8}} \left| \sum_{\substack{\zeta(4\rho)=0,\ \Re(\rho)=\frac{1}{8},\\ 0<\left|\Im(4\rho)=\gamma_{\frac{1}{8}}\right|<x^{\frac{3}{11}}}} \mathcal{D}_{1,\ \gamma_{\frac{1}{8}}} x^{i\gamma_{\frac{1}{8}}} \right|.$$

Since we expect enough cancellations to happen in the sum S_1, we are tempted to make a plausible conjecture that $S_1 = O(x^{\frac{5}{22}+\epsilon})$, but we are not in a position to establish this assertion. It is also to be noted that, trivially,

$$|\ S_1\ | \leq x^{\frac{1}{8}} \left(\max_{\substack{\zeta(4\rho)=0,\ \Re(\rho)=\frac{1}{8},\\ 0<\left|\Im(4\rho)=\gamma_{\frac{1}{8}}\right|<x^{\frac{3}{11}}}} |\ \mathcal{D}_{1,\ \gamma_{\frac{1}{8}}}\ | \right) T\log T \ll \left(\max_{\substack{\zeta(4\rho)=0,\ \Re(\rho)=\frac{1}{8},\\ 0<\left|\Im(4\rho)=\gamma_{\frac{1}{8}}\right|<x^{\frac{3}{11}}}} |\ \mathcal{D}_{1,\ \gamma_{\frac{1}{8}}}\ | \right) x^{\frac{35}{88}+\epsilon}$$

with $T = x^{\frac{3}{11}}$. Observe that $\frac{1}{3} < \frac{35}{88}$. Similarly, we do expect cancellations to happen with the sums

$$S_2 := \sum_{\substack{\zeta(5\rho)=0,\ \Re(\rho)=\frac{1}{10},\\ 0<\left|\Im(5\rho)=\gamma_{\frac{1}{10}}\right|<x^{\frac{3}{11}}}} \Big(\mathcal{D}_{2,\ \gamma_{\frac{1}{10}}} x^{\frac{1}{10}+i\gamma_{\frac{1}{10}}} \log x + \mathcal{D}_{3,\ \gamma_{\frac{1}{10}}} x^{\frac{1}{10}+i\gamma_{\frac{1}{10}}}\Big)$$

and

$$S_3 := \sum_{\substack{\zeta(5\rho)=0,\ \Re(\rho)=\frac{1}{12},\\ 0<\left|\Im(6\rho)=\gamma_{\frac{1}{12}}\right|<x^{\frac{3}{11}}}} \Big(\mathcal{D}_{4,\ \gamma_{\frac{1}{12}}} x^{\frac{1}{12}+i\gamma_{\frac{1}{12}}} (\log x)^2 + \mathcal{D}_{5,\ \gamma_{\frac{1}{12}}} x^{\frac{1}{12}+i\gamma_{\frac{1}{12}}} \log x$$

$$+ \mathcal{D}_{6,\ \gamma_{\frac{1}{12}}} x^{\frac{1}{12}+i\gamma_{\frac{1}{12}}}\Big).$$

So conjecturally, we may have $S_2 = O(x^{\frac{5}{22}+\epsilon})$ and $S_3 = O(x^{\frac{5}{22}+\epsilon})$, but we are not in a position to prove them.

For further related problems of this type, interested readers may consult Chapter 14 of [1].

2. Lemmas

Lemma 1. *For $\Re(s) > \frac{1}{2}$, we have*

$$F(s) = \frac{\zeta^2(2s)\zeta^2(3s)\zeta^2(7s)\zeta^4(8s)\zeta^5(12s)\zeta^{14}(13s)\zeta^5(14s)}{\zeta(4s)\zeta^2(5s)\zeta^3(6s)\zeta^5(10s)\zeta^6(11s)} H_1(s), \tag{5}$$

where $H_1(s)$ is a Dirichlet series which converges absolutely and uniformly for $\Re(s) > \frac{1}{15}$.

Proof. For $\Re(s) > \frac{1}{2}$, we have the Euler product for $F(s)$ as

$$F(s) = \prod_p (1 + 2p^{-2s} + 2p^{-3s} + 2p^{-4s} + \cdots). \tag{6}$$

If we let $X = p^{-s}$, then for $\Re(s) > \frac{1}{2}$, we have $\mid X \mid < 1$. To prove the lemma we split the series $1 + 2X^2 + 2X^3 + 2X^4 + \cdots$ into the product of polynomials of the form $1 - X^m$ or its inverses from $m = 2$ to $m = 14$ possibly with some error terms.

By using the Mathematica software, we get

$$\begin{aligned}
&1 + 2X^2 + 2X^3 + 2X^4 + 2X^5 + \cdots \\
&= \left(1 - X^2\right)^{-2} (1 + 2X^3 - X^4 - 2X^5) \\
&= \left(1 - X^2\right)^{-2} \left(1 - X^3\right)^{-2} \times \\
&\quad (1 - X^4 - 2X^5 - 3X^6 + 2X^7 + 4X^8 + 2X^9 - X^{10} - 2X^{11}) \\
&= \left(1 - X^2\right)^{-2} \left(1 - X^3\right)^{-2} (1 - X^4)(1 - 2X^5 - 3X^6 \\
&\quad + 2X^7 + 4X^8 - 4X^{10} + 4X^{12} - 4X^{14} + O(X^{15}) \\
&= \left(1 - X^2\right)^{-2} \left(1 - X^3\right)^{-2} (1 - X^4)(1 - X^5)^2(1 - 3X^6 + 2X^7 \\
&\quad + 4X^8 - 5X^{10} - 6X^{11} + 8X^{12} + 8X^{13} - 4X^{14} + O(X^{15})) \\
&= \left(1 - X^2\right)^{-2} \left(1 - X^3\right)^{-2} (1 - X^4)(1 - X^5)^2(1 - X^6)^3 \times \\
&\quad (1 + 2X^7 + 4X^8 - 5X^{10} - 6X^{11} + 5X^{12} + 14X^{13} + 8X^{14} + O(X^{15})) \\
&= \left(1 - X^2\right)^{-2} \left(1 - X^3\right)^{-2} (1 - X^4)(1 - X^5)^2(1 - X^6)^3(1 - X^7)^{-2} \times \\
&\quad (1 + 4X^8 - 5X^{10} - 6X^{11} + 5X^{12} + 14X^{13} + 5X^{14} + O(X^{15})) \\
&= \left(1 - X^2\right)^{-2} \left(1 - X^3\right)^{-2} (1 - X^4)(1 - X^5)^2(1 - X^6)^3(1 - X^7)^{-2}(1 - X^8)^{-4} \times \\
&\quad (1 - 5X^{10} - 6X^{11} + 5X^{12} + 14X^{13} + 5X^{14} + O(X^{15})) \\
&= \left(1 - X^2\right)^{-2} \left(1 - X^3\right)^{-2} (1 - X^4)(1 - X^5)^2(1 - X^6)^3(1 - X^7)^{-2}(1 - X^8)^{-4} \times
\end{aligned}$$

$$
\begin{aligned}
&(1-X^{10})^5(1-6X^{11}+5X^{12}+14X^{13}+5X^4+O(X^{15}))\\
&=\left(1-X^2\right)^{-2}\left(1-X^3\right)^{-2}(1-X^4)(1-X^5)^2(1-X^6)^3(1-X^7)^{-2}(1-X^8)^{-4}\times\\
&\quad(1-X^{10})^5(1-X^{11})^6(1+5X^{12}+14X^{13}+5X^{14}+O(X^{15}))\\
&=\left(1-X^2\right)^{-2}\left(1-X^3\right)^{-2}(1-X^4)(1-X^5)^2(1-X^6)^3(1-X^7)^{-2}(1-X^8)^{-4}\times\\
&\quad(1-X^{10})^5(1-X^{11})^6(1-X^{12})^{-5}(1+14X^{13}+5X^{14}+O(X^{15}))\\
&=\left(1-X^2\right)^{-2}\left(1-X^3\right)^{-2}(1-X^4)(1-X^5)^2(1-X^6)^3(1-X^7)^{-2}(1-X^8)^{-4}\times\\
&\quad(1-X^{10})^5(1-X^{11})^6(1-X^{12})^{-5}(1-X^{13})^{-14}(1+5X^{14}+O(X^{15}))\\
&=\left(1-X^2\right)^{-2}\left(1-X^3\right)^{-2}(1-X^4)(1-X^5)^2(1-X^6)^3(1-X^7)^{-2}(1-X^8)^{-4}\times\\
&\quad(1-X^{10})^5(1-X^{11})^6(1-X^{12})^{-5}(1-X^{13})^{-14}(1-X^{14})^{-5}(1+O(X^{15}). \quad (7)
\end{aligned}
$$

By substituting $X=p^{-s}$ in the above expression, by (6) we then get

$$F(s)=\frac{\zeta^2(2s)\zeta^2(3s)\zeta^2(7s)\zeta^4(8s)\zeta^5(12s)\zeta^{14}(13s)\zeta^5(14s)}{\zeta(4s)\zeta^2(5s)\zeta^3(6s)\zeta^5(10s)\zeta^6(11s)}H_1(s),$$

where $H_1(s)$ converges absolutely and uniformly for $\Re(s)>\frac{1}{15}$. □

Lemma 2. *For $\Re(s)>\frac{1}{2}$, we have*

$$F(s)=\frac{\zeta^2(2s)\zeta^2(3s)}{\zeta(4s)}H_2(s), \quad (8)$$

where $H_2(s)$ converges absolutely and uniformly for $\Re(s)>\frac{1}{5}$.

Proof. From Lemma 1, we can write

$$F(s)=\frac{\zeta^2(2s)\zeta^2(3s)}{\zeta(4s)}H_2(s),$$

where

$$H_2(s)=\frac{\zeta^2(7s)\zeta^4(8s)\zeta^5(12s)\zeta^{14}(13s)\zeta^5(14s)}{\zeta^2(5s)\zeta^3(6s)\zeta^5(10s)\zeta^6(11s)}H_1(s),$$

which converges absolutely and uniformly for $\Re(s)>\frac{1}{5}$. □

Remark 2. In the proofs of Theorems 1 and 2, we use either Lemma 1 or Lemma 2 suitably.

Lemma 3. *Let $x>0$ be large. Then under the assumption of the strong Riemann hypothesis for various zeta functions $\zeta(ls)$ appearing in $F(s)$ proved in Lemma 1 ($l=2,\ 3,\ 4,\cdots,\ 14$), we have the following residues corresponding to the poles arising from $F(s)$:*

$$\operatorname*{Res}_{s=\frac{1}{2}}\ F(s)\frac{x^s}{s}=\mathcal{C}_1x^{\frac{1}{2}}\log x+\mathcal{C}_2x^{\frac{1}{2}}, \quad (9)$$

$$\operatorname*{Res}_{s=\frac{1}{3}} F(s)\frac{x^s}{s} = \mathcal{C}_3 x^{\frac{1}{3}} \log x + \mathcal{C}_4 x^{\frac{1}{3}}, \tag{10}$$

$$\operatorname*{Res}_{s=\frac{1}{7}} F(s)\frac{x^s}{s} = \mathcal{C}_5 x^{\frac{1}{7}} \log x + \mathcal{C}_6 x^{\frac{1}{7}}, \tag{11}$$

$$\operatorname*{Res}_{s=\frac{1}{8}} F(s)\frac{x^s}{s} = \mathcal{C}_7 x^{\frac{1}{8}} (\log x)^3 + \mathcal{C}_8 x^{\frac{1}{8}} (\log x)^2 + \mathcal{C}_9 x^{\frac{1}{8}} \log x + \mathcal{C}_{10} x^{\frac{1}{8}}, \tag{12}$$

$$\operatorname*{Res}_{s=\frac{1}{12}} F(s)\frac{x^s}{s} = \mathcal{C}_{11} x^{\frac{1}{12}} (\log x)^4 + \cdots + \mathcal{C}_{14} x^{\frac{1}{12}} \log x + \mathcal{C}_{15} x^{\frac{1}{12}}, \tag{13}$$

$$\operatorname*{Res}_{s=\frac{1}{13}} F(s)\frac{x^s}{s} = \mathcal{C}_{16} x^{\frac{1}{13}} (\log x)^{13} + \cdots + \mathcal{C}_{28} x^{\frac{1}{13}} \log x + \mathcal{C}_{29} x^{\frac{1}{13}}, \tag{14}$$

$$\operatorname*{Res}_{s=\frac{1}{8}+i\gamma_{\frac{1}{8}}} F(s)\frac{x^s}{s} = \mathcal{D}_{1,\ \gamma_{\frac{1}{8}}} x^{\frac{1}{8}+i\gamma_{\frac{1}{8}}}, \tag{15}$$

$$\operatorname*{Res}_{s=\frac{1}{10}+i\gamma_{\frac{1}{10}}} F(s)\frac{x^s}{s} = \mathcal{D}_{2,\ \gamma_{\frac{1}{10}}} x^{\frac{1}{10}+i\gamma_{\frac{1}{10}}} \log x + \mathcal{D}_{3,\ \gamma_{\frac{1}{10}}} x^{\frac{1}{10}+i\gamma_{\frac{1}{10}}}, \tag{16}$$

$$\begin{aligned}\operatorname*{Res}_{s=\frac{1}{12}+i\gamma_{\frac{1}{12}}} F(s)\frac{x^s}{s} = &\mathcal{D}_{4,\ \gamma_{\frac{1}{12}}} x^{\frac{1}{12}+i\gamma_{\frac{1}{12}}} (\log x)^2 + \mathcal{D}_{5,\ \gamma_{\frac{1}{12}}} x^{\frac{1}{12}+i\gamma_{\frac{1}{12}}} \log x \\ &+ \mathcal{D}_{6,\ \gamma_{\frac{1}{12}}} x^{\frac{1}{12}+i\gamma_{\frac{1}{12}}}, \end{aligned} \tag{17}$$

where $\frac{1}{8}+\gamma_{\frac{1}{8}}$, $\frac{1}{10}+\gamma_{\frac{1}{10}}$ *and* $\frac{1}{12}+\gamma_{\frac{1}{12}}$ *are the coordinates of the simple zeros of* $\zeta(4s)$, $\zeta(5s)$ *and* $\zeta(6s)$ *on the lines* $\Re(s)=\frac{1}{8}$, $\Re(s)=\frac{1}{10}$ *and* $\Re(s)=\frac{1}{12}$ *respectively, and the* $\mathcal{C}_i$*'s, and* $\mathcal{D}_i$*'s are constants that can be evaluated explicitly.*

Proof. By Lemma 1, for $\Re(s) > \frac{1}{2}$, we have

$$F(s) = \frac{\zeta^2(2s)\zeta^2(3s)\zeta^2(7s)\zeta^4(8s)\zeta^5(12s)\zeta^{14}(13s)\zeta^5(14s)}{\zeta(4s)\zeta^2(5s)\zeta^3(6s)\zeta^5(10s)\zeta^6(11s)} H_1(s), \tag{18}$$

where $H_1(s)$ converges absolutely and uniformly for $\Re(s) > \frac{1}{15}$.

Let

$$F_1(s) := \frac{\zeta^2(3s)\zeta^2(7s)\zeta^4(8s)\zeta^5(12s)\zeta^{14}(13s)\zeta^5(14s)}{\zeta(4s)\zeta^2(5s)\zeta^3(6s)\zeta^5(10s)\zeta^6(11s)} H_1(s), \tag{19}$$

which is analytic at $s=\frac{1}{2}$, and hence, $F(s)=\zeta^2(2s)F_1(s)$ has a pole of order 2 at $s=\frac{1}{2}$. Therefore,

$$\begin{aligned}\operatorname*{Res}_{s=\frac{1}{2}} F(s)\frac{x^s}{s} &= \left(F_1(s)\frac{x^s}{s}\right)'_{s=\frac{1}{2}} \operatorname*{Res}_{s=\frac{1}{2}} \left(s-\frac{1}{2}\right)\zeta^2(2s) + F_1\left(\frac{1}{2}\right)\frac{x^{\frac{1}{2}}}{\frac{1}{2}} \operatorname*{Res}_{s=\frac{1}{2}} \zeta^2(2s) \\ &:= \mathcal{C}_1 x^{\frac{1}{2}} \log x + \mathcal{C}_2 x^{\frac{1}{2}},\end{aligned}$$

where $\mathcal{C}_1$ and $\mathcal{C}_2$ are some constants that can be evaluated explicitly. Similarly, the equalities from (10) to (14) can be proved.

Now, for any simple zero $\frac{1}{8}+i\gamma_{\frac{1}{8}}$ of $\zeta(4s)$ lying on the critical line $\Re(s)=\frac{1}{8}$, we have that

$$F_2(s) := \frac{\zeta^2(2s)\zeta^2(3s)\zeta^2(7s)\zeta^4(8s)\zeta^5(12s)\zeta^{14}(13s)\zeta^5(14s)}{\zeta^2(5s)\zeta^3(6s)\zeta^5(10s)\zeta^6(11s)} H_1(s) \tag{20}$$

is analytic at $s=\frac{1}{8}+i\gamma_{\frac{1}{8}}$ and hence, $F(s)=\frac{F_2(s)}{\zeta(4s)}$ has a simple pole at $\frac{1}{8}+i\gamma_{\frac{1}{8}}$. Therefore,

$$\begin{aligned}\operatorname*{Res}_{s=\frac{1}{8}+i\gamma_{\frac{1}{8}}} F(s)\frac{x^s}{s} &= F_2\left(\frac{1}{8}+i\gamma_{\frac{1}{8}}\right)\frac{x^{\frac{1}{8}+i\gamma_{\frac{1}{8}}}}{\frac{1}{8}+i\gamma_{\frac{1}{8}}}\operatorname*{Res}_{s=\frac{1}{8}+i\gamma_{\frac{1}{8}}}\frac{1}{\zeta(4s)}\\ &:= \mathcal{D}_{1,\ \gamma_{\frac{1}{8}}} x^{\frac{1}{8}+i\gamma_{\frac{1}{8}}},\end{aligned}$$

where $\mathcal{D}_{1,\ \gamma_{\frac{1}{8}}}$ is some constant which can be evaluated explicitly.

Similarly, for any simple zero $\frac{1}{10}+i\gamma_{\frac{1}{10}}$ of $\zeta(5s)$ lying on the critical line $\Re(s)=\frac{1}{10}$, we have that

$$F_3(s) := \frac{\zeta^2(2s)\zeta^2(3s)\zeta^2(7s)\zeta^4(8s)\zeta^5(12s)\zeta^{14}(13s)\zeta^5(14s)}{\zeta(4s)\zeta^3(6s)\zeta^5(10s)\zeta^6(11s)} H_1(s) \tag{21}$$

is analytic at $s=\frac{1}{10}+i\gamma_{\frac{1}{10}}$ and hence, $F(s)=\frac{F_3(s)}{\zeta^2(5s)}$ has a pole of order 2 at $\frac{1}{10}+i\gamma_{\frac{1}{10}}$. Therefore,

$$\begin{aligned}\operatorname*{Res}_{s=\frac{1}{10}+i\gamma_{\frac{1}{10}}} F(s)\frac{x^s}{s} &= \left(F_3(s)\frac{x^s}{s}\right)'_{s=\frac{1}{10}+i\gamma_{\frac{1}{10}}}\operatorname*{Res}_{s=\frac{1}{10}+i\gamma_{\frac{1}{10}}}\left(s-\frac{1}{10}-i\gamma_{\frac{1}{10}}\right)\frac{1}{\zeta^2(5s)}\\ &\quad+F_3\left(\frac{1}{10}+i\gamma_{\frac{1}{10}}\right)\frac{x^{\frac{1}{10}+i\gamma_{\frac{1}{10}}}}{\frac{1}{10}+i\gamma_{\frac{1}{10}}}\operatorname*{Res}_{s=\frac{1}{10}+i\gamma_{\frac{1}{10}}}\frac{1}{\zeta^2(5s)}\\ &:=\mathcal{D}_{2,\ \gamma_{\frac{1}{10}}} x^{\frac{1}{10}+i\gamma_{\frac{1}{10}}}\log x+\mathcal{D}_{3,\ \gamma_{\frac{1}{10}}} x^{\frac{1}{10}+i\gamma_{\frac{1}{10}}},\end{aligned}$$

where $\mathcal{D}_{2,\ \gamma_{\frac{1}{10}}}$ and $\mathcal{D}_{3,\ \gamma_{\frac{1}{10}}}$ are some constants that can be evaluated explicitly. Similarly, the equality (17) can be proved. This completes the proof of this lemma. □

Lemma 4 ([4, 5])**.** *For any $\frac{1}{2}\le\sigma\le 1$, and T-sufficiently large, we have*

$$\int_1^T |\zeta(\sigma+it)|^4\,dt \ll T(\log T)^4 \tag{22}$$

uniformly.

Lemma 5 ([1])**.** *There is a constant $C>0$ such that*

$$\frac{1}{\zeta(s)} \ll (\log T)^{2/3}(\log\log T)^{1/3} \tag{23}$$

in the region $\sigma \geq 1 - \dfrac{C}{(\log T)^{2/3}(\log\log T)^{1/3}}$, $T_0 < t \leq T$.

Lemma 6 ([1])**.** *There is a constant $C^* > 0$ such that $\zeta(s) \neq 0$ for* $\sigma \geq 1 - \dfrac{C^*}{(\log T)^{2/3}(\log\log T)^{1/3}}$, $|t| \geq t_0$.

Lemma 7 ([1, 3])**.** *We have $N(T) \ll T\log T$, where $N(T)$ is the number of zeros of $\zeta(s)$ in the region $\{s = \sigma + it : 0 < \sigma < 1, 0 < t \leq T\}$.*

Lemma 8 ([4])**.** *The Riemann hypothesis implies that*

$$\zeta(\sigma + it) = O(|t|^{\epsilon}) \tag{24}$$

for $\frac{1}{2} \leq \sigma \leq 1$ and $|t| \geq t_0$, and

$$\frac{1}{\zeta(\sigma+it)} = O(|t|^{\epsilon}) \tag{25}$$

for $\frac{1}{2} < \sigma \leq 1$ and $|t| \geq t_0$.

3. Proof of Theorem 1

We are now ready to prove Theorem 1.

Proof of Theorem 1. By Lemma 2, for $\Re(s) > \frac{1}{2}$, we have

$$F(s) = \frac{\zeta^2(2s)\zeta^2(3s)}{\zeta(4s)} H_2(s), \tag{26}$$

where $H_2(s)$ converges absolutely and uniformly for $\Re(s) > \frac{1}{5}$.

By applying Perron's formula (see [2]) to Equation (26), we get

$$\begin{aligned}\sum_{n\leq x} f(n) = &\sum_{\substack{n \leq x\\ n \text{ is squareful}}} 2^{\omega(n)} \\ =&\frac{1}{2\pi i}\int_{\frac{1}{2}+\frac{1}{\log x}-iT}^{\frac{1}{2}+\frac{1}{\log x}+iT} F(s)\frac{x^s}{s}ds + O\left(\frac{x^{\frac{1}{2}+\frac{1}{\log x}}(\log x)^2}{T}\right),\end{aligned}$$

where $1 \leq T \leq x$ is a parameter to be chosen later.

We move the line of integration to $\Re(s) = \frac{1}{4} - \frac{c}{(\log T)^{2/3}(\log\log T)^{1/3}}$, where $c = C^*$ as in Lemma 6. Define $g(T) := (\log T)^{2/3}(\log\log T)^{1/3}$. Then, in the rectangle formed by the vertices $\frac{1}{2} + \frac{1}{\log x} + iT$, $\frac{1}{4} - \frac{c}{g(T)} + iT$, $\frac{1}{4} - \frac{c}{g(T)} - iT$, $\frac{1}{2} + \frac{1}{\log x} - iT$, and $\frac{1}{2} + \frac{1}{\log x} + iT$, we note that $F(s)$ has two poles, one at $s = \frac{1}{2}$ of order 2 and another at $s = \frac{1}{3}$ of order 2.

Hence, by Cauchy's residue theorem and Lemma 3, we get

$$\begin{aligned}\sum_{n\le x} f(n) =& \mathcal{C}_1 x^{\frac{1}{2}}\log x + \mathcal{C}_2 x^{\frac{1}{2}} + \mathcal{C}_3 x^{\frac{1}{3}}\log x + \mathcal{C}_4 x^{\frac{1}{3}}\\ &+ \frac{1}{2\pi i}\left\{\int_{\frac{1}{4}-\frac{c}{g(T)}-iT}^{\frac{1}{4}-\frac{c}{g(T)}+iT} + \int_{\frac{1}{2}+\frac{1}{\log x}-iT}^{\frac{1}{4}-\frac{c}{g(T)}-iT} + \int_{\frac{1}{4}-\frac{c}{g(T)}+iT}^{\frac{1}{2}+\frac{1}{\log x}+iT}\right\} F(s)\frac{x^s}{s}ds\\ &+ O\left(\frac{x^{\frac{1}{2}+\frac{1}{\log x}}(\log x)^2}{T}\right)\\ :=& \mathcal{C}_1 x^{\frac{1}{2}}\log x + \mathcal{C}_2 x^{\frac{1}{2}} + \mathcal{C}_3 x^{\frac{1}{3}}\log x + \mathcal{C}_4 x^{\frac{1}{3}}\\ &+ I_1 + I_2 + I_3 + O\left(\frac{x^{\frac{1}{2}+\frac{1}{\log x}}(\log x)^2}{T}\right). \end{aligned} \tag{27}$$

The vertical line integration I_1 contributes in absolute value (by using the functional equation (Equation (3)) for $\zeta(s)$, the approximation (Equation (4)) for $\chi(s)$, and Lemmas 4 and 5):

$$\begin{aligned} I_1 \ll& \int_{-T}^{T} \left|\frac{\zeta^2\left(\frac{1}{2} - \frac{2c}{g(T)} + 2it\right)\zeta^2\left(\frac{3}{4} - \frac{3c}{g(T)} + 3it\right)}{\zeta\left(1 - \frac{4c}{g(T)} + 4it\right)}\right| \frac{x^{\frac{1}{4}-\frac{c}{g(T)}}}{t} dt\\ \ll& x^{\frac{1}{4}-\frac{c}{g(T)}} + x^{\frac{1}{4}-\frac{c}{g(T)}} \int_{10}^{T} \left|\frac{\zeta^2\left(\frac{1}{2} - \frac{2c}{g(T)} + 2it\right)\zeta^2\left(\frac{3}{4} - \frac{3c}{g(T)} + 3it\right)}{\zeta\left(1 - \frac{4c}{g(T)} + 4it\right)t}\right| dt\\ \ll& x^{\frac{1}{4}-\frac{c}{g(T)}} + x^{\frac{1}{4}-\frac{c}{g(T)}} g(T)\times\\ &\int_{10}^{T}\left|\zeta^2\left(\frac{1}{2} - \frac{2c}{g(T)} + 2it\right)\zeta^2\left(\frac{3}{4} - \frac{3c}{g(T)} + 3it\right)\right| t^{-1}dt\\ \ll& x^{\frac{1}{4}-\frac{c}{g(T)}} + x^{\frac{1}{4}-\frac{c}{g(T)}} g(T) T^{2(\frac{1}{2}-(\frac{1}{2}-\frac{2c}{g(T)}))}\times\\ &\int_{10}^{T}\left|\zeta^2\left(\frac{1}{2} + \frac{2c}{g(T)} - 2it\right)\zeta^2\left(\frac{3}{4} - \frac{3c}{g(T)} + 3it\right)\right| t^{-1}dt\\ \ll& x^{\frac{1}{4}-\frac{c}{g(T)}} + x^{\frac{1}{4}-\frac{c}{g(T)}} g(T)\exp\left\{4c\frac{\log T}{g(T)}\right\}\times\\ &\sup_{10\le T_1\le T}\left(\int_{T_1}^{2T_1}\left|\zeta\left(\frac{1}{2} + \frac{2c}{g(T)} - 2it\right)\right|^4 dt\right)^{\frac{1}{2}}\times \end{aligned}$$

$$\begin{aligned}
&\left(\int_{T_1}^{2T_1}\left|\zeta^2\left(\frac{3}{4}-\frac{3c}{g(T)}+3it\right)\right|^4 dt\right)^{\frac{1}{2}} T_1^{-1}\\
\ll& x^{\frac{1}{4}-\frac{c}{g(T)}}+x^{\frac{1}{4}-\frac{c}{g(T)}}(\log T)^{2/3}(\log\log T)^{1/3}\times\\
&\exp\left\{4c\left(\frac{\log T}{\log\log T}\right)^{\frac{1}{3}}\right\}T(\log T)^4T^{-1}\\
\ll& x^{\frac{1}{4}-\frac{c}{g(T)}}(\log\log T)^{1/3}(\log T)^{\frac{14}{3}}\exp\left\{4c\left(\frac{\log T}{\log\log T}\right)^{\frac{1}{3}}\right\}\\
\ll& x^{\frac{1}{4}-\frac{c}{g(T)}}\exp\left\{(4+\epsilon)c\left(\frac{\log T}{\log\log T}\right)^{\frac{1}{3}}\right\} \qquad (28)
\end{aligned}$$

for some small $\epsilon > 0$.

The horizontal line portions I_2 and I_3 contribute in absolute value (by using the functional equation (Equation (3)) for $\zeta(s)$, the approximation (Equation (4)) for $\chi(s)$, and Lemma 5) a total of

$$\begin{aligned}
|I_2|+|I_3| \leq& 10\int_{\frac{1}{4}-\frac{c}{g(T)}+iT}^{\frac{1}{2}+\frac{1}{\log x}+iT}\left|\frac{\zeta^2(2s)\zeta^2(3s)}{\zeta(4s)}\frac{x^s}{s}ds\right|\\
\ll& \frac{g(T)}{T}\int_{\frac{1}{4}-\frac{c}{g(T)}+iT}^{\frac{1}{2}+\frac{1}{\log x}+iT}\left|\chi^2(2s)\zeta^2(1-2s)\zeta^2(1-3s)x^s ds\right|\\
\ll& \frac{g(T)}{T}\int_{\frac{1}{4}-\frac{c}{g(T)}}^{\frac{1}{2}+\frac{1}{\log x}} T^{2(\frac{1}{2}-2\sigma)}T^{\frac{2}{3}(1-(1-2\sigma))+\frac{2}{3}(1-3\sigma)}x^\sigma d\sigma\\
\ll& \frac{g(T)}{T}\int_{\frac{1}{4}-\frac{c}{g(T)}}^{\frac{1}{2}+\frac{1}{\log x}} T^{\frac{5}{3}-\frac{14\sigma}{3}}x^\sigma d\sigma\\
\ll& g(T)T^{\frac{2}{3}}\int_{\frac{1}{4}-\frac{c}{g(T)}}^{\frac{1}{2}+\frac{1}{\log x}}\left(\frac{x}{T^{\frac{14}{3}}}\right)^\sigma d\sigma\\
\ll& (\log T)^{2/3}(\log\log T)^{1/3}T^{\frac{2}{3}}\left\{x^{\frac{1}{2}+\frac{1}{\log x}}T^{-\frac{7}{3}-\frac{14}{3}\frac{1}{\log x}}\right.\\
&\left.+x^{\frac{1}{4}-\frac{c}{(\log T)^{2/3}(\log\log T)^{1/3}}}T^{-\frac{7}{6}+\frac{14}{3}\frac{c}{(\log T)^{2/3}(\log\log T)^{1/3}}}\right\}.
\end{aligned}$$

Note that, for $T \gg x^{\frac{1}{4}}$, we have

$$|I_2|+|I_3| \ll x^{\frac{1}{4}-\frac{c}{(\log T)^{2/3}(\log\log T)^{1/3}}}T^{-\frac{1}{2}+\frac{14}{3}\frac{c}{(\log T)^{2/3}(\log\log T)^{1/3}}}(\log T)^{2/3}(\log\log T)^{1/3}. \qquad (29)$$

Therefore, for $T \gg x^{\frac{1}{4}}$, from Equations (27), (28), and (29), we get

$$\sum_{n\le x} f(n) = \mathcal{C}_1 x^{\frac{1}{2}}\log x + \mathcal{C}_2 x^{\frac{1}{2}} + \mathcal{C}_3 x^{\frac{1}{3}}\log x + \mathcal{C}_4 x^{\frac{1}{3}} + O\left(x^{\frac{1}{4}-\frac{c}{g(T)}}\exp\left\{(4+\epsilon)c\left(\frac{\log T}{\log\log T}\right)^{\frac{1}{3}}\right\}\right) + O\left(\frac{x^{\frac{1}{2}+\frac{1}{\log x}}(\log x)^2}{T}\right). \tag{30}$$

Finally, by making our choice $T = x^{\frac{1}{4}}(\log x)^2\exp\left\{-c\epsilon\left(\frac{\log x}{\log\log x}\right)^{\frac{1}{3}}\right\}$, we obtain

$$\sum_{n\le x} f(n) = \mathcal{C}_1 x^{\frac{1}{2}}\log x + \mathcal{C}_2 x^{\frac{1}{2}} + \mathcal{C}_3 x^{\frac{1}{3}}\log x + \mathcal{C}_4 x^{\frac{1}{3}} + O\left(x^{\frac{1}{4}}\exp\left\{c\epsilon\left(\frac{\log x}{\log\log x}\right)^{\frac{1}{3}}\right\}\right),$$

for some real constants $\mathcal{C}_1,\ \mathcal{C}_2,\ \mathcal{C}_3,\ \mathcal{C}_4$ that can be evaluated explicitly. This proves Theorem 1. □

4. Proof of Theorem 2

Proof of Theorem 2. By Lemma 1, for $\Re(s) > \frac{1}{2}$, we have

$$F(s) = \frac{\zeta^2(2s)\zeta^2(3s)\zeta^2(7s)\zeta^4(8s)\zeta^5(12s)\zeta^{14}(13s)\zeta^5(14s)}{\zeta(4s)\zeta^2(5s)\zeta^3(6s)\zeta^5(10s)\zeta^6(11s)}H_1(s), \tag{31}$$

where $H_1(s)$ converges absolutely and uniformly for $\Re(s) > \frac{1}{15}$.

By applying Perron's formula (see [2]) to Equation (31), we get

$$\sum_{n\le x} f(n) = \sum_{\substack{n\le x\\ n \text{ is squareful}}} 2^{\omega(n)} = \frac{1}{2\pi i}\int_{\frac{1}{2}+\frac{1}{\log x}-iT}^{\frac{1}{2}+\frac{1}{\log x}+iT} F(s)\frac{x^s}{s}ds + O\left(\frac{x^{\frac{1}{2}+\frac{1}{\log x}}(\log x)^2}{T}\right),$$

where $1 \le T \le x$ is a parameter to be chosen later.

Here, we move the line of integration to $\Re(s) = \frac{1}{14} + \frac{1}{\log\log x}$. Then, in the rectangle $\mathcal{R}$ formed by the vertices $\frac{1}{2} + \frac{1}{\log x} + iT,\ \frac{1}{14} + \frac{1}{\log\log x} + iT,\ \frac{1}{14} + \frac{1}{\log\log x} - iT,\ \frac{1}{2} + \frac{1}{\log x} - iT$, and $\frac{1}{2} + \frac{1}{\log x} + iT$, we note that $F(s)$ has poles at the points $s = \frac{1}{2}$ of order 2, $s = \frac{1}{3}$ of order 2, $s = \frac{1}{7}$ of order 2, $s = \frac{1}{8}$ of order 4, $s = \frac{1}{12}$ of order 5, $s = \frac{1}{13}$ of order 14, a simple pole at each $s = \frac{1}{8} + i\gamma_{\frac{1}{8}}$, a pole of order 2 at each $s = \frac{1}{10} + i\gamma_{\frac{1}{10}}$ and a pole of order 3 at each $s = \frac{1}{12} + i\gamma_{\frac{1}{12}}$, where $\frac{1}{8} + \gamma_{\frac{1}{8}},\ \frac{1}{10} + \gamma_{\frac{1}{10}}$ and $\frac{1}{12} + \gamma_{\frac{1}{12}}$ are the coordinates of the zeros of $\zeta(4s),\ \zeta(5s)$

and $\zeta(6s)$ on the lines $\Re(s)=\frac{1}{8}$, $\Re(s)=\frac{1}{10}$ and $\Re(s)=\frac{1}{12}$ respectively, such that $\mid\gamma_{\frac{1}{8}}\mid$, $\mid\gamma_{\frac{1}{10}}\mid$, $\mid\gamma_{\frac{1}{12}}\mid< T$.

Hence, by Cauchy's residue theorem, we get

$$\begin{aligned}\sum_{n\le x} f(n) =&\frac{1}{2\pi i}\left\{\int_{\frac{1}{14}+\frac{1}{\log\log x}-iT}^{\frac{1}{14}+\frac{1}{\log\log x}+iT}+\int_{\frac{1}{2}+\frac{1}{\log x}-iT}^{\frac{1}{14}+\frac{1}{\log\log x}-iT}+\int_{\frac{1}{14}+\frac{1}{\log\log x}+iT}^{\frac{1}{2}+\frac{1}{\log x}+iT}\right\}F(s)\frac{x^s}{s}ds\\ &+\sum_{\rho}\operatorname*{Res}_{s=\rho}\ F(s)\frac{x^s}{s}+O\left(\frac{x^{\frac{1}{2}+\frac{1}{\log x}}(\log x)^2}{T}\right)\\ :=&J_1+J_2+J_3+\sum_{\rho}\operatorname*{Res}_{s=\rho}\ F(s)\frac{x^s}{s}+O\left(\frac{x^{\frac{1}{2}+\frac{1}{\log x}}(\log x)^2}{T}\right),\end{aligned}\tag{32}$$

where in the right-hand side sum, ρ runs over all the poles of $F(s)$ inside the rectangle $\mathcal{R}$.

The vertical line integration J_1 contributes in absolute value (by using the functional equation (Equation (3)) for $\zeta(s)$, the approximation (Equation (4)) for $\chi(s)$, and Equations (24) and (25)):

$$\begin{aligned}J_1\ll&\left|\int_{\frac{1}{14}+\frac{1}{\log\log x}-iT}^{\frac{1}{14}+\frac{1}{\log\log x}+iT}F(s)\frac{x^s}{s}ds\right|\\ \ll&x^{\frac{1}{14}+\frac{1}{\log\log x}}\\ &+\int_{10}^{T}\left|\frac{\zeta^2\left(\frac{1}{7}+\frac{2}{\log x}+2it\right)\zeta^2\left(\frac{3}{14}+\frac{3}{\log x}+3it\right)x^{\frac{1}{14}+\frac{1}{\log\log x}}t^{-1+41\epsilon}}{\zeta\left(\frac{2}{7}+\frac{4}{\log x}+4it\right)\zeta^2\left(\frac{5}{14}+\frac{5}{\log x}+5it\right)\zeta^3\left(\frac{3}{7}+\frac{6}{\log x}+6it\right)}\right|dt\\ \ll&x^{\frac{1}{14}+\frac{1}{\log\log x}}+x^{\frac{1}{14}+\frac{1}{\log\log x}}T^{2\left(\frac{1}{2}-\frac{1}{7}-\frac{2}{\log\log x}\right)+2\left(\frac{1}{2}-\frac{3}{14}-\frac{3}{\log\log x}\right)}\times\\ &\quad T^{-\left(\frac{1}{2}-\frac{2}{7}-\frac{4}{\log\log x}\right)-2\left(\frac{1}{2}-\frac{5}{14}-\frac{5}{\log\log x}\right)-3\left(\frac{1}{2}-\frac{3}{7}-\frac{6}{\log\log x}\right)}\times\\ &\int_{10}^{T}\left|\frac{\zeta^2\left(\frac{6}{7}-\frac{2}{\log x}-2it\right)\zeta^2\left(\frac{11}{14}-\frac{3}{\log x}-3it\right)t^{-1+41\epsilon}}{\zeta\left(\frac{5}{7}-\frac{4}{\log x}-4it\right)\zeta^2\left(\frac{9}{14}-\frac{5}{\log x}-5it\right)\zeta^3\left(\frac{4}{7}-\frac{6}{\log x}-6it\right)}\right|dt\\ \ll&x^{\frac{1}{14}+\frac{1}{\log\log x}}+x^{\frac{1}{14}+\frac{1}{\log\log x}}T^{\frac{4}{7}+\frac{22}{\log\log x}+51\epsilon}(\log T)\\ \ll&x^{\frac{1}{14}+\frac{1}{\log\log x}}T^{\frac{4}{7}+\frac{22}{\log\log x}+52\epsilon}.\end{aligned}\tag{33}$$

Now, to find the contributions coming from the horizontal line portions J_2 and J_3 in absolute value, (to see more clearly) we divide the interval $\left[\frac{1}{14}+\frac{1}{\log\log x},\ \frac{1}{2}+\frac{1}{\log x}\right]$ into the union of the following six intervals: $\left[\frac{1}{14}+\frac{1}{\log\log x},\frac{1}{12}+\frac{1}{\log\log x}\right]$, $\left[\frac{1}{12}+\frac{1}{\log\log x},\frac{1}{10}+\frac{1}{\log\log x}\right]$, $\left[\frac{1}{10}+\frac{1}{\log\log x},\frac{1}{8}+\frac{1}{\log\log x}\right]$, $\left[\frac{1}{8}+\frac{1}{\log\log x},\frac{1}{6}+\frac{1}{\log\log x}\right]$, $\left[\frac{1}{6}+\frac{1}{\log\log x},\frac{1}{4}+\frac{1}{\log\log x}\right]$ and $\left[\frac{1}{4}+\frac{1}{\log\log x},\frac{1}{2}+\frac{1}{\log x}\right]$. Then, by using the functional equation (Equation (3)) for $\zeta(s)$, the approximation (Equation (4)) for $\chi(s)$, and Equations (24) and (25), we get

$$
\begin{aligned}
|J_2|+|J_3| \leq & 10\left|\int_{\frac{1}{14}+\frac{1}{\log\log x}+iT}^{\frac{1}{2}+\frac{1}{\log x}+iT} F(s)\frac{x^s}{s}ds\right| \\
& \ll \left\{\int_{\frac{1}{14}+\frac{1}{\log\log x}+iT}^{\frac{1}{12}+\frac{1}{\log\log x}+iT} + \int_{\frac{1}{12}+\frac{1}{\log\log x}+iT}^{\frac{1}{10}+\frac{1}{\log\log x}+iT} + \int_{\frac{1}{10}+\frac{1}{\log\log x}+iT}^{\frac{1}{8}+\frac{1}{\log\log x}+iT} + \int_{\frac{1}{8}+\frac{1}{\log\log x}+iT}^{\frac{1}{6}+\frac{1}{\log\log x}+iT}\right. \\
& \quad \left. + \int_{\frac{1}{6}+\frac{1}{\log\log x}+iT}^{\frac{1}{4}+\frac{1}{\log\log x}+iT} + \int_{\frac{1}{4}+\frac{1}{\log\log x}+iT}^{\frac{1}{2}+\frac{1}{\log x}+iT}\right\}\times \\
& \quad \left|\frac{\zeta^2(2\sigma+2iT)\zeta^2(3\sigma+3iT)}{\zeta(4\sigma+4iT)\zeta^2(5\sigma+5iT)\zeta^3(6\sigma+6iT)}\right| x^\sigma T^{-1+41\epsilon}d\sigma \\
& \ll \max_{\frac{1}{14}+\frac{1}{\log\log x}\leq\sigma\leq\frac{1}{12}+\frac{1}{\log\log x}} x^\sigma T^{2(1-5\sigma)-(\frac{1}{2}-4\sigma)-2(\frac{1}{2}-5\sigma)-3(\frac{1}{2}-6\sigma)-1+51\epsilon} \\
& \quad + \max_{\frac{1}{12}+\frac{1}{\log\log x}\leq\sigma\leq\frac{1}{10}+\frac{1}{\log\log x}} x^\sigma T^{2(1-5\sigma)-(\frac{1}{2}-4\sigma)-2(\frac{1}{2}-5\sigma)-1+51\epsilon} \\
& \quad + \max_{\frac{1}{10}+\frac{1}{\log\log x}\leq\sigma\leq\frac{1}{8}+\frac{1}{\log\log x}} x^\sigma T^{2(1-5\sigma)-(\frac{1}{2}-4\sigma)-1+51\epsilon} \\
& \quad + \max_{\frac{1}{8}+\frac{1}{\log\log x}\leq\sigma\leq\frac{1}{6}+\frac{1}{\log\log x}} x^\sigma T^{2(1-5\sigma)-1+51\epsilon} \\
& \quad + \max_{\frac{1}{6}+\frac{1}{\log\log x}\leq\sigma\leq\frac{1}{4}+\frac{1}{\log\log x}} x^\sigma T^{2(\frac{1}{2}-2\sigma)-1+51\epsilon} \\
& \quad + \max_{\frac{1}{4}+\frac{1}{\log\log x}\leq\sigma\leq\frac{1}{2}+\frac{1}{\log x}} x^\sigma T^{-1+51\epsilon} \\
& \ll \max_{\frac{1}{14}+\frac{1}{\log\log x}\leq\sigma\leq\frac{1}{12}+\frac{1}{\log\log x}} x^\sigma T^{-2+22\sigma+51\epsilon} \\
& \quad + \max_{\frac{1}{12}+\frac{1}{\log\log x}\leq\sigma\leq\frac{1}{10}+\frac{1}{\log\log x}} x^\sigma T^{-\frac{1}{2}+4\sigma+51\epsilon} \\
& \quad + \max_{\frac{1}{10}+\frac{1}{\log\log x}\leq\sigma\leq\frac{1}{8}+\frac{1}{\log\log x}} x^\sigma T^{\frac{1}{2}-6\sigma+51\epsilon} \\
& \quad + \max_{\frac{1}{8}+\frac{1}{\log\log x}\leq\sigma\leq\frac{1}{6}+\frac{1}{\log\log x}} x^\sigma T^{1-10\sigma+51\epsilon} \\
& \quad + \max_{\frac{1}{6}+\frac{1}{\log\log x}\leq\sigma\leq\frac{1}{4}+\frac{1}{\log\log x}} x^\sigma T^{-4\sigma+51\epsilon} \\
& \quad + \max_{\frac{1}{4}+\frac{1}{\log\log x}\leq\sigma\leq\frac{1}{2}+\frac{1}{\log x}} x^\sigma T^{-1+51\epsilon} \\
& \ll x^{\frac{1}{12}+\frac{1}{\log\log x}} T^{-2+\frac{22}{12}+\frac{22}{\log\log x}+51\epsilon} + x^{\frac{1}{10}+\frac{1}{\log\log x}} T^{-\frac{1}{2}+\frac{4}{10}+\frac{4}{\log\log x}+51\epsilon} \\
& \quad + x^{\frac{1}{10}+\frac{1}{\log\log x}} T^{\frac{1}{2}-\frac{6}{10}-\frac{6}{\log\log x}+51\epsilon} + x^{\frac{1}{8}+\frac{1}{\log\log x}} T^{\frac{1}{2}-\frac{6}{8}-\frac{6}{\log\log x}+51\epsilon} \\
& \quad + x^{\frac{1}{8}+\frac{1}{\log\log x}} T^{1-\frac{10}{8}-\frac{10}{\log\log x}+51\epsilon} + x^{\frac{1}{6}+\frac{1}{\log\log x}} T^{1-\frac{10}{6}-\frac{10}{\log\log x}+51\epsilon} \\
& \quad + x^{\frac{1}{6}+\frac{1}{\log\log x}} T^{-\frac{4}{6}-\frac{4}{\log\log x}+51\epsilon} + x^{\frac{1}{4}+\frac{1}{\log\log x}} T^{-\frac{4}{4}-\frac{4}{\log\log x}+51\epsilon}
\end{aligned}
$$

$$
\begin{aligned}
&+ x^{\frac{1}{2}+\frac{1}{\log x}}T^{-1+51\epsilon}\\
\ll & x^{\frac{1}{12}+\frac{1}{\log\log x}}T^{-\frac{1}{6}+\frac{22}{\log\log x}+51\epsilon} + x^{\frac{1}{10}+\frac{1}{\log\log x}}T^{-\frac{1}{10}+\frac{4}{\log\log x}+51\epsilon}\\
&+ x^{\frac{1}{10}+\frac{1}{\log\log x}}T^{-\frac{1}{10}-\frac{6}{\log\log x}+51\epsilon} + x^{\frac{1}{8}+\frac{1}{\log\log x}}T^{-\frac{1}{4}-\frac{6}{\log\log x}+51\epsilon}\\
&+ x^{\frac{1}{8}+\frac{1}{\log\log x}}T^{-\frac{1}{4}-\frac{10}{\log\log x}+51\epsilon} + x^{\frac{1}{6}+\frac{1}{\log\log x}}T^{-\frac{2}{3}-\frac{10}{\log\log x}+51\epsilon}\\
&+ x^{\frac{1}{6}+\frac{1}{\log\log x}}T^{-\frac{2}{3}-\frac{4}{\log\log x}+51\epsilon} + x^{\frac{1}{4}+\frac{1}{\log\log x}}T^{-1-\frac{4}{\log\log x}+51\epsilon}\\
&+ x^{\frac{1}{2}+\frac{1}{\log x}}T^{-1+51\epsilon}\\
\ll & x^{\frac{1}{12}+\frac{1}{\log\log x}}T^{-\frac{1}{6}+\frac{22}{\log\log x}+51\epsilon} + x^{\frac{1}{10}+\frac{1}{\log\log x}}T^{-\frac{1}{10}+\frac{4}{\log\log x}+51\epsilon}\\
&+ x^{\frac{1}{8}+\frac{1}{\log\log x}}T^{-\frac{1}{4}-\frac{6}{\log\log x}+51\epsilon} + x^{\frac{1}{6}+\frac{1}{\log\log x}}T^{-\frac{2}{3}-\frac{4}{\log\log x}+51\epsilon}\\
&+ x^{\frac{1}{2}+\frac{1}{\log x}}T^{-1+51\epsilon}.
\end{aligned}
$$

Note that, for $T \gg x^{\frac{1}{4}}$, we have

$$
|J_2| + |J_3| \ll x^{\frac{1}{2}+\frac{1}{\log x}}T^{-1+51\epsilon}. \tag{34}
$$

Therefore, for $T \gg x^{\frac{1}{4}}$, from Equations (32), (33), (34), and Lemma 3, we get

$$
\begin{aligned}
\sum_{n\le x} f(n) =& \mathcal{C}_1 x^{\frac{1}{2}}\log x + \mathcal{C}_2 x^{\frac{1}{2}} + \mathcal{C}_3 x^{\frac{1}{3}}\log x + \mathcal{C}_4 x^{\frac{1}{3}} + \mathcal{C}_5 x^{\frac{1}{7}}\log x + \mathcal{C}_6 x^{\frac{1}{7}}\\
&+ \mathcal{C}_7 x^{\frac{1}{8}}(\log x)^3 + \mathcal{C}_8 x^{\frac{1}{8}}(\log x)^2 + \mathcal{C}_9 x^{\frac{1}{8}}\log x + \mathcal{C}_{10} x^{\frac{1}{8}}\\
&+ \mathcal{C}_{11} x^{\frac{1}{12}}(\log x)^4 + \cdots + \mathcal{C}_{14} x^{\frac{1}{12}}\log x + \mathcal{C}_{15} x^{\frac{1}{12}}\\
&+ \mathcal{C}_{16} x^{\frac{1}{13}}(\log x)^{13} + \cdots + \mathcal{C}_{28} x^{\frac{1}{13}}\log x + \mathcal{C}_{29} x^{\frac{1}{13}}\\
&+ \sum_{\substack{\zeta(4\rho)=0,\ \Re(\rho)=\frac{1}{8},\\ 0<\left|\Im(4\rho)=\gamma_{\frac{1}{8}}\right|<T}} \mathcal{D}_{1,\ \gamma_{\frac{1}{8}}} x^{\frac{1}{8}+i\gamma_{\frac{1}{8}}}\\
&+ \sum_{\substack{\zeta(5\rho)=0,\ \Re(\rho)=\frac{1}{10},\\ 0<\left|\Im(5\rho)=\gamma_{\frac{1}{10}}\right|<T}} \left(\mathcal{D}_{2,\ \gamma_{\frac{1}{10}}} x^{\frac{1}{10}+i\gamma_{\frac{1}{10}}}\log x + \mathcal{D}_{3,\ \gamma_{\frac{1}{10}}} x^{\frac{1}{10}+i\gamma_{\frac{1}{10}}}\right)\\
&+ \sum_{\substack{\zeta(6\rho)=0,\ \Re(\rho)=\frac{1}{12},\\ 0<\left|\Im(6\rho)=\gamma_{\frac{1}{12}}\right|<T}} \left(\mathcal{D}_{4,\ \gamma_{\frac{1}{12}}} x^{\frac{1}{12}+i\gamma_{\frac{1}{12}}}(\log x)^2\right.\\
&\qquad \left. + \mathcal{D}_{5,\ \gamma_{\frac{1}{12}}} x^{\frac{1}{12}+i\gamma_{\frac{1}{12}}}\log x + \mathcal{D}_{6,\ \gamma_{\frac{1}{12}}} x^{\frac{1}{12}+i\gamma_{\frac{1}{12}}}\right)\\
&+ O\left(x^{\frac{1}{14}+\frac{1}{\log\log x}}T^{\frac{4}{7}+\frac{22}{\log\log x}+52\epsilon}\right) + O\left(x^{\frac{1}{2}+\frac{1}{\log x}}T^{-1+51\epsilon}\right).
\end{aligned} \tag{35}
$$

Finally, by making our choice $T = x^{\frac{3}{11}}$, we obtain

$$\begin{aligned}\sum_{n\leq x} f(n) =& \mathcal{C}_1 x^{\frac{1}{2}} \log x + \mathcal{C}_2 x^{\frac{1}{2}} + \mathcal{C}_3 x^{\frac{1}{3}} \log x + \mathcal{C}_4 x^{\frac{1}{3}} + \\ &+ \sum_{\substack{\zeta(4\rho)=0,\ \Re(\rho)=\frac{1}{8},\\ 0<\left|\Im(4\rho)=\gamma_{\frac{1}{8}}\right|<x^{\frac{3}{11}}}} \mathcal{D}_{1,\ \gamma_{\frac{1}{8}}} x^{\frac{1}{8}+i\gamma_{\frac{1}{8}}} \\ &+ \sum_{\substack{\zeta(5\rho)=0,\ \Re(\rho)=\frac{1}{10},\\ 0<\left|\Im(5\rho)=\gamma_{\frac{1}{10}}\right|<x^{\frac{3}{11}}}} \left(\mathcal{D}_{2,\ \gamma_{\frac{1}{10}}} x^{\frac{1}{10}+i\gamma_{\frac{1}{10}}} \log x + \mathcal{D}_{3,\ \gamma_{\frac{1}{10}}} x^{\frac{1}{10}+i\gamma_{\frac{1}{10}}}\right) \\ &+ \sum_{\substack{\zeta(6\rho)=0,\ \Re(\rho)=\frac{1}{12},\\ 0<\left|\Im(6\rho)=\gamma_{\frac{1}{12}}\right|<x^{\frac{3}{11}}}} \left(\mathcal{D}_{4,\ \gamma_{\frac{1}{12}}} x^{\frac{1}{12}+i\gamma_{\frac{1}{12}}} (\log x)^2 \right. \\ &\qquad \left. + \mathcal{D}_{5,\ \gamma_{\frac{1}{12}}} x^{\frac{1}{12}+i\gamma_{\frac{1}{12}}} \log x + \mathcal{D}_{6,\ \gamma_{\frac{1}{12}}} x^{\frac{1}{12}+i\gamma_{\frac{1}{12}}}\right) + O\left(x^{\frac{5}{22}+\epsilon}\right).\end{aligned}$$

This proves Theorem 2. □

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