IMPROVED UPPER BOUNDS FOR ODD PERFECT NUMBERS – PART I

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Abstract

A key ingredient in deriving upper bounds for odd perfect numbers, dependent only on their number of prime divisors, has been to establish that, for $a, d \in \mathbb{N}$, if $\{p_i\}_{i=1}^k$ is any increasing set of integers such that

$$\prod_{i=1}^{k}\left(1-\frac{1}{p_i}\right) \leq \frac{a}{a+d} < \prod_{i=1}^{k-1}\left(1-\frac{1}{p_i}\right) \tag{1}$$

then $\prod_{i=1}^k p_i \leq C(k)$ for some quantity $C(k)$ determined solely by k. Heath-Brown first proved such a bound with $C(k) = (4a)^{2^k-1}$, and this has since been improved to $C(k) = (1/a)\{(a+1)^{2^k} - (a+1)^{2^{k-1}}\} = (1/a)F_k(a+1)$ where $F_k : \mathbb{R}_{\geq 1} \to \mathbb{R}_{\geq 0}$ is defined by $F_k(x) = x^{2^k} - x^{2^{k-1}}$. While this represents the best current estimate, and is known to be sharp for the case $d = 1$, it has remained open whether – and, if so, how – it may be strengthened for cases where $d > 1$, or where the $\{p_i\}_{i=1}^k$ are required to be odd primes, not merely integers (as will be the case in applications to odd perfect numbers). In this paper we demonstrate how both these obstacles may be overcome, and begin the process of applying our results to the task of finding improved upper bounds for odd perfect numbers. In particular, we prove that for odd primes $\{p_i\}_{i=1}^k$ satisfying (1) with $a/(a+d) = 1/2$, if $p_1 \neq 3$, we must have

$$\prod_{i=1}^{k} p_i \leq \left[\left(\sqrt{\frac{8}{3^{31/32}}}\right)^{2^k} - \left(\sqrt{\frac{8}{3^{31/32}}}\right)^{2^{k-1}}\right] \approx F_k(1.66127).$$

Separately, we also derive a new bound on the smallest prime divisor, p_1, of an odd perfect number with m distinct prime divisors. This bound, that $p_1 < (3m/7) + 3$, is markedly tighter than the long-standing bound of Grün that $p_1 < (2m/3) + 2$. It also complements the recent strengthening of Grün's estimate by Zelinsky, that $p_1 < (m-1)/2$, being tighter than Zelinsky's bound for all (putative) odd perfect numbers with more than 49 distinct prime divisors.

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1. Background

A *perfect number* is a positive integer, the sum of whose divisors (excluding itself) equals itself. Equivalently, it is a positive integer, N, for which $\sigma(N) = 2N$, where $\sigma(\cdot)$ is the usual sum of divisors function.

The first few perfect numbers are $6 = 2(2^2 - 1)$ (since $1 + 2 + 3 = 6$), $28 = 2^2(2^3 - 1)$, and $496 = 2^4(2^5 - 1)$. Consistent with these examples, it has been known since antiquity that any prime of the form $p = 2^q - 1$ with q prime – which later became known as Mersenne primes after the French priest and polymath Marin Mersenne (1588-1648 AD) – gives rise to a corresponding even perfect number, N, of the form

$$N = 2^{q-1}p = 2^{q-1}(2^q - 1).$$

In light of this construction, which dates back to Euclid around 300 BC, there are currently 51 known perfect numbers, all even, corresponding to one for each known Mersenne prime. Moreover, it was established by Euler in the 1700s, having earlier been asserted by Lefèvre in 1496 and again by Descartes in 1638, that all even perfect numbers must be of this form (so that there is a one-to-one correspondence between even perfect numbers and Mersenne primes).

In contrast to the even case, the question of the existence or non-existence of odd perfect numbers remains open – representing arguably the oldest unsolved problem in mathematics. Many partial results are known, including a range of algebraic constraints on the form an odd perfect number can have, as well as computational results on the minimum size and/or number of distinct prime divisors of such a number. Although far too numerous to list in their entirety, a selection of the more noteworthy such partial results is presented in the following lemma.

Lemma 1. *Suppose $N = \prod_{i=1}^{m} p_i^{e_i}$ is an odd perfect number, for some positive integer m and associated set of positive exponents $\{e_i\}_{i=1}^{m}$ for the distinct prime divisors $\{p_i\}_{i=1}^{m}$ of N. Then the following results must hold.*

i. *(Euler) All but one of the exponents e_i must be even, while the remaining exponent must be odd. Furthermore, if e_j denotes this odd exponent, then e_j must equal one modulo four, as must the associated prime p_j. This associated prime is generally referred to as the "special prime" for N.*

ii. *(Sylvester 1888 [28]) There is no odd perfect number having fewer than five distinct prime divisors.*

iii. *(Dickson 1913 [5]) For any given m, the number of odd perfect numbers with m distinct prime divisors is finite.*

iv. *(Steuerwald 1937 [23]) The even exponents of an odd perfect number cannot all be two.*

v. *(McDaniel 1970 [16]) If N is an odd perfect number less than 10^{9118} then the exponent of at least one prime divisor of N must be at least six.*

vi. *(Kishore 1981 [13]; Hagis 1983 [9]) Any odd perfect number not divisible by three must have at least 11 distinct prime divisors.*

vii. *(Nielsen 2015 [18]) Any odd perfect number N must have at least 10 distinct prime divisors; and if N is not divisible by three then the number of distinct prime divisors must be at least 12.*

viii. *(Zelinsky 2021 [30]) If N is an odd perfect number with m distinct prime divisors then at least one of these divisors must be less than $(m-1)/2$.*

ix. *(Ochem and Rao 2012 [20]) Any odd perfect number must exceed 10^{1500} in size.*

x. *(Goto and Ohno 2008 [7]; Ianucci 1999 [11]; Ianucci 2000 [12]) The largest prime divisor of an odd perfect number must be at least 10^8, the second largest must be at least 10^4, and the third largest must be at least 100.*

Independent of efforts to obtain algebraic or computational constraints on any putative odd perfect number, an alternative approach to the odd perfect number problem has sought bounds on the potential size of such numbers, or components of them, as a function of their key characteristics. Most notably, for the purposes of this paper, over the past half-century a series of authors have succeeded in establishing absolute upper bounds on the size of an odd perfect number, $N = \prod_{i=1}^{m} p_i^{e_i}$, as a function purely of m, the number of distinct prime divisors it contains.

The first such result was derived by Pomerance [22] in 1977, who showed that if N is an odd perfect number with at most m prime divisors then

$$N \leq (4m)^{(4m)^{2^{m^2}}}.$$

In the mid-1990s this estimate was sharply improved (and generalized) by Heath-Brown [10], who established that in fact we must have

$$N \leq 4^{4^m}.$$

As he drily noted, however, while this bound is "clearly much less than that given by Pomerance", it remains "unfortunately, still too large for practical use".

Over the following 20-odd years, the elegant method of proof established by Heath-Brown has given rise to something of a cottage industry – including papers from Cook [4], Chen and Tang [3], and Nielsen ([17] and [18]) – focussed on tightening his procedure to further improve his upper bound. Most notably, Nielsen established the strengthened estimate

$$N \leq \frac{2^{2^{2m}} - 2^{2^{2m-1}}}{(2^{2^{m+1}} - 2^{2^m})} < 2^{4^m}. \tag{2}$$

As outlined in the next section, efforts to further improve Estimate (2) face a number of obstacles which have thus far stubbornly resisted resolution. In particular, the techniques used to date exploit only the oddness of the factors $\{p_i\}$, and even then only in a mild way, without exploiting their primeness at all.

To describe these obstacles, and set them in the context of Heath-Brown's estimation procedure and subsequent refinements of it, we next outline his general estimation approach. This is done to both motivate, and make clear, the innovations introduced in this paper to overcome some of these key obstacles.

These innovations form the main content of this paper and of three associated companion papers ([24], [25] and [26]). As will become apparent, these innovations allow for a substantial tightening of Estimate (2) – especially in special cases where, for example, an odd perfect number is not divisible by three.

2. Where Things Stand – Heath-Brown's Estimation Approach

To obtain his uniform upper bound for odd perfect numbers $N = \prod_{i=1}^{m} p_i^{e_i}$ with m distinct prime divisors, Heath-Brown considered the broader problem of finding bounds for (b/a)-multiply perfect numbers satisfying $\sigma(N) = (b/a)N$ where a, b are relatively prime positive integers with $b > a$ and a odd.[1] In this broader context he developed an elegant two-stage estimation approach, since refined further by authors such as Cook and Nielsen.

2.1. Heath-Brown's Procedure

Central to Heath-Brown's approach were two observations.

Observation A. If $N = \prod_{i=1}^{m} p_i^{e_i}$ satsifies $\sigma(N) = (b/a)N$ then, since

$$\begin{aligned}\sigma\left(\prod_{i=1}^{m} p_i^{e_i}\right) = \prod_{i=1}^{m} \sigma(p_i^{e_i}) &= \prod_{i=1}^{m} \left(1 + p_i + p_i^2 + \cdots + p_i^{e_i}\right)\\ &= \prod_{i=1}^{m} \left(\frac{p_i^{e_i+1} - 1}{p_i - 1}\right)\\ &= \prod_{i=1}^{m} \left[\frac{p_i^{e_i}\left(1 - \frac{1}{p_i^{e_i+1}}\right)}{\left(1 - \frac{1}{p_i}\right)}\right] ,\end{aligned}$$

[1]Throughout this paper, as well as in [24], [25] and [26], we use the term "(b/a)-multi-perfect" to mean simply that $\sigma(N)/N = b/a$, a given rational number not necessarily an integer.

we must have that

$$\prod_{i=1}^{m}\left(1-\frac{1}{p_i}\right)=\frac{a}{b}\prod_{i=1}^{m}\left(1-\frac{1}{p_i^{e_i+1}}\right)<\frac{a}{b}\,. \tag{3}$$

Observation B. We cannot have that $\prod_{i=1}^{l}\left(1-\frac{1}{p_i}\right)=a/b$ for any positive integer $l\le m$ since, after reduction to lowest terms, the left-hand side here is a fraction with at least one factor of two in the numerator, which would contradict the assumption that a is odd.[2]

In view of these two observations, if $N=\prod_{i=1}^{m}p_i^{e_i}$ satisfies $\sigma(N)=(b/a)N$ then the following three properties must always hold.

(i) There must be a unique positive integer $k\le m$ such that

$$\prod_{i=1}^{k}\left(1-\frac{1}{p_i}\right)<\frac{a}{b}<\prod_{i=1}^{k-1}\left(1-\frac{1}{p_i}\right). \tag{4}$$

(ii) Secondly, by rearranging the equality in (3), we must also then have that

$$\begin{aligned}\prod_{i=1}^{k}\left(1-\frac{1}{p_i^{e_i+1}}\right)&=\left[\frac{b}{a}\left(\prod_{i=1}^{k}\left(1-\frac{1}{p_i}\right)\right)\right]\prod_{i=k+1}^{m}\left[\left(1-\frac{1}{p_i}\right)\left(1-\frac{1}{p_i^{e_i+1}}\right)^{-1}\right]\\&\le\frac{b}{a}\left(\prod_{i=1}^{k}\left(1-\frac{1}{p_i}\right)\right)<1\end{aligned}$$

by Inequality (4), with equality if and only if $k=m$. Hence, if we define $b^*=b\prod_{i=1}^{k}(p_i-1)$ and $a^*=a\prod_{i=1}^{k}p_i$, and also specify subscripts $j_1,\dots,j_k$ such that $p_{j_1}^{e_{j_1}}<p_{j_2}^{e_{j_2}}<\cdots<p_{j_k}^{e_{j_k}}$, then there must be some unique positive integer $1\le r\le k$ such that

$$\prod_{i=1}^{r}\left(1-\frac{1}{p_{j_i}^{e_{j_i}+1}}\right)\le\left(\frac{b^*}{a^*}\right)<\prod_{i=1}^{r-1}\left(1-\frac{1}{p_{j_i}^{e_{j_i}+1}}\right). \tag{5}$$

(iii) Finally, by rearranging the identity $\sigma(N)=(b/a)N$ to extract all those factors from N that involve $\{p_{j_i}^{e_{j_i}}\}_{i=1}^{r}$, and all those factors from $\sigma(N)$ that involve $\{\sigma(p_{j_i}^{e_{j_i}})\}_{i=1}^{r}$, and then re-grouping these with the fraction b/a, we must also then have that

$$\sigma(\tilde{N})=\left(\frac{\tilde{b}}{\tilde{a}}\right)\tilde{N}$$

[2]Note that this is the sole place where Heath-Brown and subsequent authors use the requirement that the $\{p_i\}_{i=1}^{m}$ be odd primes. Even then this uses only the oddness of each p_i, not that they are also required to be prime.

where $\tilde{N} = N/(\prod_{i=1}^{r} p_{j_i}^{e_{j_i}})$ denotes the "remaining" part of N after this extraction, and the positive integers $\tilde{a}, \tilde{b}$ are defined by

$$\tilde{a} = a \prod_{i=1}^{r} \sigma\left(p_{j_i}^{e_{j_i}}\right) \qquad \text{and} \qquad \tilde{b} = b \prod_{i=1}^{r} p_{j_i}^{e_{j_i}} .$$

With these three properties in hand, Heath-Brown was able to implement his iterative two-stage estimation procedure for odd numbers $N = \prod_{i=1}^{m} p_i^{e_i}$ satisfying $\sigma(N) = (b/a)N$.

Stage one involved establishing a general upper bound on the product, $\prod_{i=1}^{k} y_i$, of any increasing set of positive integers $\{y_i\}_{i=1}^{k}$ for which the joint inequalities

$$\prod_{i=1}^{k}\left(1 - \frac{1}{y_i}\right) \leq \frac{a}{b} < \prod_{i=1}^{k-1}\left(1 - \frac{1}{y_i}\right) \tag{6}$$

hold. The relevance of such a bound in the context of odd (b/a)-multiply perfect numbers flows from Property (i) above (see, in particular, Inequality (4)). Using an inductive argument, Heath-Brown derived that in such circumstances, for any positive integers a, b with $a < b$, the product of the y_i must satisfy[3]

$$\prod_{i=1}^{k} y_i \leq (4a)^{2^k - 1}. \tag{7}$$

Next, invoking Property (ii), stage two then involved carrying over this same estimation procedure to the quantity $\prod_{i=1}^{r} p_{j_i}^{e_{j_i}+1}$, where (5) holds. Finally, Heath-Brown then used a careful iterative procedure, invoking Property (iii) above, to arrive (after a finite number of steps) at a uniform (in m) upper bound for the quantity PN, where P denotes $\prod_{i=1}^{m} p_i$. The quantity PN naturally arises in Heath-Brown's iterative procedure, as part of the stage two estimation process, from successive application of Estimate (7) to the quantities $\{y_i = p_{j_i}^{e_{j_i}+1}\}_{i=1}^{r}$, noting that Property (ii) yields that the joint inequalities required to invoke Estimate (7) for $\prod_{i=1}^{r} y_i = \prod_{i=1}^{r} p_{j_i}^{e_{j_i}+1}$ hold with respect to the fraction $(b^*/a^*) < 1$.

[3] It is worth noting that, at first glance, Heath-Brown's proof of this bound seems only to use the integrality of a and b very weakly – viz. to conclude that $(b-a)$ must be at least one. Hence, one might naively hope to improve Estimate (7) simply by dropping the requirement that a, b be integers, and applying Heath-Brown's method of proof using $\tilde{a} = a/(b-a)$ and $\tilde{b} = b/(b-a)$ in place of a and b (noting that $\tilde{a}, \tilde{b} \in \mathbb{Q}$ still satisfy $\tilde{a}/\tilde{b} = a/b$ and $\tilde{b} - \tilde{a} \geq 1$).

One quickly sees, however, that such an approach breaks down during the inductive step of Heath-Brown's argument – where he needs that, for any $1 \leq l \leq k-1$, the quantities $a' = a\prod_{i=1}^{l} y_i$ and $b' = b\prod_{i=1}^{l}(y_i - 1)$ also satisfy $(b' - a') \geq 1$. This trivially must hold if a and b are integers, but need not necessarily hold if a and b have been replaced in the working by $\tilde{a}$ and $\tilde{b}$. We mention this here because an analogous issue arises, and hence needs to be overcome, in relation to the approach we ultimately do adopt to strengthen Estimate (7) for cases where $(b-a) > 1$.

2.2. Improving Heath-Brown's Stage One Estimate

Subsequent efforts to improve Heath-Brown's upper bound for odd perfect numbers have worked to tighten both stages of his estimation procedure, as well as their iterative application. In the remainder of this paper, however, as well as its two immediate sequels ([24] and [25]), we shall focus solely on methods to strengthen Heath-Brown's stage one estimate, for sets of integers $\{y_i\}_{i=1}^k$ satisfying joint inequalities of the form (6).

Only in the last of these four companion papers, [26], do we turn to the task of further optimising stage two and the iterative aspects of Heath-Brown's procedure – to allow us to carry our improved stage one estimates through to substantially tightened overall upper bounds for odd perfect numbers.

2.2.1. Some Preliminary Definitions

To provide a convenient shorthand to help streamline our discussion of Heath-Brown stage one-type estimates, it is useful to introduce a number of definitions.

Definition 1. Suppose $a, d \in \mathbb{N}$ are given, with a and d relatively prime (so that the fraction $a/(a+d)$ is in lowest terms). For any positive integer k we will say that $\{y_i\}_{i=1}^k$ is an *HBC-admissible k-tuple* for the fraction $a/(a+d)$ if the y_i satisfy:[4] $y_i \in \mathbb{N}$ for all $i = 1, \ldots, k$; $y_1 < y_2 < \cdots < y_k$; and

$$\prod_{i=1}^{k}\left(1-\frac{1}{y_i}\right) \leq \frac{a}{a+d} < \prod_{i=1}^{k-1}\left(1-\frac{1}{y_i}\right). \tag{8}$$

Definition 2. If, in addition, the positive integers y_i in Definition 1 are required to be odd then we will say that $\{y_i\}_{i=1}^k$ is an *HBC-admissible odd k-tuple* for $a/(a+d)$; while if they are required to be prime then we will say that $\{y_i\}_{i=1}^k$ is an *HBC-admissible prime k-tuple* for $a/(a+d)$ (and similarly for other subsets of the space of HBC-admissible k-tuples).

We will also refer to a k-tuple as simply being HBC-admissible where it is clear from the context if oddness or primality (or some other condition) is also required, and where the integer k and fraction $a/(a+d)$ are also understood.

With these definitions in hand, Estimate (7) may be compactly referred to as a bound on $\prod_{i=1}^k y_i$ for HBC-admissible k-tuples $\{y_i\}_{i=1}^k$ for the fraction $a/(a+d)$. Our goal is to improve this bound, especially in cases where $d > 1$ or where we restrict our attention to HBC-admissible odd prime k-tuples.

[4]Note that this notion of admissible is unrelated to concepts of admissible k-tuples studied in other number-theoretic contexts; the prefix "HBC-", standing for "Heath-Brown-Cook-", is included to make this distinction clear.

2.2.2. Cook's Strengthening of Heath-Brown's Stage One Estimate

Estimate (7) was soon improved by Cook [4] to the tighter bound

$$\prod_{i=1}^{k} y_i \leq \frac{1}{a}\left[(a+1)^{2^k} - (a+1)^{2^{k-1}}\right] \tag{9}$$

for HBC-admissible k-tuples $\{y_i\}_{i=1}^k$ for $a/(a+d)$. Central to Cook's approach was the following lemma, a proof of which is given by Nielsen in [18] (see Lemma 1.2 of that article).

Lemma 2 ([4]). *Suppose that $\{z_i\}_{i=1}^k$ and $\{y_i\}_{i=1}^k$ are non-decreasing k-tuples of real numbers greater than one such that*

$$\prod_{i=1}^{l} z_i \leq \prod_{i=1}^{l} y_i \quad \text{for all } 1 \leq l \leq k. \tag{10}$$

Then we must have

$$\prod_{i=1}^{k}(1-1/z_i) \leq \prod_{i=1}^{k}(1-1/y_i),$$

with equality if and only if $z_i = y_i$ for all $i \geq 1$.

Remark 1. Note that geometrically this lemma says that, given some comparison set of non-decreasing real numbers $\{z_i\}_{i=1}^k$, the quantity $\prod_{i=1}^k(1-1/y_i)$ cannot be forced to the left of the corresponding quantity $\prod_{i=1}^k(1-1/z_i)$ on the number line, unless at least one of the partial products, $\prod_{i=1}^l y_i$, is smaller than its counterpart, $\prod_{i=1}^l z_i$.

By choosing an appropriate set of "calibrating" numbers (our terminology), Cook then showed that this is enough to enable an inductive proof of Estimate (9). Specifically, for any given k Cook considered the calibrating set of increasing integers $\{z_i\}_{i=1}^k$ given by

$$z_i = \begin{cases} (a+1)^{2^{i-1}} + 1 & \text{for } i = 1, 2, \ldots, k-1 \\ (a+1)^{2^{k-1}} & \text{for } i = k \end{cases}$$

so that, by construction,

$$\prod_{i=1}^{k}\left(1 - \frac{1}{z_i}\right) = \frac{a}{a+1} \geq \frac{a}{a+d} \tag{11}$$

for any integer $d \geq 1$ (with equality if and only if $d = 1$).

Then if $y_i = z_i$ for all $1 \leq i \leq k$ we have, by direct calculation, that

$$a\prod_{i=1}^{k} y_i = a\prod_{i=1}^{k} z_i = (a+1)^{2^k} - (a+1)^{2^{k-1}},$$

so Estimate (9) holds.

On the other hand, if the k-tuples $\{y_i\}_{i=1}^{k}$ and $\{z_i\}_{i=1}^{k}$ are not identical then, comparing (11) with the HBC-admissibility condition (8), Lemma 2 yields that we must have $\prod_{i=1}^{l} y_i < \prod_{i=1}^{l} z_i$ for some $1 \leq l \leq k$. Yet then if $l = k$ we are once again done; while if $l < k$ then we may consider the "auxiliary problem", obtained by re-arranging (8), of bounding $\prod_{i=l+1}^{k} y_i$ where the $\{y_i\}_{i=l+1}^{k}$ satisfy

$$\prod_{i=l+1}^{k}\left(1-\frac{1}{y_i}\right) \leq \frac{\tilde{a}}{\tilde{a}+\tilde{d}} < \prod_{i=l+1}^{k-1}\left(1-\frac{1}{y_i}\right), \tag{12}$$

and where the numerator and denominator of the "auxiliary fraction" in (12), $\tilde{a}/(\tilde{a}+\tilde{d})$, are given by $\tilde{a} = a\prod_{i=1}^{l} y_i$ and $(\tilde{a}+\tilde{d}) = (a+d)\prod_{i=1}^{l}(y_i - 1)$.

Yet then, by the inductive assumption, Cook's improved bound holds for this auxiliary problem, yielding

$$\tilde{a}\prod_{i=l+1}^{k} y_i \leq (\tilde{a}+1)^{2^{k-l}} - (\tilde{a}+1)^{2^{k-l-1}}.$$

Hence, Estimate (9) also holds in this case (using that $\prod_{i=1}^{l} y_i < \prod_{i=1}^{l} z_i$ and that, by construction, $a\prod_{i=1}^{l} z_i = (a+1)^{2^l} - 1$).

2.2.3. Extending Cook's Approach

Estimate (9) is stronger than (7), and is sharp for fractions $a/(a+d)$ where $d = 1$. For cases where $d > 1$, however, Cook's approach does not exploit this in any way to tighten the bound on $\prod_{i=1}^{k} y_i$ in such situations. Indeed, we see that in cases where $d > 1$ Cook's procedure only uses the geometric constraint implicit in Lemma 2 very weakly – with his choice of calibrating k-tuple tailored to force $\prod_{i=1}^{k}(1 - 1/y_i)$ to remain all the way to the right of $a/(a+1)$, not merely to the right of $a/(a+d)$, unless one of the conditions (10) is violated.

This geometric perspective strongly suggests that, in cases where $d > 1$, it might be worthwhile, in seeking a stronger bound on $\prod_{i=1}^{k} y_i$, to consider the use of a different calibrating k-tuple – chosen so that (amongst other things) $\prod_{i=1}^{k}(1 - 1/z_i)$ equals $a/(a+d)$ rather than $a/(a+1)$. This is the key insight which allows us to obtain such a strengthened bound, for fractions $a/(a+d)$ where $d > 1$.

This use of a different calibrating k-tuple should *not* be done, however, in expectation of being able to push through the *whole* of Cook's inductive approach, to obtain a strengthened version of Estimate (9) for fractions with $d > 1$.

The complication is that, in the latter part of Cook's argument, he was able, by induction, to assume Bound (9) holds for any auxiliary $(k-l)$-tuple (for some $l \geq 1$) that is HBC-admissible for the associated auxiliary fraction $(a\prod_{i=1}^{l} y_i)/((a+d)\prod_{i=1}^{l}(y_i-1))$. In the $d > 1$ case, however, there is no guarantee that, for this auxiliary fraction, the difference between denominator and numerator need remain greater than or equal to d.[5]

Given this obstacle, the key to strengthening (9) in cases where $d > 1$ is the following:

(a) use a different calibrating k-tuple, carefully tailored to the fraction $a/(a+d)$, in the part of Cook's approach where he invokes Lemma 2;

(b) then simply invoke Estimate (9), even though it is only sharp for fractions where the denominator and numerator differ by one, in handling the auxiliary bounding problem which the estimation procedure then naturally gives rise to.

This is exactly the approach we use in Section 3 below to establish a new Heath-Brown stage one-type estimate that, while coinciding with Estimate (9) for fractions of the form $a/(a+1)$, is markedly stronger for fractions where $d > 1$.

3. Main Result

Definition 3. Suppose $a, d \in \mathbb{N}$ are given, with a and d relatively prime (so that the fraction $a/(a+d)$ is in lowest terms). For any such fraction $a/(a+d)$ and any $k \in \mathbb{N}$ let α and $\{n_i = n_i(a,d)\}_{i=1}^{k}$ be the positive real numbers defined by

$$\alpha = (a+d)/d = 1 + \frac{a}{d} \tag{13}$$

and

$$n_i = \begin{cases} 1 + \alpha^{2^{i-1}} & \text{for } i = 1, 2, \ldots, k-1 \\ \alpha^{2^{k-1}} & \text{for } i = k. \end{cases} \tag{14}$$

The key properties of these numbers for our purposes, and hence the reason for their choice, are set out in the following lemma.

Lemma 3. *The real numbers n_i simultaneously satisfy*

$$\prod_{i=1}^{k}\left(1 - \frac{1}{n_i}\right) = \frac{a}{a+d} \tag{15}$$

[5]This complication is exactly analogous to the issue discussed earlier in footnote 3.

and also

$$\prod_{i=1}^{j} n_i = \frac{d}{a}\left(\alpha^{2^j} - 1\right) \qquad \text{for each } j = 1, \ldots, k-1; \tag{16}$$

$$\prod_{i=1}^{k} n_i = \frac{d}{a}\left(\alpha^{2^k} - \alpha^{2^{k-1}}\right). \tag{17}$$

Proof. We first establish identities (16) and(17). To see (16), observe that by direct computation, for any $j = 1, \ldots, k-1$, we have

$$\begin{aligned}\prod_{i=1}^{j} n_i &= (1+\alpha)(1+\alpha^2)\ldots(1+\alpha^{2^{j-1}}) \\ &= 1 + \alpha + \alpha^2 + \alpha^3 + \cdots + \alpha^{2^j-2} + \alpha^{2^j-1} = \frac{\alpha^{2^j} - 1}{\alpha - 1} = \frac{d}{a}\left(\alpha^{2^j} - 1\right)\end{aligned}$$

since $(\alpha - 1) = a/d$, as desired. As for (17), it then follows that

$$\prod_{i=1}^{k} n_i = n_k \prod_{i=1}^{k-1} n_i = \alpha^{2^{k-1}}\left[\frac{d}{a}\left(\alpha^{2^{k-1}} - 1\right)\right] = \frac{d}{a}\left(\alpha^{2^k} - \alpha^{2^{k-1}}\right)$$

as claimed.

Finally, to see identity (15), by (17) and the definition of the n_i we have

$$\begin{aligned}\prod_{i=1}^{k}\left(1 - \frac{1}{n_i}\right) &= \frac{\prod_{i=1}^{k}(n_i - 1)}{\prod_{i=1}^{k} n_i} \\ &= \frac{\alpha\alpha^2\alpha^4\ldots\alpha^{2^{k-2}}\left(\alpha^{2^{k-1}} - 1\right)}{\left[\frac{d}{a}\left(\alpha^{2^{k-1}} - 1\right)\alpha^{2^{k-1}}\right]} \\ &= \frac{a}{d}\left(\frac{\alpha^{2^{k-1}-1}}{\alpha^{2^{k-1}}}\right) = \frac{a}{d}\left(\frac{1}{\alpha}\right) = \frac{a}{d}\left(\frac{d}{a+d}\right) = \frac{a}{a+d}\end{aligned}$$

as claimed. □

We are now in a position to state our main theorem.

Theorem 1. *Suppose $\{y_i\}_{i=1}^{k}$ is an increasing sequence of positive integers such that*

$$\prod_{i=1}^{k}\left(1 - \frac{1}{y_i}\right) \leq \frac{a}{a+d} < \prod_{i=1}^{k-1}\left(1 - \frac{1}{y_i}\right). \tag{18}$$

Then there exists at least one $\kappa \in \{1, \ldots, k\}$ such that

$$\prod_{i=1}^{\kappa} y_i \leq \prod_{i=1}^{\kappa} n_i. \tag{19}$$

Moreover, let κ_{max} denote the largest κ such that (19) holds; and also, for any $\kappa = 1, 2, \ldots, k$, define[6]

$$\alpha_{(\kappa)} = \begin{cases} (a+d)/d^{(1-1/2^\kappa)} = \alpha d^{1/2^\kappa} & \text{for } \kappa = 1, 2, \ldots, k-1 \\ (a+d)/d = \alpha & \text{for } \kappa = k\ . \end{cases} \tag{20}$$

Then

$$a \prod_{i=1}^{k} y_i \leq d^{\,\eta} \left(\alpha_{(\kappa_{max})}^{2^k} - \alpha_{(\kappa_{max})}^{2^{k-1}} \right) = d^{\,\eta} F_k \left(\alpha_{(\kappa_{max})} \right) \tag{21}$$

where η is defined to equal one if $\kappa_{max} = k$ and zero if $\kappa_{max} < k$.

Before turning to the proof of Theorem 1, it is helpful to note the following two observations.

Remark 2. In case $d = 1$ in Theorem 1, then $\alpha_{(\kappa)} = \alpha = (a+1)$ for any κ, and hence (21) just becomes the "usual" bound

$$a \prod_{i=1}^{k} y_i \leq (a+1)^{2^k} - (a+1)^{2^{k-1}}.$$

Remark 3. In case $k = 1$ in Theorem 1, then we must have $\kappa_{\max} = k = 1$, so $\alpha_{(\kappa_{\max})} = (a+d)/d = \alpha$. Hence (21) becomes the estimate that $ay_1 \leq d(\alpha^2 - \alpha) = d\alpha(\alpha - 1) = a\alpha$ or, in other words, $y_1 \leq \alpha$. Yet this is easily seen directly since, for $k = 1$, the left-hand half of the HBC-admissibility condition (18) implies

$$1 - \frac{1}{y_1} \leq \frac{a}{a+d} = 1 - \frac{d}{a+d} = 1 - \frac{1}{\alpha}\ ,$$

which clearly holds if and only if $y_1 \leq \alpha$.

Proof. If $k = 1$, it follows from Remark 3 that Estimate (21), and hence also the claim embodied in (19), must hold. So now suppose $k \geq 2$.

In this case, to see the first claim of the theorem, suppose (19) were not true for any κ. Then we would have

$$\prod_{i=1}^{\kappa} y_i > \prod_{i=1}^{\kappa} n_i \quad \text{for all } \kappa = 1, 2, \ldots, k.$$

[6] We use the notation $\alpha_{(\kappa)}$ here rather than α_κ because we wish to reserve the latter for a different concept – see Definition 3.7 of [24].

Yet then, by Lemma 2 and identity (15) from Lemma 3, we would have

$$\prod_{i=1}^{k}\left(1-\frac{1}{y_i}\right) > \prod_{i=1}^{k}\left(1-\frac{1}{n_i}\right) = \frac{a}{a+d}\ ,$$

which would contradict (18).[7]

To see the second claim, let $\kappa_{\max}$ be as defined. Then if $\kappa_{\max} = k$, and noting $\alpha_{(\kappa_{\max})} = \alpha$ in this case, we immediately have by identity (17) that

$$a\prod_{i=1}^{k} y_i \le a\prod_{i=1}^{k} n_i = d\left(\alpha^{2^k} - \alpha^{2^{k-1}}\right)$$

which is the desired bound (21) in this case.

On the other hand, if $\kappa_{\max} \neq k$, then $\kappa_{\max} \in \{1, 2, \ldots, k-1\}$ and so we will have

$$\prod_{i=1}^{\kappa_{\max}} y_i \le \prod_{i=1}^{\kappa_{\max}} n_i = \frac{d}{a}\left(\alpha^{2^{\kappa_{\max}}} - 1\right) \tag{22}$$

by Lemma 3, and

$$\prod_{i=\kappa_{\max}+1}^{k}\left(1-\frac{1}{y_i}\right) \le \frac{a\left(\prod_{i=1}^{\kappa_{\max}} y_i\right)}{a\left(\prod_{i=1}^{\kappa_{\max}} y_i\right)+d^*} < \prod_{i=\kappa_{\max}+1}^{k-1}\left(1-\frac{1}{y_i}\right) \tag{23}$$

where d^* is a positive integer given by

$$d^* = (a+d)\prod_{i=1}^{\kappa_{\max}}(y_i - 1) - a\prod_{i=1}^{\kappa_{\max}} y_i\ .$$

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[7]To invoke Lemma 2 here requires that the sequence $\{n_i\}_{i=1}^k$ be non-decreasing. It is readily checked, however, that: (i) $\{n_i\}_{i=1}^k$ is non-decreasing if and only if $n_k \ge n_{k-1}$, which holds if and only if $1 + \alpha^{2^{k-2}} \le \alpha^{2^{k-1}}$, or equivalently $\alpha \ge ((1+\sqrt{5})/2)^{1/2^{k-2}}$; and (ii) for $y_1 \ge 3$ the fraction $a/(a+d)$ must, by the left-hand inequality of the HBC-admissibility condition (18), satisfy $a/(a+d) \ge (2/3)(3/4)\ldots((k+1)/(k+2)) = 2/(k+2)$, whence in turn α must satsify $\alpha \ge (k+2)/k = 1 + 2/k$.

Yet it is easily seen that, for all $k \ge 2$, $(1+2/k)^{2^{k-2}} \ge 1 + 2^{k-2}(2/k) = 1 + (2^{k-1}/k) \ge 2 > (1+\sqrt{5})/2$, and hence $1 + 2/k > ((1+\sqrt{5})/2)^{1/2^{k-2}}$; so it follows that for $k \ge 2$ the sequence $\{n_i\}_{i=1}^k$ must indeed be non-decreasing for $y_1 \ge 3$, as required.

On the other hand, for $k \ge 2$ we cannot have $y_1 = 1$, as this would lead to a contradiction to the right-hand inequality in (18); while if $y_1 = 2$ then we automatically have $y_1 < 1 + \alpha = n_1$, since $\alpha = 1 + a/d > 1$, so the first claim of the theorem holds directly in this case without the need to invoke Lemma 2.

But then, just by the "usual" estimate applied to (23), we will have

$$\begin{aligned}\left(a\prod_{i=1}^{\kappa_{\max}} y_i\right)\prod_{i=\kappa_{\max}+1}^{k} y_i &\le F_{k-\kappa_{\max}}\left(1+a\prod_{i=1}^{\kappa_{\max}} y_i\right)\\ &\le F_{k-\kappa_{\max}}\left(1+a\prod_{i=1}^{\kappa_{\max}} n_i\right)\\ &= F_{k-\kappa_{\max}}\left(d\alpha^{2^{\kappa_{\max}}}-d+1\right)\\ &\le F_{k-\kappa_{\max}}\left(d\alpha^{2^{\kappa_{\max}}}\right)\\ &= F_{k-\kappa_{\max}}\left(\left(d^{1/2^{\kappa_{\max}}}\alpha\right)^{2^{\kappa_{\max}}}\right) = F_{k-\kappa_{\max}}\left(\alpha_{(\kappa_{\max})}^{2^{\kappa_{\max}}}\right)\end{aligned}$$

where we have twice used the monotonicity of $F_{k-\kappa_{\max}}(\cdot)$ (in deriving the second and fourth lines), and also twice invoked Estimate (22) (to obtain the second and third lines). Hence, directly from the definition of $F_{k-\kappa_{\max}}(\cdot)$, we obtain

$$a\prod_{i=1}^{k} y_i \le \left(\alpha_{(\kappa_{\max})}^{2^{\kappa_{\max}}}\right)^{2^{k-\kappa_{\max}}} - \left(\alpha_{(\kappa_{\max})}^{2^{\kappa_{\max}}}\right)^{2^{k-\kappa_{\max}-1}} = \left(\alpha_{(\kappa_{\max})}^{2^{k}} - \alpha_{(\kappa_{\max})}^{2^{k-1}}\right)$$

which is again the desired bound (21) in this case. □

With the proof of Theorem 1 in hand, we are now in a position to establish two immediate corollaries of this theorem, based on the following observation.

Remark 4. Observe that, since $d \ge 1$, we have

$$\frac{(a+d)}{d} = \alpha = \alpha_{(k)} \le \alpha_{(k-1)} \le \cdots \le \alpha_{(3)} \le \alpha_{(2)} \le \alpha_{(1)} = \frac{(a+d)}{\sqrt{d}}$$

with strict inequalities at every stage if $d > 1$. Since $F_r(x) = x^{2^r} - x^{2^{r-1}}$ is a monotonically increasing function of x for all $r \in \mathbb{N}$ and $x \ge 1$ (see Lemma 9 of Appendix A), we therefore in turn have

$$F_k(\alpha_{(k)}) \le F_k(\alpha_{(k-1)}) \le \cdots \le F_k(\alpha_{(2)}) \le F_k(\alpha_{(1)})\,.$$

Moreover, the first inequality may be strengthened since, if we further define a quantity $\bar{\alpha}_{(k)}$ lying between $\alpha_{(k)}$ and $\alpha_{(k-1)}$ by

$$\bar{\alpha}_{(k)} = (a+d)/d^{(1-1/2^k)} = \alpha d^{1/2^k}\ , \tag{24}$$

then

$$\begin{aligned} dF_k(\alpha_{(k)}) &= d\left[\alpha^{2^k} - \alpha^{2^{k-1}}\right] \\ &= d\left[\left(\frac{1}{d}\right)\left(\alpha d^{1/2^k}\right)^{2^k} - \left(\frac{1}{\sqrt{d}}\right)\left(\alpha d^{1/2^k}\right)^{2^{k-1}}\right] \\ &\le \left[\bar{\alpha}_{(k)}^{2^k} - \bar{\alpha}_{(k)}^{2^{k-1}}\right] = F_k(\bar{\alpha}_{(k)})\,. \end{aligned}$$

Hence, we even have

$$dF_k(\alpha_{(k)}) \le F_k(\bar{\alpha}_{(k)}) \le F_k(\alpha_{(k-1)}) \le \cdots \le F_k(\alpha_{(2)}) \le F_k(\alpha_{(1)})\,, \tag{25}$$

with strict inequalities throughout unless $d = 1$.

In view of (25), we immediately have the following results.

Corollary 1. *Suppose a, d and $\{y_i\}_{i=1}^k$ are as in Theorem 1. Then*

$$a\prod_{i=1}^k y_i \le \left[\left(\frac{a+d}{\sqrt{d}}\right)^{2^k} - \left(\frac{a+d}{\sqrt{d}}\right)^{2^{k-1}}\right]. \tag{26}$$

Proof. This follows directly from (21) and (25), noting that $\alpha_{(1)} = (a+d)/\sqrt{d}$. $\square$

Corollary 2. *Suppose a, d and $\{y_i\}_{i=1}^k$ are as in Theorem 1, with $k \ge 2$, and let $\lfloor(1+\alpha)\rfloor = \lfloor n_1 \rfloor$ denote the greatest integer less than or equal to $(1+\alpha)$. Then unless $y_1 = \lfloor(1+\alpha)\rfloor$, the product $\prod_{i=1}^k y_i$ must in fact satisfy the stronger bound*

$$a\prod_{i=1}^k y_i \le \left[\left(\frac{a+d}{d^{3/4}}\right)^{2^k} - \left(\frac{a+d}{d^{3/4}}\right)^{2^{k-1}}\right]. \tag{27}$$

Proof. Under the hypotheses of Theorem 1, we cannot (for $k \ge 2$) have $y_1 \le \alpha = (a+d)/d$, or we would have $\prod_{i=1}^{k-1}(1-1/y_i) \le 1 - 1/y_1 \le 1 - d/(a+d) = a/(a+d)$, contradicting the right-hand inequality in (18). On the other hand, if $y_1 > 1+\alpha = n_1$ then (19) does not hold for $\kappa = 1$, and hence we must have $\kappa_{\max} \ge 2$. Combining these two observations, it follows immediately from (25) and Theorem 1 that, unless $y_1 = \lfloor(1+\alpha)\rfloor$, Estimate (27) must hold. $\square$

The following example illustrates the strength of Corollaries 1 and 2, and hence of the more general bound (21) in Theorem 1.

Example 1. Consider the case $a = 7$ and $d = 3$ (i.e., the fraction 7/10). Then, for any HBC-admissible $\{y_i\}_{i=1}^k$ for this fraction, Corollary 1 gives us the bound

$$\prod_{i=1}^k y_i \le \frac{1}{7}\left[\left(\frac{10}{\sqrt{3}}\right)^{2^k} - \left(\frac{10}{\sqrt{3}}\right)^{2^{k-1}}\right] \approx \frac{1}{7}\left[5.7735^{2^k} - 5.7735^{2^{k-1}}\right], \tag{28}$$

which is a much tighter bound than the “standard” estimate

$$\prod_{i=1}^{k} y_i \leq \frac{1}{7}\left[8^{2^k} - 8^{2^{k-1}}\right]. \tag{29}$$

Moreover, by Corollary 2 we must have that, if $y_1 \neq 4$, then

$$\prod_{i=1}^{k} y_i \leq \frac{1}{7}\left[\left(\frac{10}{3^{3/4}}\right)^{2^k} - \left(\frac{10}{3^{3/4}}\right)^{2^{k-1}}\right] \approx \frac{1}{7}\left[4.38691^{2^k} - 4.38691^{2^{k-1}}\right]. \tag{30}$$

Estimates (28) and (30) represent substantial improvements over the “standard” bound, (29), for this example. With only a little additional work, however, we can significantly tighten them still further, based on the following important general observation about the proof of Theorem 1.

Remark 5. In the latter part of the proof of Theorem 1 we simply used the “standard” (i.e., difference equals one) estimate, applied to (23). In particular situations, however, we will often be able to do vastly better when $d^* > 1$ – that is, get a much tighter bound – by instead iteratively using our improved estimate (i.e., Theorem 1) at this point in the estimation process. In other words, in many circumstances it will be more useful to think of the proof of Theorem 1 as a *process* for getting better upper bounds for $\prod_{i=1}^{k} y_i$, rather than thinking of Theorem 1 (or Corollaries 1 or 2) simply as “formal results to be invoked without further ado”.

This principle will be fruitfully invoked repeatedly in the proof of Lemma 8 in Section 5 below. However, as an immediate concrete illustration of it, consider again the task of finding an upper bound for the product of the components of any HBC-admissible k-tuple for the fraction 7/10. For presentational simplicity, suppose also that $k \geq 3$ here, to avoid having to deal with additional subcases in the event $k = 1$ or 2.

Example 1 Revisited. Consider again the fraction 7/10, for which we have $a = 7$ and $d = 3$. For this fraction we then have $\alpha = 10/3$, and hence in turn

$$\begin{cases} n_1 = 1 + \alpha = 13/3 \approx 4.333 \\ n_2 = 1 + \alpha^2 = 109/9 \approx 12.111 \end{cases}$$

and also $n_1 n_2 = (13/3)(109/9) \approx 52.481$.

Then for any HBC-admissible k-tuple $\{y_1, \ldots, y_k\}$ for 7/10, we clearly cannot have $y_1 \leq 3$ (since $7/10 > 2/3$). Moreover, if $y_1 = 4$ then we would have that $\{y_2, \ldots, y_k\}$ is an HBC-admissible $(k-1)$-tuple for $(7/10)(4/3) = 14/15$, so we would get directly from the "usual" bound that

$$\prod_{i=1}^{k} y_i \le 4\left[\frac{1}{14}\left(15^{2^{k-1}} - 15^{2^{k-2}}\right)\right]$$
$$= \frac{2}{7}\left(15^{2^{k-1}} - 15^{2^{k-2}}\right) \approx \frac{2}{7}\left(3.87298^{2^k} - 3.87298^{2^{k-1}}\right)$$

which is a marked improvement over simply applying the "usual" estimate to $\{y_i\}_{i=1}^k$ for 7/10, namely (29).

Next, if $y_1 \ge 5$ then $\kappa_{\max}(7/10) \ge 2$, so the general bound from Corollary 1 that

$$\prod_{i=1}^{k} y_i \le \frac{1}{7}\left[\left(\frac{10}{\sqrt{3}}\right)^{2^k} - \left(\frac{10}{\sqrt{3}}\right)^{2^{k-1}}\right] \approx \frac{1}{7}\left[5.7735^{2^k} - 5.7735^{2^{k-1}}\right]$$

can be improved (as per Corollary 2) to

$$\prod_{i=1}^{k} y_i \le \frac{1}{7}\left[\left(\frac{10}{3^{3/4}}\right)^{2^k} - \left(\frac{10}{3^{3/4}}\right)^{2^{k-1}}\right] \approx \frac{1}{7}\left[4.38691^{2^k} - 4.38691^{2^{k-1}}\right].$$

Moreover, if $y_1 = 5$ then $\{y_2, \ldots, y_k\}$ is an HBC-admissible $(k-1)$-tuple for $(7/10)(5/4) = 7/8$, so we would get directly from the "usual" bound that

$$\prod_{i=1}^{k} y_i \le 5\left[\frac{1}{7}\left(8^{2^{k-1}} - 8^{2^{k-2}}\right)\right] \approx \frac{5}{7}\left(2.82843^{2^k} - 2.82843^{2^{k-1}}\right).$$

On the other hand, if $y_1 \ge 6$ and if $y_1 y_2 \le n_1 n_2 < 53$ then the only two possibilities are $y_1 = 6, y_2 = 7$ or $y_1 = 6, y_2 = 8$. Yet if $y_2 = 8$ then $\{y_1, y_3, \ldots, y_k\}$ is an HBC-admissible $(k-1)$-tuple for $(7/10)(8/7) = 4/5$, so we would get directly from the "usual" bound that

$$\prod_{i=1}^{k} y_i \le 8\left[\frac{1}{4}\left(5^{2^{k-1}} - 5^{2^{k-2}}\right)\right] \approx 2\left(2.23607^{2^k} - 2.23607^{2^{k-1}}\right);$$

while if $y_1 = 6, y_2 = 7$ then $\{y_3, \ldots, y_k\}$ is an HBC-admissible $(k-2)$-tuple for $(7/10)(6/5)(7/6) = 49/50$, so we would get directly from the "usual" bound that

$$\prod_{i=1}^{k} y_i \le 42\left[\frac{1}{49}\left(50^{2^{k-2}} - 50^{2^{k-3}}\right)\right] \approx \frac{6}{7}\left(2.65915^{2^k} - 2.65915^{2^{k-1}}\right).$$

Finally, however, if we have instead that $y_1 \ge 6 > n_1$ and $y_1 y_2 > n_1 n_2$ then $\kappa_{\max}(7/10) \ge 3$, and so we have (from Theorem 1 and (25)) the general bound

$$\prod_{i=1}^{k} y_i \le \frac{1}{7}\left[\left(\frac{10}{3^{7/8}}\right)^{2^k} - \left(\frac{10}{3^{7/8}}\right)^{2^{k-1}}\right] \approx \frac{1}{7}\left(3.82401^{2^k} - 3.82401^{2^{k-1}}\right).$$

Thus overall, combining all of the above, we obtain that without restriction, for $k \geq 3$, we must have

$$\prod_{i=1}^{k} y_i \leq \frac{2}{7}\left(\sqrt{15}^{\,2^k} - \sqrt{15}^{\,2^{k-1}}\right) \approx \frac{2}{7}\left(3.87298^{2^k} - 3.87298^{2^{k-1}}\right).$$

Moreover, this estimate is actually *sharp* (for HBC-admissible k-tuples without any restriction upon the y_i), since the right-hand side bound here can clearly be achieved by taking $y_1 = 4$ and $\{y_2, \ldots, y_k\}$ to be the product-maximising HBC-admissible $(k-1)$-tuple for 14/15 (viz. $y_2 = 15 + 1 = 16$, $y_3 = 15^2 + 1 = 226$, $y_4 = 15^4 + 1 = 50626$, $\ldots$, $y_k = 15^{2^{k-2}}$).

The following lemma summarizes these results.

Lemma 4. *Suppose $k \geq 3$. For any HBC-admissible k-tuple $\{y_i\}_{i=1}^{k}$ for the fraction 7/10 we must have*

$$\prod_{i=1}^{k} y_i \leq \frac{2}{7}\left(\sqrt{15}^{\,2^k} - \sqrt{15}^{\,2^{k-1}}\right) \approx \frac{2}{7}\left(3.87298^{2^k} - 3.87298^{2^{k-1}}\right); \qquad (31)$$

and this estimate cannot be further improved for the general class of HBC-admissible k-tuples.

Clearly, however, if we were to further restrict ourselves in Lemma 4 to, say, the space of HBC-admissible *prime* k-tuples, we might hope to further tighten Estimate (31) considerably. In the specific context of developing improved bounds for odd perfect numbers, the remainder of this paper is devoted to beginning the process of deriving tighter upper bounds for $\prod_{i=1}^{k} y_i$ for HBC-admissible odd prime k-tuples $\{y_i\}_{i=1}^{k}$ for the fraction 1/2. Initially this is done in the context of odd perfect numbers N for which 3 does not divide N – see Section 5 below. These ideas are then developed further in the next two companion papers to this one, [24] and [25], entitled *Improved Upper Bounds for Odd Perfect Numbers – Parts II and III.*

In particular, in [24] we obtain new bounds for $\prod_{i=1}^{k} y_i$, for odd prime HBC-admissible k-tuples for 1/2 for which $y_1 \neq 3$. These bounds are not universally stronger than those of this paper, but are dramatically sharper for all values of k up to even very high levels.

In [25] we then extend these ideas to obtain an improved general bound for $\prod_{i=1}^{k} y_i$ for odd prime HBC-admissible k-tuples for 1/2 without any constraint on y_1. For the first time, these papers, and the analysis in Section 5 below, exploit the primality of the y_i in a strong way to obtain improvements to the stage one estimates of Heath-Brown, Cook, and Nielsen. Note, however, that in these settings the primality of the y_i is not made use of in an algebraic sense, but in the sense that it allows the bounding task in each case to be transferred to one or more "auxiliary problems" where the fractions involved have differences $d > 1$ between their numerator and denominator. This enables the strengthening of the "standard" bound in such circumstances, provided by Theorem 1, to be exploited.

Finally, in [26] – the fourth paper in this series – we show how these various improved stage one estimates for $\prod_{i=1}^{k} y_i$ may then be extended to substantially improved, albeit still very large, upper bounds for any odd perfect number N itself, purely in terms of its number of distinct prime divisors.

4. A Brief Digression – Strengthening Grün's Bound for p_1

Before embarking on the program just outlined, we briefly digress to establish a result that is of independent interest in the study of odd perfect numbers – namely an elementary new upper bound on the maximum possible size of the smallest prime divisor, p_1, of an odd perfect number with m distinct prime divisors. Although not essential to the argument, we use this strengthened estimate in our proof of Lemma 7 below.

For nearly 70 years the best known such upper bound was the 1952 estimate of Grün [8] that $p_1 < (2m/3) + 2$. This was recently strengthened by Zelinsky [30] to the bound $p_1 < (m-1)/2$.

Using an entirely different method, we show here that p_1 must also satisfy the bound $p_1 < (3m/7) + 3$. This bound is always tighter than that of Grün, while it is stronger than Zelinsky's for any odd perfect number with more than 49 distinct prime divisors (and hence for all but a finite number of putative odd perfect numbers). For example, for an odd perfect number with (say) 100 distinct prime divisors, Grün's bound shows that the smallest prime divisor, p_1, must satisfy $p_1 \le 68$; Zelinsky's bound strengthens this to $p_1 \le 49$; while our estimate establishes that in fact we must have $p_1 \le 45$, a further tightening.

The crux of our result is the final part of the following lemma.

Lemma 5. *Suppose that $\{y_1, \ldots, y_k\}$ is an HBC-admissible k-tuple for $1/2$, so that*

$$\prod_{i=1}^{k}\left(1-\frac{1}{y_i}\right) \le \frac{1}{2} < \prod_{i=1}^{k-1}\left(1-\frac{1}{y_i}\right). \tag{32}$$

Then the following hold:

(i) If we require only that the y_i be integers we must have $y_1 \le k+1$.

(ii) If we further require that the y_i be odd integers we must have the tighter bound (cf. Grün's estimate) that $y_1 < 2(k+3)/3 = (2k/3) + 2$.

(iii) Finally, if we require that the y_i be prime we must have the even tighter bound that:[8]

$$y_1 < (3k/7) + 3\,. \tag{33}$$

Remark 6. Bound (33) is still far from sharp since, for example, if $k \le 6$ we must actually have $y_1 \le 3$; if $k \le 14$ we must have $y_1 \le 5$; and if $k \le 26$ we must have $y_1 \le 7$. The latter observation follows since, if $y_1 > 7$ and $k \le 26$, then

$$\prod_{i=1}^{k}\left(1-\frac{1}{y_i}\right) \ge \left(\frac{10}{11}\right)\left(\frac{12}{13}\right)\left(\frac{16}{17}\right)\cdots\left(\frac{108}{109}\right)\left(\frac{112}{113}\right) \approx 0.5021;$$

while the other two follow in similar fashion.

Proof. To see the first claim, note that from the left-hand inequality of (32) we must have

$$\begin{aligned}\frac{1}{2} \ge \prod_{i=1}^{k}\left(\frac{y_i-1}{y_i}\right) &\ge \left(\frac{y_1-1}{y_1}\right)\left(\frac{y_1}{y_1+1}\right)\left(\frac{y_1+1}{y_1+2}\right)\cdots\left(\frac{y_1+k-2}{y_1+k-1}\right)\\ &= \frac{y_1-1}{y_1+k-1}\end{aligned}$$

which implies $y_1 + (k-1) \ge 2y_1 - 2$, and hence $y_1 \le k+1$ as claimed.

As for the second claim, following Grün's approach and proceeding in similar fashion we must in this case have, from the square of the left-hand inequality of (32), that

$$\begin{aligned}\frac{1}{4} \ge \left(\prod_{i=1}^{k}\left(\frac{y_i-1}{y_i}\right)\right)^2 &\ge \left(\frac{y_1-1}{y_1}\right)^2\left(\frac{y_1+1}{y_1+2}\right)^2\left(\frac{y_1+3}{y_1+4}\right)^2\cdots\left(\frac{y_1+2k-3}{y_1+2k-2}\right)^2\\ &> \left[\left(\frac{y_1-2}{y_1-1}\right)\left(\frac{y_1-1}{y_1}\right)\right]\left[\left(\frac{y_1}{y_1+1}\right)\left(\frac{y_1+1}{y_1+2}\right)\right]\cdots\\ &\qquad \left[\left(\frac{y_1+2k-4}{y_1+2k-3}\right)\left(\frac{y_1+2k-3}{y_1+2k-2}\right)\right]\\ &= \frac{y_1-2}{y_1+2k-2}\,,\end{aligned}$$

which holds if and only if $y_1 + 2k - 2 > 4y_1 - 8$ or, equivalently, $y_1 < 2(k+3)/3$, as claimed.

[8]To illustrate the relative strength of these various results, for $k = 26$ the sharp bound is that $y_1 \le 7$ for a prime HBC-admissible k-tuple (cf. Remark 6). This compares with a bound of only $y_1 \le 27$ for a general HBC-admissible k-tuple from Lemma 5(i); $y_1 \le 19$ for the case of an odd HBC-admissible k-tuple from Lemma 5(ii); and $y_1 \le 14$ from our general estimate for the prime HBC-admissible case, Lemma 5(iii).

Finally, to see the third claim, for HBC-admissible prime k-tuples, note that the required bound clearly holds if $y_1 \leq 3$, so we may assume without loss of generality in what follows that $y_1 > 3$. In this case, we must then have $y_1 \equiv \pm 1 \pmod 6$.

Suppose first that we had $y_1 \equiv -1 \pmod 6$. Then by considering the cube of the left-hand inequality in (32) we would have that, for k even:

$$\begin{aligned}
\frac{1}{8} \geq & \left[\left(\frac{y_1-1}{y_1}\right)^3 \left(\frac{y_1+1}{y_1+2}\right)^3\right] \left[\left(\frac{y_1+5}{y_1+6}\right)^3 \left(\frac{y_1+7}{y_1+8}\right)^3\right] \cdots \\
& \quad \left[\left(\frac{y_1+3k-7}{y_1+3k-6}\right)^3 \left(\frac{y_1+3k-5}{y_1+3k-4}\right)^3\right] \\
& > \left[\left(\frac{y_1-3}{y_1-2}\right)\left(\frac{y_1-2}{y_1-1}\right)\left(\frac{y_1-1}{y_1}\right)\left(\frac{y_1}{y_1+1}\right)\left(\frac{y_1+1}{y_1+2}\right)\left(\frac{y_1+2}{y_1+3}\right)\right] \times \\
& \quad \left[\left(\frac{y_1+3}{y_1+4}\right) \cdots \left(\frac{y_1+8}{y_1+9}\right)\right] \times \cdots \times \left[\left(\frac{y_1+3k-9}{y_1+3k-8}\right) \cdots \left(\frac{y_1+3k-4}{y_1+3k-3}\right)\right] \\
& = \frac{y_1-3}{y_1+3k-3}\,.
\end{aligned} \tag{34}$$

Here, within each set of square brackets, we have used the trivial inequality that, for any $x > 3$,

$$\left(\frac{x-1}{x}\right)^3 > \left(\frac{x-3}{x-2}\right)\left(\frac{x-2}{x-1}\right)\left(\frac{x-1}{x}\right),$$

and the only slightly less trivial inequality that, for any $x > 2$,

$$\left(\frac{x-1}{x}\right)^3 > \left(\frac{x-2}{x-1}\right)\left(\frac{x-1}{x}\right)\left(\frac{x}{x+1}\right). \tag{35}$$

To see Inequality (35) simply observe that

$$\left(\frac{x-1}{x}\right)^2 = \frac{x^2-2x+1}{x^2} > \frac{x^2-2x}{x^2-1} = \left(\frac{x-2}{x-1}\right)\left(\frac{x}{x+1}\right)$$

from which (35) follows immediately.

Returning to (34), it is also easily checked that the exact same inequality must hold, in the event $y_1 \equiv -1 \pmod 6$, for k odd.

On the other hand, if we instead had $y_1 \equiv +1 \pmod 6$ then, by again considering the cube of the left-hand inequality in (32), we would have that, for k even,

$$\begin{aligned}\frac{1}{8} &\geq \left[\left(\frac{y_1-1}{y_1}\right)^3\left(\frac{y_1+3}{y_1+4}\right)^3\right]\left[\left(\frac{y_1+5}{y_1+6}\right)^3\left(\frac{y_1+9}{y_1+10}\right)^3\right]\cdots \\ &\quad \left[\left(\frac{y_1+3k-7}{y_1+3k-6}\right)^3\left(\frac{y_1+3k-3}{y_1+3k-2}\right)^3\right] \\ &> \left[\left(\frac{y_1-2}{y_1-1}\right)\left(\frac{y_1-1}{y_1}\right)\left(\frac{y_1}{y_1+1}\right)\left(\frac{y_1+1}{y_1+2}\right)\left(\frac{y_1+2}{y_1+3}\right)\left(\frac{y_1+3}{y_1+4}\right)\right]\times \\ &\quad \left[\left(\frac{y_1+4}{y_1+5}\right)\cdots\left(\frac{y_1+9}{y_1+10}\right)\right]\times\cdots\times\left[\left(\frac{y_1+3k-8}{y_1+3k-7}\right)\cdots\left(\frac{y_1+3k-3}{y_1+3k-2}\right)\right] \\ &= \frac{y_1-2}{y_1+3k-2} > \frac{y_1-3}{y_1+3k-3}\,,\end{aligned}$$

and similarly for k odd.

Hence, overall, we must always have, for any HBC-admissible prime k-tuple for $1/2$, that $(1/8) > (y_1-3)/(y_1+3k-3)$, which immediately implies the desired bound on y_1. $\square$

To see the applicability of Lemma 5 to odd perfect numbers, it remains only to recall (see Property (i) in Subsection 2.1) that, if $N = \prod_{i=1}^m p_i^{e_i}$ is an odd perfect number, then there must be some $k \in \{1, 2, \ldots, m\}$ such that $\{p_i\}_{i=1}^k$ is an HBC-admissible odd prime k-tuple for $1/2$.[9] Hence, Part (iii) of Lemma 5 applies and we obtain the following result.

Lemma 6. *Suppose that $N = \prod_{i=1}^m p_i^{e_i}$ is an odd perfect number. Then the smallest prime divisor of N, p_1, must satisfy*

$$p_1 < (3m/7) + 3\,. \tag{36}$$

Remark 7. For any odd perfect number $N = \prod_{i=1}^m p_i^{e_i}$, Estimate (36) is a noticeable improvement over Grün's long-standing upper bound for p_1, and also a further improvement over Zelinsky's recent strengthening of Grün's bound if $m > 49$. Even so, this bound still increases linearly with m. Using recent results of Dusart [6], however, it is possible to establish that, in the asymptotic case as $m \to \infty$, p_1 may actually be bounded by a quantity that grows much more slowly in m (almost like $\sqrt{m}$). Further details are provided in Appendix B.

[9] Equally, since the proofs of the various bounds in Lemma 5 only ever used the left-hand inequality in (32), it would suffice to note Observation A in Subsection 2.1 that, if $N = \prod_{i=1}^m p_i^{e_i}$ is odd perfect, then $\prod_{i=1}^m (1 - 1/p_i) < 1/2$.

5. Improved Bounds for $\prod_{i=1}^{k} p_i$ where $p_1 > 3$

We are now in a position to return to the program outlined at the end of Section 3.

Lemma 7. *Suppose $\{p_i\}_{i=1}^{k}$ is an HBC-admissible prime k-tuple for $1/2$, so that*

$$\prod_{i=1}^{k}\left(1-\frac{1}{p_i}\right) \le \frac{1}{2} < \prod_{i=1}^{k-1}\left(1-\frac{1}{p_i}\right), \tag{37}$$

and suppose $p_1 \ge 5$ (so $k \ge 7$, by Remark 6). Then

$$\prod_{i=1}^{k} p_i \le \left[\left(\sqrt{\frac{8}{3^{7/8}}}\right)^{2^k} - \left(\sqrt{\frac{8}{3^{7/8}}}\right)^{2^{k-1}}\right] \approx F_k(1.74906). \tag{38}$$

Moreover, if $p_1 \neq 5$, we have the tighter uniform bounds that if $6 \le p_1 < 18$ then

$$\prod_{i=1}^{k} p_i \le \left[\left(\sqrt{\frac{12}{5^{15/16}}}\right)^{2^k} - \left(\sqrt{\frac{12}{5^{15/16}}}\right)^{2^{k-1}}\right] \approx F_k(1.62910)\,; \tag{39}$$

while if $18 \le p_1 < 258$ then

$$\prod_{i=1}^{k} p_i \le \left[\left(\sqrt{\frac{512}{255^{31/32}}}\right)^{2^k} - \left(\sqrt{\frac{512}{255^{31/32}}}\right)^{2^{k-1}}\right] \approx F_k(1.54514)\,; \tag{40}$$

and if $p_1 \ge 258$ then

$$\prod_{i=1}^{k} p_i \le \left[\left(\sqrt{2^{5/4}\left(\frac{2^{16}+1}{2^{16}}\right)}\right)^{2^k} - \left(\sqrt{2^{5/4}\left(\frac{2^{16}+1}{2^{16}}\right)}\right)^{2^{k-1}}\right] \approx F_k(1.54222)\,. \tag{41}$$

Proof. Observe that by (37) we have

$$\prod_{i=2}^{k}\left(1-\frac{1}{p_i}\right) \le \frac{p_1}{2(p_1-1)} = \frac{p_1}{p_1+(p_1-2)} < \prod_{i=2}^{k-1}\left(1-\frac{1}{p_i}\right).$$

Hence $\{p_2,\ldots,p_k\}$ is an HBC-admissible $(k-1)$-tuple for the fraction $a/(a+d)$ where $a = p_1$, $d = p_1 - 2$, $p_1 \ge 5$ and $7 \le p_2 < \cdots < p_{k-1} < p_k$.

Then, for this fraction $a/(a+d)$ we have that the associated α and $\{n_i\}_{i=1}^{k-1}$ are given by

$$\alpha = \frac{a+d}{d} = \frac{2p_1-2}{p_1-2} = 2 + \frac{2}{p_1-2} \le \frac{8}{3}$$

and $n_i = 1 + \alpha^{2^{i-1}}$ for all $i = 1, \ldots, k-2$; so, in particular,

$$n_1 = 1 + \alpha \le 11/3 \quad \text{and} \quad n_2 = 1 + \alpha^2 \le 73/9$$

and thus

$$n_1 n_2 \le \left(\frac{11}{3}\right)\left(\frac{73}{9}\right) \approx 29.741 \,.$$

Therefore, since $p_2 \ge 7$ and $p_3 \ge 11$, we immediately have that $p_2 > n_1$ and $p_2 p_3 > n_1 n_2$; and hence, for our HBC-admissible $(k-1)$-tuple $\{p_2, \ldots, p_k\}$ for $p_1/(p_1 + (p_1 - 2))$, we must always have $\kappa_{\max} \ge 3$.

It then immediately follows from Theorem 1 that we have the estimate

$$\begin{aligned} \prod_{i=1}^{k} p_i = p_1 \prod_{i=2}^{k} p_i &\le \left[\left(\frac{2(p_1-1)}{(p_1-2)^{7/8}}\right)^{2^{k-1}} - \left(\frac{2(p_1-1)}{(p_1-2)^{7/8}}\right)^{2^{k-2}}\right] \\ &= \left[\left(\sqrt{\frac{2(p_1-1)}{(p_1-2)^{7/8}}}\right)^{2^{k}} - \left(\sqrt{\frac{2(p_1-1)}{(p_1-2)^{7/8}}}\right)^{2^{k-1}}\right] \end{aligned} \tag{42}$$

which, in the case $p_1 = 5$, yields

$$\prod_{i=1}^{k} p_i \le \left[\left(\sqrt{\frac{8}{3^{7/8}}}\right)^{2^k} - \left(\sqrt{\frac{8}{3^{7/8}}}\right)^{2^{k-1}}\right] \approx F_k(1.74906) \,, \tag{43}$$

a significant improvement over the "standard" bound of $F_k(2) = (2^{2^k} - 2^{2^{k-1}})$.

For arbitrary p_1, however, we see that (42) is not a tight enough bound to get a uniform estimate of the desired sort, independent of p_1, since the right-hand side of (42) eventually grows slowly as p_1 increases (on the order of $(\sqrt{2}p_1^{1/16})^{2^k} \sim (256p_1)^{2^{k-4}}$).[10] Hence, to get an improved *uniform* bound of the form sought, regardless of p_1, we need to fine-tune (42), by showing that as the size of p_1 increases we can also establish a correspondingly stronger lower bound for the $\kappa_{\max}$ value for our associated auxiliary problem (so that, as p_1 grows, so also, in due course, must the lower bound for $\kappa_{\max}$, sufficient to keep $\prod_{i=1}^{k} p_i$ suitably contained).

To this end, suppose now that $p_1 > 5$, and then for any fixed $\omega \in \mathbb{Z}_{\ge 0}$ define $A_\omega : \mathbb{R}_{>2} \to \mathbb{R}$ by

$$A_\omega(x) = \frac{2(x-1)}{(x-2)^{1-1/2^{\omega+3}}} \,. \tag{44}$$

[10] Of course, by Grün's result (see Lemma 5) or, better still, our own sharpening of his estimate in the previous section, it follows that for p_1 to be large and for (37) to hold would require k to be commensurately *enormous*. Conversely then, even for quite large k Estimate (42) would already, with little further work, yield notably improved uniform bounds for $\prod_{i=1}^{k} p_i$ for HBC-admissible prime k-tuples for $1/2$ with $p_1 \ge 5$. Rather than pursue this line of reasoning here, however, we defer exploration of this idea to [24], where it is taken up in a much more carefully optimized fashion.

Also, subdivide the set of integers $\mathbb{Z}_{\geq 4}$ into disjoint subsets $\Omega_0, \Omega_1, \Omega_2, \ldots$ defined by $\Omega_\omega = [2^{2^\omega} + 2, 2^{2^{\omega+1}} + 2)$ for $\omega = 0, 1, 2, \ldots$ (so $\Omega_0 = [4, 6)$, $\Omega_1 = [6, 18)$, $\Omega_2 = [18, 258)$, and so on).

Then by our assumption that $p_1 > 5$ we must have that $p_1 \notin \Omega_0$. Moreover, by Lemma 5 we must have, for any k, that $p_1 < (3k/7) + 3$, and hence $p_1 \notin \Omega_\omega$ for any integer $\omega \geq \omega_k$ where $\omega_k = \log_2(\log_2((3k/7) + 1))$. For if we had $p_1 \in \Omega_\omega$ for some such ω we would have

$$\frac{3k}{7} + 3 > p_1 \geq 2^{2^\omega} + 2 \geq 2^{2^{\omega_k}} + 2 = \frac{3k}{7} + 3,$$

a contradiction. Hence, for any k, if $p_1 \in \Omega_\omega$ then the following must hold:

(i) We must have that $1 \leq \omega < \omega_k$.

(ii) We must also have that $p_1 - 2 \geq 2^{2^\omega}$ by the definition of Ω_ω, and hence $\alpha = 2 + 2/(p_1 - 2) \leq 2(1 + 1/2^{2^\omega})$ for our auxiliary $(k-1)$-tuple $\{p_i\}_{i=2}^k$ for $p_1/(p_1 + (p_1 - 2))$. Therefore, for any $j \leq k - 1$ we have, by Lemma 3, that

$$\prod_{i=1}^{j} n_i \leq \frac{d}{a}\left(\alpha^{2^j} - 1\right) < \alpha^{2^j} \leq 2^{2^j}\left(1 + \frac{1}{2^{2^\omega}}\right)^{2^j}, \tag{45}$$

with equality in the first inequality unless $j = k - 1$.

We now consider three cases.

`Case 1:` Suppose $\omega = 1$, so $p_1 \in [6, 18)$. In this case we already have that $n_1 < p_2$ and $n_1 n_2 < p_2 p_3$ while, by Estimate (45),

$$n_1 n_2 n_3 \leq 2^8\left(1 + \frac{1}{4}\right)^8 \approx 1,526.9 < p_2 p_3 p_4$$

since here $p_2 > p_1 \geq 6$ so $p_2 \geq 11$ (since p_1, p_2 are prime), and thus also $p_3 \geq 13$ and $p_4 \geq 17$, whence $p_2 p_3 p_4 \geq 11 \cdot 13 \cdot 17 = 2431$. Hence we have

$$\prod_{i=1}^{j} n_i < \prod_{i=1}^{j} p_{i+1} \quad \text{for all } j \leq \omega + 2\,.$$

`Case 2:` Suppose $\omega = 2$, so $p_1 \in [18, 258)$. In this case $p_1 \geq 18$ so, by primality of the p_i, we must in fact have $p_1 \geq 19$ and hence $p_2 \geq 23$, $p_3 \geq 29$, $p_4 \geq 31$ and $p_5 \geq 37$. Thus here, in addition to already having that $n_1 < p_2$ and $n_1 n_2 < p_2 p_3$, we also have by Estimate (45) that

$$n_1 n_2 n_3 \leq 2^8\left(1 + \frac{1}{16}\right)^8 \approx 415.8 < 23 \cdot 29 \cdot 31 \leq p_2 p_3 p_4$$

and similarly

$$n_1n_2n_3n_4 \le 2^{16}\left(1+\frac{1}{16}\right)^{16} \approx 172,879.3 < 765,049 = 23\cdot 29\cdot 31\cdot 37 \le p_2p_3p_4p_5.$$

Thus once again we have

$$\prod_{i=1}^{j} n_i < \prod_{i=1}^{j} p_{i+1} \quad \text{for all } j \le \omega+2\,.$$

`Case 3:` Suppose $3 \le \omega < \omega_k$. Finally, in this case, in addition to already having that $n_1 < p_2$ and $n_1n_2 < p_2p_3$, we clearly have:

1. for any $3 \le j \le \omega$,

$$\prod_{i=1}^{j} n_i \le 2^{2^\omega}\left(1+\frac{1}{2^{2^\omega}}\right)^{2^\omega} < 2.2^{2^\omega} < 2p_1 < \prod_{i=1}^{j} p_{i+1}\,;$$

2. for $j = \omega+1$,

$$\prod_{i=1}^{j} n_i < 2^{2^{\omega+1}}\left(1+\frac{1}{2^{2^\omega}}\right)^{2^{\omega+1}} < 2.2^{2^{\omega+1}} = 2(2^{2^\omega})^2 < 2p_1^2 < p_2p_3p_4 < \prod_{i=1}^{j} p_{i+1};$$

3. for $j = \omega+2$,

$$\prod_{i=1}^{j} n_i < 2^{2^{\omega+2}}\left(1+\frac{1}{2^{2^\omega}}\right)^{2^{\omega+2}}$$
$$< 2.2^{2^{\omega+2}} = 2(2^{2^\omega})^4 < 2p_1^4 < p_2p_3p_4p_5p_6 \le \prod_{i=1}^{j} p_{i+1}\,.$$

Thus once again we have

$$\prod_{i=1}^{j} n_i < \prod_{i=1}^{j} p_{i+1} \quad \text{for all } j \le \omega+2\,.$$

Hence, combining all three cases above, we have established that for all possible ω,

$$2^{2^\omega}+2 \le p_1 < 2^{2^{\omega+1}}+2 \quad \text{implies} \quad \kappa_{\max} \ge \omega+3\,. \tag{46}$$

Note that this result says exactly that, as p_1 increases, the minimum possible value for $\kappa_{\max}$ also periodically increases.[11]

[11] In the working to obtain (46) it is implicitly required, since the n_i are only defined here for $1 \le i \le k-1$, that we must have $\omega+2 \le k-1$ when $p_1 \in \Omega_\omega$. However, it is easy to see that this must indeed hold since, when $p_1 \ge 7$ (as here), it follows from Remark 6 that we must have that $k \ge 15$. Yet then we will clearly have $\omega+2 < \omega_k+2 = 2+\log_2(\log_2((3k/7)+1)) < k-1$, as required.

Now suppose $p_1 \in \Omega_\omega$, for some $1 \le \omega < \omega_k$. Then by (46) in Theorem 1 we have

$$\prod_{i=1}^{k} p_i = p_1 \prod_{i=2}^{k} p_i \le \left[\left(\frac{2(p_1-1)}{(p_1-2)^{1-1/2^{\omega+3}}} \right)^{2^{k-1}} - \left(\frac{2(p_1-1)}{(p_1-2)^{1-1/2^{\omega+3}}} \right)^{2^{k-2}} \right]$$
$$= (A_\omega(p_1))^{2^{k-1}} - (A_\omega(p_1))^{2^{k-2}} = F_{k-1}\left(A_\omega(p_1)\right). \tag{47}$$

Yet then, by Lemma 10 in Appendix A, either (a) or (b) below must hold:

(a) We have $\omega \ge 3$. Then the function $A_\omega(p_1)$ is monotonically increasing as p_1 ranges across Ω_ω. Hence, since $F_{k-1}(\cdot)$ is also monotonically increasing, $F_{k-1}\left(A_\omega(p_1)\right)$ is bounded above by the value it tends towards at the right-hand endpoint of Ω_ω, namely $F_{k-1}(A_\omega(2^{2^{\omega+1}}+2))$.

(b) We have $\omega \in \{1,2\}$. Then $A_\omega(p_1)$ is monotonically decreasing initially and then monotonically increasing as p_1 ranges across Ω_ω. Hence, $F_{k-1}\left(A_\omega(p_1)\right)$ is bounded by the larger of the achievable values it tends towards at the endpoints of Ω_ω, namely $F_{k-1}(A_\omega(p_{\min}))$ and $F_{k-1}(A_\omega(p_{\max}))$ where $p_{\min}$ and $p_{\max}$ are the smallest and largest primes in the interval Ω_ω, respectively.

Considering first the former case, we thus have that if $p_1 \in \Omega_\omega$ for $3 \le \omega < \omega_k$ then

$$\begin{aligned}
F_{k-1}\left(A_\omega(p_1)\right) &< F_{k-1}\left(A_\omega(2^{2^{\omega+1}}+2)\right) \\
&= F_{k-1}\left(\frac{2(2^{2^{\omega+1}}+1)}{(2^{2^{\omega+1}})^{(2^{\omega+3}-1)/2^{\omega+3}}}\right) \\
&= F_{k-1}\left(\frac{2(2^{2^{\omega+1}}+1)}{2^{(2^{\omega+3}-1)/4}}\right) \\
&= F_{k-1}\left(2^{5/4}\left(\frac{2^{2^{\omega+1}}+1}{2^{2^{\omega+1}}}\right)\right) \\
&\le F_{k-1}\left(2^{5/4}\left(\frac{2^{2^4}+1}{2^{2^4}}\right)\right) \\
&= F_k\left(\sqrt{2^{5/4}\left(\frac{2^{16}+1}{2^{16}}\right)}\right) \approx F_k(1.54222)\,,
\end{aligned} \tag{48}$$

which is a uniform estimate of the desired sort.

Next, for $\omega = 2$, since the largest and smallest primes in the interval $\Omega_2 = [18, 258)$

are 257 and 19, respectively, it follows that if $p_1 \in \Omega_2$ then

$$F_{k-1}(A_2(p_1)) \le F_{k-1}(\max\{A_2(19), A_2(257)\}) = F_{k-1}\left(\max\left\{\frac{2(18)}{17^{31/32}}, \frac{2(256)}{255^{31/32}}\right\}\right)$$
$$= F_k\left(\sqrt{\frac{512}{255^{31/32}}}\right) \approx F_k(1.54514)\,. \tag{49}$$

And similarly, for $\omega = 1$, since the largest and smallest primes in the interval $\Omega_1 = [6, 18)$ are 17 and 7, respectively, it follows that if $p_1 \in \Omega_1$ then

$$F_{k-1}\left(A_1(p_1)\right) \le F_{k-1}\left(\max\{A_1(7), A_1(17)\}\right) = F_{k-1}\left(\max\left\{\frac{2(6)}{5^{15/16}}, \frac{2(16)}{15^{15/16}}\right\}\right)$$
$$= F_k\left(\sqrt{\frac{12}{5^{15/16}}}\right) \approx F_k(1.62910)\,. \tag{50}$$

Yet finally then, drawing together Estimates (43), (47), (48), (49) and (50), we have exactly the desired bounds of Lemma 7. □

Lemma 7 makes a start on using Theorem 1 to obtain strengthened Heath-Brown stage one-type estimates for $\prod_{i=1}^k p_i$ for HBC-admissible odd prime k-tuples $\{p_i\}_{i=1}^k$ for the fraction 1/2. We conclude this paper by showing that Estimate (38) of Lemma 7, for the case $p_1 = 5$, can be further improved through careful supplementary analysis of the possible options for p_2. What follows is an illustration of Remark 5, that it may often be more useful to think of Theorem 1 in terms of a *process* for getting better upper bounds for $\prod_{i=1}^k y_i$, rather than simply as a result to be invoked without further ado.

Lemma 8. *Suppose $\{p_i\}_{i=1}^k$ is an HBC-admissible prime k-tuple for 1/2, so that*

$$\prod_{i=1}^{k}\left(1 - \frac{1}{p_i}\right) \le \frac{1}{2} < \prod_{i=1}^{k-1}\left(1 - \frac{1}{p_i}\right) \tag{51}$$

and suppose $p_1 = 5$. Then

$$\prod_{i=1}^{k} p_i \le \left[\left(\sqrt{\frac{8}{3^{31/32}}}\right)^{2^k} - \left(\sqrt{\frac{8}{3^{31/32}}}\right)^{2^{k-1}}\right] \approx F_k(1.66127). \tag{52}$$

Proof. We split our analysis into two cases – first where $p_1 = 5$ and $p_2 = 7$, and second where $p_1 = 5$ but $p_2 \ne 7$, so $p_2 \ge 11$. Consider first the case where $p_1 = 5$ and $p_2 = 7$. Then here, by (51), we have that

$$\prod_{i=3}^{k}\left(1 - \frac{1}{p_i}\right) \le \frac{1}{2}\left(\frac{5}{4}\right)\left(\frac{7}{6}\right) = \frac{35}{48} < \prod_{i=3}^{k-1}\left(1 - \frac{1}{p_i}\right)$$

so $\{p_i\}_{i=3}^k$ is an HBC-admissible odd prime $(k-2)$-tuple for the fraction $a^*/(a^*+d^*)$ where $a^* = 35$, $d^* = 13$ and $p_3 \geq 11$.

Then for this fraction we have that the associated α and n_i values, here denoted α^* and $\{n_i^*\}_{i=1}^{k-2}$, are given by

$$\alpha^* = (a^* + d^*)/d^* = 48/13 \approx 3.692$$

and

$$n_i^* = 1 + (\alpha^*)^{2^i} \quad \text{for all } i = 1, \ldots, k-3$$

so, in particular, $n_1^* = 1 + \alpha^* \approx 4.692$, $n_2^* = 1 + (\alpha^*)^2 \approx 14.633$ and thus $n_1^* n_2^* \approx 68.663$. Thus, we immediately have $p_3 > n_1^*$ and $p_3 p_4 \geq 11 \cdot 13 = 143 > n_1^* n_2^*$ so that, for our HBC-admissible $(k-2)$-tuple for the fraction 35/48, we must have $\kappa_{\max} \geq 3$. Thence, by Theorem 1, in this case

$$\prod_{i=1}^{k} p_i = 5.7 \prod_{i=3}^{k} p_i \leq \left[\left(\frac{48}{13^{7/8}} \right)^{2^{k-2}} - \left(\frac{48}{13^{7/8}} \right)^{2^{k-3}} \right] \approx F_k(1.50188)\,. \tag{53}$$

Now consider the alternative case where $p_1 = 5$ but $p_2 \neq 7$, so $p_2 \geq 11$. Here note first that, if we go back to our earlier auxiliary setting for the case $p_1 = 5$ from the proof of Lemma 7 (i.e., use that $\{p_i\}_{i=2}^k$ must be an HBC-admissible odd $(k-1)$-tuple for 5/8), then $\alpha = 8/3$ for this setting. Hence the associated $\{n_i\}_{i=1}^{k-1}$ satisfy $n_1 = 11/3$, $n_2 = 73/9$, $n_3 = 1+(8/3)^4 \approx 51.568$, $n_4 = 1+(8/3)^8 \approx 2,558.113$ and $n_5 = 1 + (8/3)^{16} \approx 6,538,826.029$.

Hence, here we have that

$$\begin{aligned} n_1 &\approx 3.667 < 11 \leq p_2 \\ n_1 n_2 &\approx 29.741 < 11 \cdot 13 \leq p_2 p_3 \\ n_1 n_2 n_3 &\approx 1,533.668 < 11 \cdot 13 \cdot 17 \leq p_2 p_3 p_4, \end{aligned}$$

while also

$$n_1 n_2 n_3 n_4 \approx 3,923,294 < 41 \cdot 43 \cdot 47 \cdot 53, \tag{54}$$

and similarly

$$n_1 n_2 n_3 n_4 n_5 \approx 2.56537 \times 10^{13} < 467 \cdot 479 \cdot 487 \cdot 491 \cdot 499. \tag{55}$$

It then follows directly that, for this auxiliary setting, we must have $\kappa_{\max} \geq 4$, a strengthening of our previous bound that $\kappa_{\max} \geq 3$ for this case, so we immediately obtain the improved estimate (compared with (43))

$$\prod_{i=1}^{k} p_i = 5 \prod_{i=2}^{k} p_i \leq \left[\left(\frac{8}{3^{15/16}} \right)^{2^{k-1}} - \left(\frac{8}{3^{15/16}} \right)^{2^{k-2}} \right] \approx F_k(1.69003)\,.$$

However, we can tighten this still further by noting that if we had $p_2 \geq 41$ then, in view of (54), we would have $\kappa_{\max} \geq 5$ for our auxiliary setting, and hence would obtain

$$\prod_{i=1}^{k} p_i \leq \left[\left(\sqrt{\frac{8}{3^{31/32}}} \right)^{2^k} - \left(\sqrt{\frac{8}{3^{31/32}}} \right)^{2^{k-1}} \right] \approx F_k(1.66127)\,. \tag{56}$$

Yet on the other hand, if we had that $11 \leq p_2 < 41$, this would leave us with just the eight possibilities $p_2 = 11, 13, 17, 19, 23, 29, 31$ or 37.

To handle these eight cases, observe that (51) with $p_1 = 5$ implies that

$$\prod_{i=3}^{k} \left(1 - \frac{1}{p_i}\right) \leq \frac{1}{2}\left(\frac{5}{4}\right)\left(\frac{p_2}{p_2 - 1}\right) = \frac{5p_2}{5p_2 + (3p_2 - 8)} < \prod_{i=3}^{k-1} \left(1 - \frac{1}{p_i}\right),$$

so that $\{p_3, \ldots, p_k\}$ would be an HBC-admissible $(k-2)$-tuple for the fraction $a^{**}/(a^{**} + d^{**})$ where $a^{**} = 5p_2$, $d^{**} = 3p_2 - 8$ and $p_3 \geq 13$. Then clearly a^{**} and d^{**} do not share p_2 as a common factor. However, they do share 5 as a common factor for $p_2 = 11$ and $p_2 = 31$ – so in these two cases we redefine $a^{**} = p_2$ and $d^{**} = (3p_2 - 8)/5$. Then with these choices the fraction $a^{**}/(a^{**} + d^{**})$ is in lowest terms in all eight cases, taking the values 11/16, 65/96, 85/128, 95/144, 115/176, 145/224, 31/48 and 185/288 for $p_2 = 11, \ldots, 37$, respectively.

Moreover, for the fraction $a^{**}/(a^{**} + d^{**})$ we then have that the associated α and n_i values, here denoted α^{**} and $\{n_i^{**}\}_{i=1}^{k-2}$, are given by

$$\alpha^{**} = \frac{a^{**} + d^{**}}{d^{**}} = \frac{8(p_2 - 1)}{(3p_2 - 8)} = \frac{8}{3} + \frac{40}{3(3p_2 - 8)}$$

and

$$n_i^{**} = 1 + (\alpha^{**})^{2^{i-1}} \quad \text{for all } i = 1, \ldots, k-3\,.$$

Hence for $p_2 \geq 11$ we have $\alpha^{**} \leq (8/3) + (40/75) = 16/5$, so $n_1^{**} = 1 + \alpha^{**} \leq 21/5$, $n_2^{**} = 1 + (\alpha^{**})^2 \leq 281/25$ and thus $n_1^{**} < 13 \leq p_3$ and $n_1^{**} n_2^{**} \leq 47.208 < 13 \cdot 17 \leq p_3 p_4$. For our HBC-admissible odd prime $(k-2)$-tuple for the fraction $5p_2/8(p_2-1)$, we must therefore always have $\kappa_{\max} \geq 3$ (for any $p_2 \geq 11$); so, by Theorem 1, we immediately get the bound

$$\prod_{i=1}^{k} p_i = p_1 p_2 \prod_{i=3}^{k} p_i \leq \delta \left[(\chi_{p_2})^{2^{k-2}} - (\chi_{p_2})^{2^{k-3}} \right] = \delta F_k(\chi_{p_2}^{1/4}) \tag{57}$$

where

$$\chi_{p_2} = \frac{8(p_2 - 1)}{(3p_2 - 8)^{7/8}} \qquad \text{for } p_2 = 13, 17, 19, 23, 29, 37$$

$$\chi_{p_2} = \frac{8(p_2 - 1)}{5^{1/8}(3p_2 - 8)^{7/8}} \qquad \text{for } p_2 = 11, 31$$

and where

$$\delta = \begin{cases} 1 & \text{for } p_2 = 13, 17, 19, 23, 29, 37 \\ p_1 = 5 & \text{for } p_2 = 11, 31. \end{cases}$$

Yet then, for our eight possible p_2 values in the range $11 \le p_2 \le 37$, we have respectively that, by direct calculation,

$$\chi_{p_2}^{1/4} = \begin{cases} (16/5^{7/8})^{1/4} \approx 1.40647 \\ (96/31^{7/8})^{1/4} \approx 1.47684 \\ (128/43^{7/8})^{1/4} \approx 1.47734 \\ (144/49^{7/8})^{1/4} \approx 1.47863 \\ (176/61^{7/8})^{1/4} \approx 1.48196 \\ (224/79^{7/8})^{1/4} \approx 1.48750 \\ (48/17^{7/8})^{1/4} \approx 1.41628 \\ (288/103^{7/8})^{1/4} \approx 1.49465\,. \end{cases} \tag{58}$$

(Note that the reason for the marked drop-off in the $\chi_{p_2}^{1/4}$ values for $p_2 = 11$ and $p_2 = 31$, compared with the other six cases, reflects that in these two cases some cancellation was possible at the outset in the original fraction $a^{**}/(a^{**} + d^{**})$, reducing both numerator and denominator by a factor of five. This then results in tighter bounds from (57) because the bounds we get from Theorem 1 are not scaling-invariant!)

Finally then, bringing together Estimates (56), (57) and (58), it follows immediately that for $p_1 = 5$ but $p_2 \neq 7$ we must always have

$$\prod_{i=1}^{k} p_i \le \left[\left(\sqrt{\frac{8}{3^{31/32}}} \right)^{2^k} - \left(\sqrt{\frac{8}{3^{31/32}}} \right)^{2^{k-1}} \right] \approx F_k(1.66127)\,,$$

and this, combined with Estimate (53) for the case $p_1 = 5, p_2 = 7$, completes the proof of the lemma. □

Remark 8. We could, of course, endeavour to further push the general bound in Lemma 8 down to $\prod_{i=1}^{k} p_i \le F_k(\beta)$, where β is even closer to $\sqrt{(8/3)} \approx 1.63299$, by repeating the procedure used above to treat the cases $p_1 = 5, 11 \le p_2 < 41$ to similarly strengthen the handling of the cases $p_1 = 5, 41 \le p_2 < 467$ (while using (55) to allow for an improved estimate also for the cases $p_1 = 5, p_2 \ge 467$). However, we do not pursue this here, contenting ourselves for the present with Lemma 8. Instead, in [24] we take up a different approach for obtaining tighter Heath-Brown stage one-type bounds for $\prod_{i=1}^{k} p_i$ for HBC-admissible odd prime k-tuples for $1/2$ for which $p_1 \neq 3$.

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A. Appendix – Two Technical Lemmas

In this appendix we provide the proofs of a number of results which were deferred from the main body of the paper, because presenting them there would have been too much of a digression from the main flow of the argument at each point.

Lemma 9. *For any $r \in \mathbb{N}$ define $F_r : \mathbb{R}_{\geq 1} \to \mathbb{R}_{\geq 0}$ by*

$$F_r(x) = x^{2^r} - x^{2^{r-1}}.$$

Then for all $r \in \mathbb{N}$ and for all $x \in \mathbb{R}_{\geq 1}$:

i. *$F_r(x)$ is monotonically increasing in x (cf. Lemma 1.3 of [18]);*

ii. *$F_r(x)$ is monotonically non-decreasing in r (and monotonically increasing if $x > 1$);*

iii. *$F_v(F_w(x)) \leq F_{v+w}(x)$ for any $v, w \in \mathbb{N}$ (and x such that $F_w(x) \geq 1$);*

iv. *for all $\lambda > 0$, $x \geq 1$ such that $\lambda x \geq 1$ we have*

$$F_r(\lambda x) \begin{cases} \leq \lambda^{2^r} F_r(x) & \text{if } \lambda \leq 1 \\ > \lambda^{2^r} F_r(x) & \text{if } \lambda > 1 . \end{cases}$$

Proof. To see the first claim, observe that for any fixed r, $F_r'(x) = 2^r x^{2^r - 1} - 2^{r-1} x^{2^{r-1}-1} = 2^{r-1} x^{2^{r-1}-1}[2x^{2^{r-1}} - 1] > 0$ for all $x \geq 1$.

As for the second claim, for any fixed $x \geq 1$ we have that, for any $r \in \mathbb{N}$, $F_{r+1}(x) = x^{2^{r+1}} - x^{2^r} = x^{2^r}(x^{2^r} - 1) \geq x^{2^r} - 1 \geq x^{2^r} - x^{2^{r-1}} = F_r(x)$, with strict inequalities throughout if $x > 1$, as required.

To see the third claim, observe that by the monotonicity of $F_v(\cdot)$,

$$\begin{aligned} F_v(F_w(x)) &= F_v\left(x^{2^w} - x^{2^{w-1}}\right) \\ &\leq F_v\left(x^{2^w}\right) \\ &= \left(x^{2^w}\right)^{2^v} - \left(x^{2^w}\right)^{2^{v-1}} \\ &= x^{2^{v+w}} - x^{2^{v+w-1}} \\ &= F_{v+w}(x) \end{aligned}$$

as desired.

Finally, to see the fourth claim observe that, for $\lambda < 1$, then

$$\begin{aligned} F_r(\lambda x) = (\lambda x)^{2^r} - (\lambda x)^{2^{r-1}} &= \lambda^{2^r}\left[x^{2^r} - x^{2^{r-1}}/\lambda^{2^{r-1}}\right] \\ &< \lambda^{2^r}\left[x^{2^r} - x^{2^{r-1}}\right] \\ &= \lambda^{2^r} F_r(x) \end{aligned}$$

as asserted; while for $\lambda > 1$ the same calculation shows that $F_r(\lambda x) > \lambda^{2^r} F_r(x)$, again as claimed. □

Lemma 10. *For any fixed $\omega \in \mathbb{R}_{>-3}$, the function $A_\omega : \mathbb{R}_{>2} \to \mathbb{R}$ defined by*

$$A_\omega(x) = \frac{2(x-1)}{(x-2)^{1-1/2^{\omega+3}}}$$

is a decreasing function of x on the interval $2 < x < 1 + 2^{\omega+3}$, and an increasing function of x on the interval $x > 1 + 2^{\omega+3}$, with a global minimum at $x = 1 + 2^{\omega+3}$.

Proof. Simply differentiate $A_\omega(x)$. □

B. Appendix – An Asymptotic Version of Grün's Bound

In this appendix we derive an asymptotic version of Grün's bound for p_1, in which this bound grows (roughly) like $\sqrt{k}$ rather than linearly in k. This result represents a strengthening of an earlier asymptotic bound of Norton [19], using a method similar to his (but making use of stronger information on the distribution of primes, from Dusart [6], than was available at that time).[12] In [30], Zelinsky also recently established similar (but more general) improvements of Norton's asymptotic estimates.

To obtain our asymptotic bound, let $\{\rho_i\}_{i=1}^{\infty}$ denote the set of all primes, labelled in ascending order (so $\rho_1 = 2, \rho_2 = 3, \rho_3 = 5$, and so on); and let $\gamma \approx 0.5772157$ denote Euler's constant.

B.1. Preliminaries – Dusart's Estimates

By Theorem 5.9 of [6] we have that, for any $n, k \in \mathbb{N}$ with $n \geq n_0 = 168{,}065$ (so that $\rho_n > 2{,}278{,}382$),

$$e^\gamma \ln(\rho_{n+k}) \prod_{\rho \leq \rho_{n+k}} \left(\frac{\rho - 1}{\rho}\right) \geq 1 - \frac{0.2}{\ln^3(\rho_{n+k})} > 1 - \frac{0.2}{\ln^3(\rho_n)}$$

and

$$\frac{1}{e^\gamma \ln(\rho_n)} \prod_{\rho \leq \rho_n} \left(\frac{\rho}{\rho - 1}\right) \geq 1 - \frac{0.2}{\ln^3(\rho_n)}$$

whence

$$\frac{\ln(\rho_n)}{\ln(\rho_{n+k})}\left(1 - \frac{0.2}{\ln^3(\rho_n)}\right)^2 < \prod_{\rho \leq \rho_n} \left(\frac{\rho}{\rho - 1}\right) \prod_{\rho \leq \rho_{n+k}} \left(\frac{\rho - 1}{\rho}\right) = \prod_{\rho_{n+1} \leq \rho \leq \rho_{n+k}} \left(\frac{\rho - 1}{\rho}\right). \tag{59}$$

[12]The author thanks the anonymous referee of this paper for alerting him to Norton's work.

Furthermore, there clearly exists an $n_1 \in \mathbb{N}$ (a quick calculation yields that $n_1 = 33$ will do) such that, for all $n \geq n_1$,

$$\ln(\ln(n)) - 1 + \frac{[(\ln(\ln(n)) - 2.1)]}{\ln(n)} \geq 0 \, .$$

It then follows from Proposition 5.16 of Dusart [6] that, for any such n,

$$\ln(\rho_n) \geq \ln(n \ln(n)) = \ln(n) + \ln(\ln(n)) \; ; \tag{60}$$

while Lemma 5.10 of [6] correspondingly gives, for $n \geq 4$,

$$\rho_n / \ln(\rho_n) \leq n \tag{61}$$

and

$$\rho_n \leq en \ln(n) \, . \tag{62}$$

B.2. An Asymptotic Version of Grün's Bound

Now suppose $\{p_i\}_{i=1}^k$ is an HBC-admissible odd prime k-tuple for 1/2, so

$$\prod_{i=1}^{k} \left(1 - \frac{1}{p_i}\right) \leq \frac{1}{2} < \prod_{i=1}^{k-1} \left(1 - \frac{1}{p_i}\right) \, , \tag{63}$$

and suppose $p_1 = \rho_{n+1}$, so then in turn $p_2 \geq \rho_{n+2}, \ldots, p_{k-1} \geq \rho_{n+k-1}$ and $p_k \geq \rho_{n+k}$.

Then by the left-hand inequality of (63) – which must actually be strict since p_k must occur in the denominator of the left-hand quantity – we must have

$$\prod_{\rho_{n+1} \leq \rho \leq \rho_{n+k}} \left(\frac{\rho - 1}{\rho}\right) \leq \prod_{i=1}^{k} \left(1 - \frac{1}{p_i}\right) < \frac{1}{2} \, .$$

Hence, if $n \geq n_0$, we must (by (59)) have

$$\frac{\ln(\rho_n)}{\ln(\rho_{n+k})} \left(1 - \frac{0.4}{\ln^3(\rho_n)}\right) < \frac{\ln(\rho_n)}{\ln(\rho_{n+k})} \left(1 - \frac{0.2}{\ln^3(\rho_n)}\right)^2 < \frac{1}{2} \, ,$$

and therefore

$$2\ln(\rho_n) - \frac{0.8}{\ln^2(\rho_n)} < \ln(\rho_{n+k}) \, . \tag{64}$$

On the other hand, however, by Estimate (62) we have (for $n + k \geq 4$)

$$\begin{aligned} \ln(\rho_{n+k}) &< \ln(n+k) + \ln(\ln(n+k)) + 1 \\ &= \ln(n) + \ln(1 + k/n) + \ln(\ln(n)) + \ln\left(1 + \frac{\ln(1 + k/n)}{\ln(n)}\right) + 1 \, . \end{aligned} \tag{65}$$

So now set $\lambda = (k+n)/n^2 \in \mathbb{R}$, so $k = \lambda n^2 - n$. Then, for $n \geq n_1$, Estimates (60) and (65) yield that

$$\begin{aligned}\ln(\rho_{n+k}) &< \ln(n) + \ln(\lambda n) + \ln(\ln(n)) + \ln\left(1 + \frac{\ln(\lambda n)}{\ln(n)}\right) + 1 \\ &= [2\ln(n) + 2\ln(\ln(n))] + 1 + \ln(\lambda) + \ln\left(2 + \frac{\ln(\lambda)}{\ln(n)}\right) - \ln(\ln(n)) \\ &\leq 2\ln(\rho_n) + 1 + \ln(\lambda) + \ln\left(2 + \frac{\ln(\lambda)}{\ln(n)}\right) - \ln(\ln(n))\end{aligned}$$

whence, by Estimate (64), we must, if $n \geq \max(n_0, n_1)$, have

$$0 < \left(1 + \frac{0.8}{\ln^2(\rho_n)} + \ln(\lambda) + \ln\left(2 + \frac{\ln(\lambda)}{\ln(n)}\right)\right) - \ln(\ln(n))\,. \tag{66}$$

Now, for any specified $\varepsilon > 0$, let $n_2(\varepsilon) \in \mathbb{N}$ be such that, for all $n \geq n_2(\varepsilon)$,

$$\ln^{1-\varepsilon}(n) + \frac{1}{n} \leq 2\ln^{1-\varepsilon}(n) \leq n \tag{67}$$

and

$$1 + \frac{0.8}{\ln^2(\rho_n)} + \ln(2) + \ln(3) - \varepsilon\ln(\ln(n)) < 0\,. \tag{68}$$

Then, if $n \geq n_3(\varepsilon) = \max(n_0, n_1, n_2(\varepsilon)) \in \mathbb{N}$, we must have

$$\lambda > \ln^{1-\varepsilon}(n) + \frac{1}{n}\,.$$

For otherwise, by (67) we would have $\ln(\lambda) \leq \ln(2) + (1-\varepsilon)\ln(\ln(n))$ and $\ln(\lambda) \leq \ln(n)$. Yet then, by Inequality (66), we would have $0 < 1 + [0.8/\ln^2(\rho_n)] + \ln(2) + \ln(3) - \varepsilon\ln(\ln(n))$, contradicting (68).

So now, let $n_4 = n_4(\varepsilon) \in \mathbb{N}$ be such that $n^2\ln^{1-\varepsilon}(n) \geq (n+1)^2\ln^{1-2\varepsilon}(n+1)$ for all $n \geq n_4$, and let $n_5(\varepsilon) = \max(n_3(\varepsilon), n_4(\varepsilon))$; and then, if $n \geq n_5(\varepsilon)$ we must have

$$\begin{aligned}k = \lambda n^2 - n > n^2\ln^{1-\varepsilon}(n) &\geq (n+1)^2\ln^{1-2\varepsilon}(n+1) \\ &= (n+1)^{1+2\varepsilon}((n+1)\ln(n+1))^{1-2\varepsilon}\,.\end{aligned}$$

Yet then, since $(n+1) \geq \rho_{n+1}/\ln(\rho_{n+1})$ and $(n+1)\ln(n+1) \geq \rho_{n+1}/e$ by Estimates (61) and (62) respectively, we must have that, if $n \geq n_5(\varepsilon)$, then

$$\begin{aligned}k > \left(\frac{\rho_{n+1}}{\ln(\rho_{n+1})}\right)^{1+2\varepsilon}\left(\frac{\rho_{n+1}}{e}\right)^{1-2\varepsilon} &= \frac{\rho_{n+1}^2}{e^{1-2\varepsilon}\ln^{1+2\varepsilon}(\rho_{n+1})} \\ &= \frac{p_1^2}{e^{1-2\varepsilon}\ln^{1+2\varepsilon}(p_1)} > \frac{p_1^2}{e\ln^{1+2\varepsilon}(p_1)}\,;\end{aligned}$$

while obviously $p_1 \le \rho_{n_5(\varepsilon)}$ if $n < n_5(\varepsilon)$.

This establishes that, for any given $\varepsilon > 0$, there is a $k_0(\varepsilon) \in \mathbb{N}$ – simply take $k_0(\varepsilon)$ to be the least integer greater than $\rho^2_{n_5(\varepsilon)}/e\ln^{1+2\varepsilon}(\rho_{n_5(\varepsilon)})$ – such that, for any $k \ge k_0(\varepsilon)$, if $\{p_i\}_{i=1}^k$ is an HBC-admissible odd prime k-tuple for 1/2 then p_1 must satisfy

$$\frac{p_1^2}{e\ln^{1+2\varepsilon}(p_1)} < k\ . \tag{69}$$

In other words, asymptotically there is a Grün-style upper bound for p_1 in terms of k that almost varies only as $\sqrt{k}$, rather than depending linearly on k as Grün's original bound does (or as does our strengthening of his bound in Section 4).

C. Appendix – Bounds for the Radical of an Odd Perfect Number

In the study of odd perfect numbers, $N = \prod_{i=1}^m p_i^{e_i}$, there is considerable literature on upper bounds for the quantity $\mathrm{rad}(N) = \prod_{i=1}^m p_i$, the *radical* of N. Notable recent contributions include Luca and Pomerance [15], Klurman [14] and Ochem and Rao [21]. These papers all establish *relative* bounds for $\mathrm{rad}(N)$ in terms of a power of N (for example, Luca and Pomerance proved that $\mathrm{rad}(N) < 2N^{17/26}$).[13]

In the case of odd perfect numbers where $\prod_{i=1}^m(1-\frac{1}{p_i}) \le \frac{1}{2} < \prod_{i=1}^{m-1}(1-\frac{1}{p_i})$, the results in this paper contribute to that literature – since, in that case, the improved first-stage Heath-Brown estimate provided by Theorem 1 becomes exactly an absolute bound for $\mathrm{rad}(N)$. For other (putative) odd perfect numbers, however, for which $\prod_{i=1}^k(1-\frac{1}{p_i}) \le \frac{1}{2} < \prod_{i=1}^{k-1}(1-\frac{1}{p_i})$ for some $k < m$, this identification between $\prod_{i=1}^k p_i$ and $\mathrm{rad}(N)$ breaks down, and further work is required to translate the results here into bounds for $\mathrm{rad}(N)$.

It turns out, happily, that in these cases it is still possible to derive new upper bounds for $\mathrm{rad}(N)$ using the results of this paper and those of [26] (in which the focus is on optimizing stage two and the iterative aspects of Heath-Brown's procedure for estimating N). These new bounds are *absolute* ones, rather than relative to the size of N; full details are provided in [27].

[13] Related techniques and insights have also been used by various authors to obtain relative upper bounds for the largest few prime divisors of N in terms of a power of N – see, for example, the result of Acquaah and Konyagin [1] that $p_m < (3N)^{1/3}$ and analogous bounds for p_{m-1} and p_{m-2} in Zelinsky [29] and Bibby *et al.* [2].