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SUBSET AND SUBSEQUENCE SUMS WITH BOUNDED NUMBERS OF TERMS

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Abstract

Let A be a nonempty finite set of integers, and let α and β be nonnegative integers such that $\alpha + \beta \leq |A|$, where $|A|$ denotes the cardinality of the set A. Let $\Sigma_{\alpha}^{\beta}(A)$ denote the set of those integers which can be represented as a sum of a subset of A with at least α elements and at most $|A| - \beta$ elements. The usual sets of subsums $\Sigma(A)$ and $\Sigma_0(A)$ are special cases of $\Sigma_{\alpha}^{\beta}(A)$ for $(\alpha, \beta) = (1, 0)$ and $(\alpha, \beta) = (0, 0)$, respectively. If $\beta = 0$, then we denote $\Sigma_{\alpha}^{0}(A)$ simply by $\Sigma_{\alpha}(A)$. We establish the optimal lower bound for the cardinality of $\Sigma_{\alpha}^{\beta}(A)$. We also prove inverse theorems for the set of subsums $\Sigma_{\alpha}^{\beta}(A)$ which characterize the sets $A \subseteq \mathbb{Z}$ for which $|\Sigma_{\alpha}^{\beta}(A)|$ achieves the optimal lower bound. These results generalize the various direct and inverse theorems for $\Sigma_{\alpha}(A)$ proved recently by Bhanja and Pandey. Furthermore, we prove direct and inverse theorems for the subsequence sums $\Sigma_{\alpha}^{\beta}(\mathscr{A})$ in $\mathbb{Z}$ for an arbitrary finite sequence of integers $\mathscr{A}$ which generalize the results obtained for the set of subsums $\Sigma_{\alpha}^{\beta}(A)$ and also solve two open problems of Bhanja and Pandey related to the set of subsums $\Sigma_{\alpha}(\mathscr{A})$.

1. Introduction

Throughout the paper, let G denote an additive abelian group, and let $|S|$ denote the cardinality of the set $S \subseteq G$. For a nonzero integer c and a set $S \subseteq G$, the dilated set $\{cs : s \in S\}$ is denoted by $c * S$, and we simply write $-S$ for $(-1) * S$. Let A be a nonempty finite subset of G. For nonnegative integers α and β with $\alpha + \beta \leq |A|$, define

$$\Sigma_{\alpha}^{\beta}(A) = \{\sigma(B) : B \subseteq A \text{ and } \alpha \leq |B| \leq |A| - \beta\},$$

where $\sigma(B)$ denotes the sum of all the elements of the set B. The usual sets of subsums $\Sigma(A)$ and $\Sigma_0(A)$ are special cases of $\Sigma_{\alpha}^{\beta}(A)$ for $(\alpha, \beta) = (1, 0)$ and $(\alpha, \beta) = (0, 0)$, respectively. If $\beta = 0$, then $\Sigma_{\alpha}^{0}(A)$ is simply denoted by $\Sigma_{\alpha}(A)$.

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Estimation of the optimal lower bound for the cardinality of $\Sigma_{\alpha}^{\beta}(A)$ in terms of α, β and $|A|$ is one of the important problems, called the *direct problem*. Another important problem of interest is the characterization of the sets A for which $|\Sigma_{\alpha}^{\beta}(A)|$ achieves the optimal lower bound, called the *inverse problem*. These problems are extremely important in additive combinatorics and have many applications in zero-sum problems (see [3, 4, 9, 14, 15, 16, 23, 25, 26] and the references given therein).

Nathanson [23] proved direct and inverse results for the sumset $\Sigma(A)$ in the additive group of integers $\mathbb{Z}$. Balandraud [3] studied the direct problems for $\Sigma(A)$ and $\Sigma_0(A)$ in the finite prime field $\mathbb{F}_p$, where p is a prime number. The direct and inverse problems for $\Sigma_{\alpha}(A)$ in $\mathbb{Z}$ have been studied recently by Bhanja and Pandey [5, 6] and by Dwivedi and Mistri [13]. The lower bound for the cardinality of the set of subsums $\Sigma_{\alpha}(A)$ in $\mathbb{F}_p$ was obtained by Balandraud [4]. For a set $A \subseteq \mathbb{F}_p$ such that $A \cap (-A) = \emptyset$, Balandraud [4] conjectured that

$$|\Sigma_{\alpha}^{\beta}(A)| \geq \min\left\{p, \frac{|A|(|A|+1)}{2} - \frac{\alpha(\alpha+1)}{2} - \frac{\beta(\beta+1)}{2} + 1\right\},$$

unless

$$A = \lambda * \{1, -2, 3, \ldots, |A|\}$$

with $0 \neq \lambda \in \mathbb{F}_p$, $\frac{|A|(|A|+1)}{2} = p + 4$ and $(\alpha, \beta) \in \{(1,1), (1,2), (2,1)\}$.

Motivated by this conjecture, we study the direct and inverse problems for $\Sigma_{\alpha}^{\beta}(A)$ in $\mathbb{Z}$. In Section 2, we study the direct problem and obtain the optimal lower bound for $|\Sigma_{\alpha}^{\beta}(A)|$ considering the following cases:

(a) the set A contains only positive integers,

(b) the set A contains only nonnegative integers including zero,

(c) the set contains both positive and negative integers,

(d) the set A contains positive integers, negative integers and zero.

In Section 3, we study the inverse problem for $\Sigma_{\alpha}^{\beta}(A)$. The results in this section characterize the sets A for which $|\Sigma_{\alpha}^{\beta}(A)|$ achieves the optimal lower bound. In Section 4, we generalize the definition of the set of subsums $\Sigma_{\alpha}^{\beta}(A)$ to the set of subseuence sums $\Sigma_{\alpha}^{\beta}(\mathscr{A})$ for a sequence $\mathscr{A}$ in G. We also establish several direct and inverse theorems for $\Sigma_{\alpha}^{\beta}(\mathscr{A})$, which also generalize and solve two open problems of Bhanja and Pandey [6, Open problems (1) and (2), Section 4] .

We remark that the various known direct and inverse theorems for $\Sigma(A)$, $\Sigma_0(A)$ and $\Sigma_{\alpha}(A)$ [23, 5, 6, 13] can be obtained as special cases of the direct and inverse theorems for $\Sigma_{\alpha}^{\beta}(A)$ or $\Sigma_{\alpha}^{\beta}(\mathscr{A})$ proved in Section 2, Section 3 and Section 4.

The proofs of the direct and inverse theorems for $\Sigma_{\alpha}^{\beta}(A)$ and $\Sigma_{\alpha}^{\beta}(\mathscr{A})$ require several preliminary results (see Subsection 1.1) for the generalized h-fold sumset defined as follows. Given a nonempty finite set $A \subseteq G$ and an ordered $|A|$-tuple

$\bar{\mathbf{r}} = (r_a : a \in A)$ of positive integers associated with the set A, we define the *generalized h-fold sumset* $h^{(\bar{\mathbf{r}})}A$ as follows:

$$h^{(\bar{\mathbf{r}})}A = \left\{ \sum_{a\in A} s_a a : s_a \in \mathbb{Z}, 0 \le s_a \le r_a, \text{ and } \sum_{a\in A} s_a = h \right\}.$$

If $r_a = r$ for each $a \in A$, then $h^{(\bar{\mathbf{r}})}A$ is simply denoted by $h^{(r)}A$. The direct and inverse problems for $h^{(r)}A$ have been studied by Mistri and Pandey [18] in $\mathbb{Z}$ and by Monopoli [22] in $\mathbb{F}_p$ (see [21] also). Yang and Chen [28] have studied the direct and inverse problems for the sumset $h^{(\bar{\mathbf{r}})}A$ in $\mathbb{Z}$.

The classical *h-fold sumset* hA and the *restricted h-fold sumset* $h\hat{}A$ are special cases of this sumset for $r = h$ and $r = 1$, respectively. These sumsets have been studied extensively in the literature (see [1, 2, 8, 10, 11, 12, 24, 27] and the references given therein).

Facts 1. The following facts allow us to consider the sumset $\Sigma_\alpha^\beta(A)$ only for the pairs (α, β) satisfying $1 \le \alpha \le |A| - 1$ and $0 \le \beta \le |A| - 1$.

(i) It is easy to see that $\Sigma_\alpha^\beta(A) = \alpha\hat{}A$ if $\alpha + \beta = |A|$. Since the direct and inverse theorems are well known for the restricted h-fold sumset in $\mathbb{Z}$ [23], we always assume that $\alpha + \beta \le |A| - 1$, and so $0 \le \alpha \le |A| - 1$ and $0 \le \beta \le |A| - 1$.

(ii) It is easy to verify that $\Sigma_\alpha^\beta(A) = \sigma(A) - \Sigma_\beta^\alpha(A)$, and thus $|\Sigma_\alpha^\beta(A)| = |\Sigma_\beta^\alpha(A)|$.

(iii) Furthermore, $\Sigma_0^\beta(A) = \Sigma_1^\beta(A)$ if $0 \in \Sigma_1^\beta(A)$, and $\Sigma_0^\beta(A) = \Sigma_1^\beta(A) \cup \{0\}$ if $0 \notin \Sigma_1^\beta(A)$. Therefore, we consider only positive values of α.

Since in the definition of the sumset $h^{(\bar{\mathbf{r}})}A$, the relative order of the elements of the set A is taken into consideration, from now onwards, while using the sumset $h^{(\bar{\mathbf{r}})}A$ in a statement or in a proof, we will assume that the order of the elements in the set A is fixed.

1.1. Notation and Preliminary Results

Here we fix some notation which will be used throughout the paper. For integers a and b, where $a \le b$, we denote the interval of integers $\{n \in \mathbb{Z} : a \le n \le b\}$ by $[a, b]$. For a function f, we take $\sum_{i=u}^{v} f(i) = 0$, whenever u and v are integers such that $u > v$.

With slight deviation from the notation used by Yang and Chen [28], we use the following notation as in Dwivedi and Mistri [13]. Given positive integers h and k, and an ordered k-tuple $\bar{\mathbf{r}} = (r_0, r_1, \ldots, r_{k-1})$ of positive integers, let $\mu = \mu(\bar{\mathbf{r}}, h)$ be the largest integer and $\eta = \eta(\bar{\mathbf{r}}, h)$ be the least integers such that

$$\sum_{j=0}^{\mu-1} r_j \le h \quad \text{and} \quad \sum_{j=\eta+1}^{k-1} r_j \le h,$$

respectively. Now define

$$\delta = \delta(\bar{\mathbf{r}}, h) = h - \sum_{j=0}^{\mu-1} r_j \quad \text{and} \quad \theta = \theta(\bar{\mathbf{r}}, h) = h - \sum_{j=\eta+1}^{k-1} r_j.$$

Furthermore, define

$$L(\bar{\mathbf{r}}, h) = \left(\sum_{j=\eta+1}^{k-1} j r_j - \sum_{j=0}^{\mu-1} j r_j \right) + \eta\theta - \mu\delta + 1.$$

A k-term *arithmetic progression* in $\mathbb{Z}$ is a set of the form $\{a, a+d, \ldots, a+(k-1)d\}$ for some integer a and a nonzero integer d. We will require the direct and inverse theorems for $h^{(\bar{\mathbf{r}})}A$ due to Yang and Chen [28] to prove the direct and inverse theorems for $\Sigma_{\alpha}^{\beta}(A)$ and $\Sigma_{\alpha}^{\beta}(\mathscr{A})$. For the sake of completeness, we state these results here.

Theorem 2 ([28]). *Let $A = \{a_0, a_1, \ldots, a_{k-1}\}$ be a set of integers with $a_0 < a_1 < \cdots < a_{k-1}$, where k is a positive integer. Let $\bar{\mathbf{r}} = (r_0, r_1, \ldots, r_{k-1})$ be an ordered k-tuple of positive integers, and h be an integer satisfying $2 \leq h \leq \sum_{j=0}^{k-1} r_j$. Then*

$$|h^{(\bar{\mathbf{r}})}A| \geq L(\bar{\mathbf{r}}, h).$$

This lower bound is best possible.

Theorem 3 ([28]). *Let $k \geq 5$ be an integer. Let $\bar{\mathbf{r}} = (r_0, \ldots, r_{k-1})$ be an ordered k-tuple of positive integers, and let h be an integer satisfying*

$$2 \leq h \leq \sum_{j=0}^{k-1} r_j - 2.$$

If A is a set of k integers, then

$$|h^{(\bar{\mathbf{r}})}A| = L(\bar{\mathbf{r}}, h)$$

if and only if A is a k-term arithmetic progression.

Theorem 4 ([28]). *Let $A = \{a_0, a_1, a_2\}$ be a set of integers with $a_0 < a_1 < a_2$ and $\bar{\mathbf{r}} = (r_0, r_1, r_2)$ be an ordered 3-tuple of positive integers. Suppose that h is an integer with $2 \leq h \leq r_0 + r_1 + r_2 - 2$. Then*

(i) for $r_1 = 1$, we have $|h^{(\bar{\mathbf{r}})}A| = L(\bar{\mathbf{r}}, h)$;

(ii) for $r_1 \geq 2$, we have $|h^{(\bar{\mathbf{r}})}A| = L(\bar{\mathbf{r}}, h)$ if and only if A is a 3-term arithmetic progression.

Theorem 5 ([28]). *Let $A = \{a_0, a_1, a_2, a_3\}$ be a set of integers with $a_0 < a_1 < a_2 < a_3$ and $\bar{\mathbf{r}} = (r_0, r_1, r_2, r_3)$ be an ordered 4-tuple of positive integers. Suppose that h is an integer with $2 \leq h \leq r_0 + r_1 + r_2 + r_3 - 2$. Then*

(i) for $r_1 = r_2 = 1$, we have $|h^{(\bar{\mathbf{r}})}A| = L(\bar{\mathbf{r}}, h)$ if and only if $a_1 - a_0 = a_3 - a_2$;

(ii) for $r_1 \geq 2$ or $r_2 \geq 2$, we have $|h^{(\bar{\mathbf{r}})}A| = L(\bar{\mathbf{r}}, h)$ if and only if A is a 4-term arithmetic progression.

To prove some inverse theorems for the set of subsums $\Sigma_\alpha(A)$, Dwivedi and Mistri [13] expressed this subsums as a certain generalized h-fold sumset. We extend this idea to the set of subsums $\Sigma_\alpha^\beta(A)$, and also for the set of subsequence sums $\Sigma_\alpha^\beta(\mathscr{A})$ for a sequence $\mathscr{A}$ (see Section 4). The following lemmas which can be proved easily by simple set-theoretic arguments are crucial for the proof of direct and inverse theorems for $\Sigma_\alpha^\beta(A)$.

Lemma 1. *Let $A = \{a_1, \ldots, a_k\}$ be a nonempty finite subset of G with $0 \notin A$, where k is a positive integer. Let α and β be integers such that $0 \leq \alpha \leq k-1$, $0 \leq \beta \leq k-1$, and $\alpha + \beta \leq k-1$. Let $A_0 = \{a_0, a_1, \ldots, a_k\} \subseteq G$, where $a_0 = 0$, and let $\bar{\mathbf{r}} = (k - \alpha - \beta, \underbrace{1, \ldots, 1}_{k \text{ times}})$. Then*

$$\Sigma_\alpha^\beta(A) = (k-\beta)^{(\bar{\mathbf{r}})} A_0.$$

Lemma 2. *Let $A = \{a_0, a_1, \ldots, a_{k-1}\}$ be a nonempty finite subset of G with $a_0 = 0$, where k is a positive integer. Let α and β be integers such that $1 \leq \alpha \leq k$, $0 \leq \beta \leq k-1$, and $\alpha + \beta \leq k$. Let $\bar{\mathbf{r}} = (k - \alpha - \beta + 1, \underbrace{1, \ldots, 1}_{k-1 \text{ times}})$. Then*

$$\Sigma_\alpha^\beta(A) = (k-\beta)^{(\bar{\mathbf{r}})} A.$$

Let $\pi : [1, k] \to [1, k]$ be a permutation, where k is a positive integer. Following the notation in [13], for a set $A = \{a_1, a_2, \ldots, a_k\}$ and an ordered k-tuple $\bar{\mathbf{r}} = (r_1, r_2, \ldots, r_k)$ of positive integers, we write

$$A_\pi = \{a_{\pi(1)}, a_{\pi(2)}, \ldots, a_{\pi(k)}\}$$

and

$$\bar{\mathbf{r}}_\pi = (r_{\pi(1)}, r_{\pi(2)}, \ldots, r_{\pi(k)}).$$

Note that the order of the elements in the set A is assumed to be fixed in the definition of $h^{(\bar{\mathbf{r}})}A$. In the proofs, sometimes we will require to consider the elements of the set A in a different order. In that situation, we will need the following obvious lemma to apply the above results.

Lemma 3 ([13]). *Let $A = \{a_1, a_2, \ldots, a_k\}$ be an ordered nonempty finite subset of G, where k is a positive integer. Let $\bar{\mathbf{r}} = (r_1, r_2, \ldots, r_k)$ an ordered k-tuple of positive integers. Let $h \geq 2$ be an integer, and let π be a permutation of $[1, k]$. Then*

$$h^{(\bar{\mathbf{r}})}A = h^{(\bar{\mathbf{r}}_\pi)}A_\pi.$$

2. Direct Theorems for Subsums $\Sigma_\alpha^\beta(A)$

Theorem 6. *Let $k \geq 2$ be an integer. Let α and β be integers such that*

$$1 \leq \alpha \leq k-1, 0 \leq \beta \leq k-1, \text{ and } \alpha + \beta \leq k-1.$$

If A is a set of k positive integers, then

$$|\Sigma_\alpha^\beta(A)| \geq \frac{k(k+1)}{2} - \frac{\alpha(\alpha+1)}{2} - \frac{\beta(\beta+1)}{2} + 1. \tag{2.1}$$

If A is a set of k nonnegative integers and $0 \in A$, then

$$|\Sigma_\alpha^\beta(A)| \geq \frac{k(k-1)}{2} - \frac{\alpha(\alpha-1)}{2} - \frac{\beta(\beta-1)}{2} + 1. \tag{2.2}$$

The lower bounds in (2.1) and (2.2) are best possible.

We remark that Theorem 6 is a special case of a result of Bhanja [7, Theorem 6 and Corollary 7]. But the proof presented here is original and the idea of the proof enables us to prove some new direct theorems.

Proof of Theorem 6. First assume that A is a set of $k \geq 2$ positive integers. Let $A = \{a_1, \ldots, a_k\}$, and let $A_0 = \{a_0, a_1, \ldots, a_k\}$, where $a_0 = 0$. Let

$$\bar{\mathbf{r}} = (r_0, r_1, \ldots, r_k),$$

where $r_0 = k - \alpha - \beta$ and $r_1 = r_2 = \cdots = r_k = 1$. Then Lemma 1 implies that

$$\Sigma_\alpha^\beta(A) = (k-\beta)^{(\bar{\mathbf{r}})}A_0.$$

It is easy to see that $\mu = \alpha + 1$ and $\eta = \beta$. Therefore,

$$\delta = (k-\beta) - \sum_{j=0}^{\alpha} r_j = (k-\beta) - (k-\beta) = 0$$

and

$$\theta = (k-\beta) - \sum_{j=\beta+1}^{k} r_j = (k-\beta) - (k-\beta) = 0.$$

Hence

$$\begin{aligned}L(\bar{\mathbf{r}}, k-\beta) &= \left(\sum_{j=\beta+1}^{k} jr_j - \sum_{j=0}^{\alpha} jr_j\right) + 0 - 0 + 1\\ &= \left(\sum_{j=\beta+1}^{k} j - \sum_{j=1}^{\alpha} j\right) + 1\\ &= \frac{k(k+1)}{2} - \frac{\alpha(\alpha+1)}{2} - \frac{\beta(\beta+1)}{2} + 1.\end{aligned}$$

Therefore, it follows from Theorem 2 that

$$|\Sigma_{\alpha}^{\beta}(A)| = |(k-\beta)^{(\bar{\mathbf{r}})} A_0| \geq L(\bar{\mathbf{r}}, k-\beta) = \frac{k(k+1)}{2} - \frac{\alpha(\alpha+1)}{2} - \frac{\beta(\beta+1)}{2} + 1.$$

We can see that the lower bound in (2.1) is best possible by taking the set $A = [1, k]$, where $k \geq 2$. This proves the first part of the theorem.

Now assume that A is a set of $k \geq 2$ nonnegative integers with $0 \in A$. Let $A = \{a_0, a_1, \ldots, a_{k-1}\}$, where $0 = a_0 < a_1 < \cdots < a_{k-1}$. Let $\bar{\mathbf{r}} = (r_0, r_1, \ldots, r_{k-1})$, where $r_0 = k - \alpha - \beta + 1$ and $r_1 = r_2 = \cdots = r_{k-1} = 1$. It follows from Lemma 2 that

$$\Sigma_{\alpha}^{\beta}(A) = (k-\beta)^{(\bar{\mathbf{r}})} A.$$

It is easy to see that $\mu = \alpha$ and $\eta = \beta - 1$. Therefore,

$$\delta = (k-\beta) - \sum_{j=0}^{\alpha-1} r_j = 0 \text{ and } \theta = (k-\beta) - \sum_{j=\beta}^{k-1} r_j = 0.$$

Hence

$$\begin{aligned}L(\bar{\mathbf{r}}, k-\beta) &= \left(\sum_{j=\beta}^{k-1} jr_j - \sum_{j=0}^{\alpha-1} jr_j\right) + 0 - 0 + 1\\ &= \left(\sum_{j=\beta}^{k-1} j - \sum_{j=1}^{\alpha-1} j\right) + 1\\ &= \frac{k(k-1)}{2} - \frac{\alpha(\alpha-1)}{2} - \frac{\beta(\beta-1)}{2} + 1.\end{aligned}$$

Therefore, it follows from Theorem 2 that

$$|\Sigma_{\alpha}^{\beta}(A)| = |(k-\beta)^{(\bar{\mathbf{r}})} A| \geq L(\bar{\mathbf{r}}, k-\beta) = \frac{k(k-1)}{2} - \frac{\alpha(\alpha-1)}{2} - \frac{\beta(\beta-1)}{2} + 1.$$

We can see that the lower bound in (2.2) is best possible by taking the set $A = [0, k-1]$, where $k \geq 2$. This proves the second part of the theorem and completes the proof. □

Theorem 7. *Let A be a finite set containing p positive integers and n negative integers, where $1 \leq n \leq p$. Let $k = p + n$, and let α and β be integers such that $1 \leq \alpha \leq k-1$, $0 \leq \beta \leq k-1$, and $\alpha + \beta \leq k-1$. Then*

$$|\Sigma_{\alpha}^{\beta}(A)| \geq \mathcal{L}(\alpha, \beta, A), \tag{2.3}$$

where $\mathcal{L}(\alpha, \beta, A)$ is defined as follows.

1. *If $1 \leq \alpha < k - \beta < n \leq p$, then*

$$\mathcal{L}(\alpha, \beta, A) = \frac{p(p+1)}{2} + \frac{n(n+1)}{2} - \frac{(\beta - n)(\beta - n + 1)}{2} - \frac{(\beta - p)(\beta - p + 1)}{2} + 1.$$

2. *If either $1 \leq \alpha < n \leq k - \beta < p$ or $1 \leq \alpha = n < k - \beta \leq p$, then*

$$\mathcal{L}(\alpha, \beta, A) = \frac{p(p+1)}{2} + \frac{n(n+1)}{2} - \frac{(\beta - n)(\beta - n + 1)}{2} + 1.$$

3. *If either $1 \leq \alpha < n \leq p \leq k - \beta$ or $1 \leq \alpha = n < p < k - \beta$ or $1 \leq \alpha = n = p < k - \beta$, then*

$$\mathcal{L}(\alpha, \beta, A) = \frac{p(p+1)}{2} + \frac{n(n+1)}{2} + 1.$$

4. *If $1 \leq n < \alpha < k - \beta \leq p$, then*

$$\mathcal{L}(\alpha, \beta, A) = \frac{p(p+1)}{2} + \frac{n(n+1)}{2} - \frac{(\beta - n)(\beta - n + 1)}{2} - \frac{(\alpha - n)(\alpha - n + 1)}{2} + 1.$$

5. *If either $1 \leq n < \alpha < p < k - \beta$ or $1 \leq n < \alpha = p < k - \beta$, then*

$$\mathcal{L}(\alpha, \beta, A) = \frac{p(p+1)}{2} + \frac{n(n+1)}{2} - \frac{(\alpha - n)(\alpha - n + 1)}{2} + 1.$$

6. *If $1 \leq n \leq p < \alpha < k - \beta$, then*

$$\mathcal{L}(\alpha, \beta, A) = \frac{p(p+1)}{2} + \frac{n(n+1)}{2} - \frac{(\alpha - n)(\alpha - n + 1)}{2} - \frac{(\alpha - p)(\alpha - p + 1)}{2} + 1.$$

The lower bound in (2.3) is best possible.

Proof. Let

$$A = \{-b_n, -b_{n-1}, \ldots, -b_1, a_1, \ldots, a_p\}$$

and

$$A_0 = \{-b_n, -b_{n-1}, \ldots, -b_1, 0, a_1, \ldots, a_p\},$$

where $-b_n < -b_{n-1} < \ldots < -b_1 < 0 < a_1 < \ldots < a_p$. Let $k = |A| = p + n$, $k_0 = |A_0| = p + n + 1$, and

$$\bar{\mathbf{r}} = (r_0, r_1, \ldots, r_{n-1}, r_n, r_{n+1}, \ldots, r_{n+p}),$$

where $r_0 = r_1 = \cdots = r_{n-1} = r_{n+1} = \cdots = r_{n+p} = 1$ and $r_n = k - \alpha - \beta$. Then it follows from Lemma 1 and Lemma 3 that $\Sigma_\alpha^\beta(A) = (k-\beta)^{(\bar{\mathbf{r}})} A_0$. Therefore, it follows from Theorem 2 that

$$|\Sigma_\alpha^\beta(A)| = |(k-\beta)^{(\bar{\mathbf{r}})} A_0| \geq L(\bar{\mathbf{r}}, k-\beta).$$

Hence it suffices to prove that $L(\bar{\mathbf{r}}, k-\beta) = \mathcal{L}(\alpha, \beta, A)$.
`Case 1`: $1 \leq \alpha < k - \beta < n \leq p$. In this case, we have $1 \leq \alpha < n \leq p < \beta$. We can easily determine that $\mu = k - \beta$, $\eta = \beta$, and $\delta = \theta = 0$. Hence

$$\begin{aligned} L(\bar{\mathbf{r}}, k-\beta) &= \left(\sum_{j=\beta+1}^{p+n} j r_j - \sum_{j=0}^{k-\beta-1} j r_j \right) + 1 \\ &= \left(\sum_{j=\beta+1}^{k} j - \sum_{j=1}^{k-\beta-1} j \right) + 1 \\ &= \frac{p(p+1)}{2} + \frac{n(n+1)}{2} - \frac{(\beta-n)(\beta-n+1)}{2} - \frac{(\beta-p)(\beta-p+1)}{2} + 1. \end{aligned}$$

`Case 2(i)`: $1 \leq \alpha < n \leq k - \beta < p$. In this case, we have $1 \leq \alpha < n < \beta \leq p$. We can easily determine that $\mu = n$, $\eta = \beta$, $\delta = p - \beta$, and $\theta = 0$. Hence

$$\begin{aligned} L(\bar{\mathbf{r}}, k-\beta) &= \left(\sum_{j=\beta+1}^{k} j r_j - \sum_{j=0}^{n-1} j r_j \right) + 0 - n(p-\beta) + 1 \\ &= \left(\sum_{j=\beta+1}^{k} j - \sum_{j=1}^{n-1} j \right) - pn + \beta n + 1 \\ &= \frac{p(p+1)}{2} + \frac{n(n+1)}{2} - \frac{(\beta-n)(\beta-n+1)}{2} + 1. \end{aligned}$$

`Case 2(ii)`: $1 \leq \alpha = n < k - \beta \leq p$. In this case, we have $0 \leq \beta < p$ and $1 \leq n \leq \beta < p$. We can easily determine that $\mu = n + 1$, $\eta = \beta$, and $\delta = \theta = 0$. Hence

$$\begin{aligned} L(\bar{\mathbf{r}}, k-\beta) &= \left(\sum_{j=\beta+1}^{k} j r_j - \sum_{j=0}^{n} j r_j \right) + 1 \\ &= \left(\sum_{j=\beta+1}^{k} j - \sum_{j=1}^{n-1} j - n r_n \right) + 1 \\ &= \frac{p(p+1)}{2} + \frac{n(n+1)}{2} - \frac{(\beta-n)(\beta-n+1)}{2} + 1. \end{aligned}$$

`Case 3(i)`: $1 \le \alpha < n \le p \le k - \beta)$. In this case, we have $0 \le \beta \le n \le p$. We can easily determine that $\mu = \eta = n$, $\delta = p - \beta$, and $\theta = n - \beta$. Hence

$$\begin{aligned} L(\bar{\mathbf{r}}, k-\beta) &= \left(\sum_{j=n+1}^{p+n} jr_j - \sum_{j=0}^{n-1} jr_j \right) + n(n-\beta) - n(p-\beta) + 1 \\ &= \left(\sum_{j=n+1}^{p+n} j - \sum_{j=1}^{n-1} j \right) + n^2 - pn + 1 \\ &= \frac{p(p+1)}{2} + \frac{n(n+1)}{2} + 1. \end{aligned}$$

`Case 3(ii)`: $1 \le \alpha = n < p < k - \beta$. In this case, we have $0 \le \beta < n$ and $0 \le \beta < p$. We can easily determine that $\mu = n + 1$, $\eta = n$, $\delta = 0$, and $\theta = n - \beta$. Hence

$$\begin{aligned} L(\bar{\mathbf{r}}, k-\beta) &= \left(\sum_{j=n+1}^{k} jr_j - \sum_{j=0}^{n} jr_j \right) + n(n-\beta) - 0 + 1 \\ &= \left(\sum_{j=n+1}^{p+n} j - \sum_{j=1}^{n-1} j - nr_n \right) + n^2 - \beta n + 1 \\ &= \frac{p(p+1)}{2} + \frac{n(n+1)}{2} + 1. \end{aligned}$$

`Case 3(iii)`: $1 \le \alpha = n = p < k - \beta$. We can easily determine that $\mu = n + 1$, $\eta = n - 1$, and $\delta = \theta = 0$. Hence

$$\begin{aligned} L(\bar{\mathbf{r}}, k-\beta) &= \left(\sum_{j=n}^{k} jr_j - \sum_{j=0}^{n} jr_j \right) + 1 \\ &= nr_n + \sum_{j=n+1}^{p+n} j - \sum_{j=1}^{n-1} j - nr_n + 1 \\ &= \frac{p(p+1)}{2} + \frac{n(n+1)}{2} + 1. \end{aligned}$$

`Case 4`: $1 \le n < \alpha < k - \beta \le p$. In this case, we have $1 \le n \le \beta < p$. We can easily determine that $\mu = \alpha + 1$, $\eta = \beta$, and $\delta = \theta = 0$. Hence

$$\begin{aligned} L(\bar{\mathbf{r}}, k-\beta) &= \left(\sum_{j=\beta+1}^{k} jr_j - \sum_{j=0}^{\alpha} jr_j \right) + 1 \\ &= \left(\sum_{j=\beta+1}^{k} j - \sum_{j=1}^{n-1} j - nr_n - \sum_{j=n+1}^{\alpha} j \right) + 1 \\ &= \frac{p(p+1)}{2} + \frac{n(n+1)}{2} - \frac{(\beta-n)(\beta-n+1)}{2} - \frac{(\alpha-n)(\alpha-n+1)}{2} + 1. \end{aligned}$$

`Case 5(i)`: $1 \le n < \alpha < p < k-\beta$. In this case, we have $0 \le \beta < n$ and $0 \le \beta < p$. We can easily determine that $\mu = \alpha+1$, $\eta = n$, $\delta = 0$, and $\theta = n-\beta$. Hence

$$\begin{aligned} L(\bar{\mathbf{r}}, k-\beta) &= \left(\sum_{j=n+1}^{k} jr_j - \sum_{j=0}^{\alpha} jr_j\right) + n(n-\beta) + 1 \\ &= \left(\sum_{j=n+1}^{n+p} j - \sum_{j=1}^{n-1} j - nr_n - \sum_{j=n+1}^{\alpha} j\right) + n^2 - \beta n + 1 \\ &= \frac{p(p+1)}{2} + \frac{n(n+1)}{2} - \frac{(\alpha-n)(\alpha-n+1)}{2} + 1. \end{aligned}$$

`Case 5(ii)`: $1 \le n < \alpha = p < k-\beta$. In this case, we have $0 \le \beta < n$ and $0 \le \beta < p$. We can easily determine that $\mu = \alpha+1$, $\eta = n-1$, and $\delta = \theta = 0$. Hence

$$\begin{aligned} L(\bar{\mathbf{r}}, k-\beta) &= \left(\sum_{j=n}^{k} jr_j - \sum_{j=0}^{\alpha} jr_j\right) + 1 \\ &= nr_n + \sum_{j=n+1}^{p+n} j - \sum_{j=1}^{n-1} j - nr_n - \sum_{j=n+1}^{\alpha} j + 1 \\ &= \frac{p(p+1)}{2} + \frac{n(n+1)}{2} - \frac{(\alpha-n)(\alpha-n+1)}{2} + 1. \end{aligned}$$

`Case 6`: $1 \le n \le p < \alpha < k-\beta$. In this case, we have $0 \le \beta < n$ and $0 \le \beta < p$. We can easily determine that $\mu = \alpha+1$, $\eta = k-\alpha-1$, and $\delta = \theta = 0$. Hence

$$\begin{aligned} L(\bar{\mathbf{r}}, k-\beta) &= \left(\sum_{j=k-\alpha}^{k} jr_j - \sum_{j=0}^{\alpha} jr_j\right) + 1 \\ &= \sum_{j=k-\alpha}^{n-1} j + nr_n + \sum_{j=n+1}^{k} j - \sum_{j=1}^{n-1} j - nr_n - \sum_{j=n+1}^{\alpha} j + 1 \\ &= \frac{p(p+1)}{2} + \frac{n(n+1)}{2} - \frac{(\alpha-n)(\alpha-n+1)}{2} - \frac{(\alpha-p)(\alpha-p+1)}{2} + 1. \end{aligned}$$

Combining all the cases, we get $L(\bar{\mathbf{r}}, k-\beta) = \mathcal{L}(\alpha, \beta, A)$, which proves the inequality (2.3). We can see that the lower bound in (2.3) is best possible by taking the set $A = [-n, p] \setminus \{0\}$. This completes the proof. □

Theorem 8. *Let A be a finite set containing p positive integers, n negative integers and zero, where $1 \le n \le p$. Let $k = p+n+1$, and let α and β be integers such that $1 \le \alpha \le k-1$, $0 \le \beta \le k-1$ and $\alpha+\beta \le k-1$. Then*

$$|\Sigma_{\alpha}^{\beta}(A)| \ge \mathcal{L}_0(\alpha, \beta, A), \tag{2.4}$$

where $\mathcal{L}_0(\alpha, \beta, A)$ is defined as follows.

1. *If* $1 \leq \alpha < k - \beta < n \leq p$, *then*

$$\mathcal{L}_0(\alpha, \beta, A) = \frac{p(p+1)}{2} + \frac{n(n+1)}{2} - \frac{(\beta - n)(\beta - n - 1)}{2} - \frac{(\beta - p)(\beta - p - 1)}{2} + 1.$$

2. *If either* $1 \leq \alpha < n \leq k - \beta < p$ *or* $1 \leq \alpha = n < k - \beta \leq p$, *then*

$$\mathcal{L}_0(\alpha, \beta, A) = \frac{p(p+1)}{2} + \frac{n(n+1)}{2} - \frac{(\beta - n)(\beta - n - 1)}{2} + 1.$$

3. *If either* $1 \leq \alpha < n \leq p \leq k - \beta$ *or* $1 \leq \alpha = n < p < k - \beta$ *or* $1 \leq \alpha = n = p < k - \beta$, *then*

$$\mathcal{L}_0(\alpha, \beta, A) = \frac{p(p+1)}{2} + \frac{n(n+1)}{2} + 1.$$

4. *If* $1 \leq n < \alpha < k - \beta \leq p$, *then*

$$\mathcal{L}_0(\alpha, \beta, A) = \frac{p(p+1)}{2} + \frac{n(n+1)}{2} - \frac{(\beta - n)(\beta - n - 1)}{2} - \frac{(\alpha - n)(\alpha - n - 1)}{2} + 1.$$

5. *If either* $1 \leq n < \alpha < p < k - \beta$ *or* $1 \leq n < \alpha = p < k - \beta$, *then*

$$\mathcal{L}_0(\alpha, \beta, A) = \frac{p(p+1)}{2} + \frac{n(n+1)}{2} - \frac{(\alpha - n)(\alpha - n - 1)}{2} + 1.$$

6. *If* $1 \leq n \leq p < \alpha < k - \beta$, *then*

$$\mathcal{L}_0(\alpha, \beta, A) = \frac{p(p+1)}{2} + \frac{n(n+1)}{2} - \frac{(\alpha - n)(\alpha - n - 1)}{2} - \frac{(\alpha - p)(\alpha - p - 1)}{2} + 1.$$

The lower bound in (2.4) *is best possible.*

Proof. Let

$$A = \{-b_n, -b_{n-1}, \ldots, -b_1, 0, a_1, \ldots, a_p\},$$

where $-b_n < -b_{n-1} < \ldots < -b_1 < 0 < a_1 < \ldots < a_p$. Let $k = |A| = p + n + 1$ and

$$\bar{\mathbf{r}} = (r_0, r_1, \ldots, r_{n-1}, r_n, r_{n+1}, \ldots, r_{n+p}),$$

where $r_0 = r_1 = \cdots = r_{n-1} = r_{n+1} = \cdots = r_{n+p} = 1$ and $r_n = k - \alpha - \beta + 1$. Then it follows from Lemma 2 and Lemma 3 that $\Sigma_\alpha^\beta(A) = (k - \beta)^{(\bar{\mathbf{r}})} A$. Therefore, it follows from Theorem 2 that

$$|\Sigma_\alpha^\beta(A)| = |(k - \beta)^{(\bar{\mathbf{r}})} A| \geq L(\bar{\mathbf{r}}, k - \beta).$$

Hence it suffices to prove that $L(\bar{\mathbf{r}}, k - \beta) = \mathcal{L}_0(\alpha, \beta, A)$.

`Case 1`: $1 \le \alpha < k-\beta < n \le p$. In this case, we have $1 \le \alpha < n \le p < \beta$. We can easily determine that $\mu = k-\beta$, $\eta = \beta - 1$, and $\delta = \theta = 0$. Hence

$$\begin{aligned}
L(\bar{\mathbf{r}}, k-\beta) &= \left(\sum_{j=\beta}^{k-1} jr_j - \sum_{j=0}^{k-\beta-1} jr_j\right) + 1 \\
&= \left(\sum_{j=\beta}^{k-1} j - \sum_{j=1}^{k-\beta-1} j\right) + 1 \\
&= \frac{p(p+1)}{2} + \frac{n(n+1)}{2} - \frac{(\beta-n)(\beta-n-1)}{2} - \frac{(\beta-p)(\beta-p-1)}{2} + 1.
\end{aligned}$$

`Case 2(i)`: $1 \le \alpha < n \le k-\beta < p$. In this case, we have $1 \le \alpha < n < \beta \le p+1$. We can easily determine that $\mu = n$, $\eta = \beta - 1$, $\delta = p - \beta + 1$, and $\theta = 0$. Hence

$$\begin{aligned}
L(\bar{\mathbf{r}}, k-\beta) &= \left(\sum_{j=\beta}^{k-1} jr_j - \sum_{j=0}^{n-1} jr_j\right) + 0 - n(p-\beta+1) + 1 \\
&= \left(\sum_{j=\beta}^{k-1} j - \sum_{j=1}^{n-1} j\right) - pn + \beta n - n + 1 \\
&= \frac{p(p+1)}{2} + \frac{n(n+1)}{2} - \frac{(\beta-n)(\beta-n-1)}{2} + 1.
\end{aligned}$$

`Case 2(ii)`: $1 \le \alpha = n < k-\beta \le p$. In this case, we have $1 \le \alpha = n < n+1 \le \beta \le p$. Now the computation is the same as in Case 2. We can easily determine that $\mu = n$, $\eta = \beta - 1$, $\delta = p - \beta + 1$, and $\theta = 0$. Hence

$$\begin{aligned}
L(\bar{\mathbf{r}}, k-\beta) &= \left(\sum_{j=\beta}^{k-1} jr_j - \sum_{j=0}^{n-1} jr_j\right) + 0 - n(p-\beta+1) + 1 \\
&= \left(\sum_{j=\beta}^{k-1} j - \sum_{j=1}^{n-1} j\right) - pn + \beta n - n + 1 \\
&= \frac{p(p+1)}{2} + \frac{n(n+1)}{2} - \frac{(\beta-n)(\beta-n-1)}{2} + 1.
\end{aligned}$$

`Case 3(i)`: $1 \le \alpha < n \le p \le k-\beta$. In this case, we have $0 \le \beta \le n+1 \le p+1$. We can easily determine that $\mu = \eta = n$, $\delta = p - \beta + 1$, and $\theta = n - \beta + 1$. Hence

$$\begin{aligned}
L(\bar{\mathbf{r}}, k-\beta) &= \left(\sum_{j=n+1}^{p+n} jr_j - \sum_{j=0}^{n-1} jr_j\right) + n(n-\beta+1) - n(p-\beta+1) + 1 \\
&= \left(\sum_{j=n+1}^{p+n} j - \sum_{j=1}^{n-1} j\right) + n^2 - pn + 1 \\
&= \frac{p(p+1)}{2} + \frac{n(n+1)}{2} + 1.
\end{aligned}$$

`Case 3(ii)`: $1 \leq \alpha = n < p < k - \beta$. In this case, we have $0 \leq \beta \leq n < p$. We can easily determine that $\mu = n$, $\eta = n$, $\delta = p - \beta + 1$, and $\theta = n - \beta + 1$. Now all computations are the same as in Case 3. Hence

$$\begin{aligned} L(\bar{\mathbf{r}}, k - \beta) &= \left(\sum_{j=n+1}^{p+n} jr_j - \sum_{j=0}^{n-1} jr_j \right) + n(n - \beta + 1) - n(p - \beta + 1) + 1 \\ &= \left(\sum_{j=n+1}^{p+n} j - \sum_{j=1}^{n-1} j \right) + n^2 - pn + 1 \\ &= \frac{p(p+1)}{2} + \frac{n(n+1)}{2} + 1. \end{aligned}$$

`Case 3(iii)`: $1 \leq \alpha = n = p < k - \beta$. We can easily determine that $\mu = n$, $\eta = n$, $\delta = p - \beta + 1$, and $\theta = n - \beta + 1$. Now all computations are the same as in Case 5. Hence

$$\begin{aligned} L(\bar{\mathbf{r}}, k - \beta) &= \left(\sum_{j=n+1}^{p+n} jr_j - \sum_{j=0}^{n-1} jr_j \right) + n(n - \beta + 1) - n(p - \beta + 1) + 1 \\ &= \left(\sum_{j=n+1}^{p+n} j - \sum_{j=1}^{n-1} j \right) + n^2 - pn + 1 \\ &= \frac{p(p+1)}{2} + \frac{n(n+1)}{2} + 1. \end{aligned}$$

`Case 4`: $1 \leq n < \alpha < k - \beta \leq p$. In this case, we have $1 \leq n < n + 1 \leq \beta \leq p$. We can easily determine that $\mu = \alpha$, $\eta = \beta - 1$, and $\delta = \theta = 0$. Hence

$$\begin{aligned} L(\bar{\mathbf{r}}, k - \beta) &= \left(\sum_{j=\beta}^{k-1} jr_j - \sum_{j=0}^{\alpha-1} jr_j \right) + 1 \\ &= \frac{p(p+1)}{2} + \frac{n(n+1)}{2} - \frac{(\beta - n)(\beta - n - 1)}{2} - \frac{(\alpha - n)(\alpha - n - 1)}{2} + 1. \end{aligned}$$

`Case 5(i)`: $1 \leq n < \alpha < p < k - \beta$. In this case, we have $0 \leq \beta \leq n \leq p$. We can easily determine that $\mu = \alpha$, $\eta = n$, $\delta = 0$, and $\theta = n - \beta + 1$. Hence

$$\begin{aligned} L(\bar{\mathbf{r}}, k - \beta) &= \left(\sum_{j=n+1}^{k-1} jr_j - \sum_{j=0}^{\alpha-1} jr_j \right) + n(n - \beta + 1) + 1 \\ &= \frac{p(p+1)}{2} + \frac{n(n+1)}{2} - \frac{(\alpha - n)(\alpha - n - 1)}{2} + 1. \end{aligned}$$

`Case 5(ii)`: $1 \leq n < \alpha = p < k - \beta$. In this case, we have $0 \leq \beta \leq n < p$. We can easily determine that $\mu = \alpha$, $\eta = n$, $\delta = 0$, and $\theta = n - \beta + 1$. Now all computations are the same as in Case 8. Hence

$$\begin{aligned} L(\bar{\mathbf{r}}, k-\beta) &= \left(\sum_{j=n+1}^{k-1} jr_j - \sum_{j=0}^{\alpha-1} jr_j\right) + n(n-\beta+1) + 1 \\ &= \frac{p(p+1)}{2} + \frac{n(n+1)}{2} - \frac{(\alpha-n)(\alpha-n-1)}{2} + 1. \end{aligned}$$

Case 6: $1 \leq n \leq p < \alpha < k-\beta$. In this case, we have $0 \leq \beta \leq n \leq p < \alpha$. We can easily determine that $\mu = \alpha$, $\eta = k-\alpha-1$, and $\delta = \theta = 0$. Hence

$$\begin{aligned} L(\bar{\mathbf{r}}, k-\beta) &= \left(\sum_{j=k-\alpha}^{k-1} jr_j - \sum_{j=0}^{\alpha-1} jr_j\right) + 1 \\ &= \frac{p(p+1)}{2} + \frac{n(n+1)}{2} - \frac{(\alpha-n)(\alpha-n-1)}{2} - \frac{(\alpha-p)(\alpha-p-1)}{2} + 1. \end{aligned}$$

Combining all the cases, we get $L(\bar{\mathbf{r}}, k-\beta) = \mathcal{L}_0(\alpha, \beta, A)$, which proves the inequality (2.4). We can see that the lower bound in (2.4) is best possible by taking the set $A = [-n, p]$. This completes the proof. □

Remark 1. The lower bounds in Theorem 7 and Theorem 8 are obtained under the assumption that $n \leq p$. If $n > p$, then we can find the corresponding lower bound by replacing the set A by $-A$ and applying the above theorems.

3. Inverse Theorems for Subsums $\Sigma_{\alpha}^{\beta}(A)$

Theorem 9. *Let $k \geq 3$ be an integer. Let α and β be integers such that $1 \leq \alpha \leq k-2$, $0 \leq \beta \leq k-2$, and $\alpha + \beta \leq k-1$.*

If A is a set of k positive integers such that

$$|\Sigma_{\alpha}^{\beta}(A)| = \frac{k(k+1)}{2} - \frac{\alpha(\alpha+1)}{2} - \frac{\beta(\beta+1)}{2} + 1, \tag{3.1}$$

then

$$A = d * [1, k]$$

for some positive integer d except in the case $k = 3$ when we have

$$A = \{a_1, a_2, a_1 + a_2\},$$

where $0 < a_1 < a_2$.

If A is a set of k nonnegative integers such that $0 \in A$ and

$$|\Sigma_{\alpha}^{\beta}(A)| = \frac{k(k-1)}{2} - \frac{\alpha(\alpha-1)}{2} - \frac{\beta(\beta-1)}{2} + 1, \tag{3.2}$$

then

$$A = d * [0, k-1]$$

for some positive integer d except in the cases $k = 3$ and $k = 4$ when we have $A = \{0, a_1, a_2\}$ and $A = \{0, a_1, a_2, a_1 + a_2\}$, respectively, where $0 < a_1 < a_2$.

We remark that Theorem 9 is a special case of a result of Bhanja [7, Theorem 9 and Corollary 10]. But the following proof presented here is original and the idea of the proof enables us to prove some new inverse theorems. Moreover, Theorem 9 and Corollary 10 of Bhanja [7] are valid for $k \geq 6$ and $k \geq 7$, respectively. But Theorem 9 gives complete description for $k \geq 3$.

Proof of Theorem 9. First assume that the set A contains only positive integers. Write $A = \{a_1, \ldots, a_k\}$ and $A_0 = \{a_0, a_1, \ldots, a_k\}$, where $0 = a_0 < a_1 < \cdots < a_k$. Let $\bar{\mathbf{r}} = (r_0, r_1, \ldots, r_k)$, where $r_0 = k - \alpha - \beta$ and $r_1 = \cdots = r_k = 1$. Then it follows from Lemma 1 that $\Sigma_\alpha^\beta(A) = (k-\beta)^{(\bar{\mathbf{r}})} A_0$. Therefore,

$$|(k-\beta)^{(\bar{\mathbf{r}})} A_0| = |\Sigma_\alpha^\beta(A)| = \frac{k(k+1)}{2} - \frac{\alpha(\alpha+1)}{2} - \frac{\beta(\beta+1)}{2} + 1 = L(\bar{\mathbf{r}}, k-\beta).$$

Now if $k = 3$, then it follows from Theorem 5 that

$$|(k-\beta)^{(\bar{\mathbf{r}})} A_0| = L(\bar{\mathbf{r}}, k-\beta) = \frac{k(k+1)}{2} - \frac{\alpha(\alpha+1)}{2} - \frac{\beta(\beta+1)}{2} + 1$$

if and only if $a_1 - a_0 = a_3 - a_2$, which implies that $a_3 = a_1 + a_2$. Therefore, $A = \{a_1, a_2, a_1 + a_2\}$.

If $k \geq 4$, then it follows from Theorem 3 that

$$|(k-\beta)^{(\bar{\mathbf{r}})} A_0| = L(\bar{\mathbf{r}}, k) = \frac{k(k+1)}{2} - \frac{\alpha(\alpha+1)}{2} - \frac{\beta(\beta+1)}{2} + 1$$

if and only if A_0 is an arithmetic progression. Hence

$$a_1 - a_0 = a_2 - a_1 = \cdots = a_k - a_{k-1},$$

which implies that $a_i = i a_1$ for $i = 1, \ldots, k$. Therefore, $A = a_1 * [1, k]$.

Now assume that $0 \in A$ and write

$$A = \{a_0, a_1, \ldots, a_{k-1}\},$$

where $0 = a_0 < a_1 < \cdots < a_{k-1}$. Let $\bar{\mathbf{r}} = (r_0, r_1, \ldots, r_{k-1})$, where $r_0 = k - \alpha - \beta + 1$ and $r_1 = \cdots = r_{k-1} = 1$. Then it follows from Lemma 2 that

$$\Sigma_\alpha^\beta(A) = (k-\beta)^{(\bar{\mathbf{r}})} A.$$

Therefore,

$$|(k-\beta)^{(\bar{\mathbf{r}})} A| = |\Sigma_\alpha^\beta(A)| = \frac{k(k-1)}{2} - \frac{\alpha(\alpha-1)}{2} - \frac{\beta(\beta-1)}{2} + 1 = L(\bar{\mathbf{r}}, k-\beta).$$

Now if $k = 3$, then it follows from Theorem 4 that any set A with three elements satisfies

$$|(k-\beta)^{(\bar{\mathbf{r}})}A| = L(\bar{\mathbf{r}}, k-\beta) = \frac{k(k-1)}{2} - \frac{\alpha(\alpha-1)}{2} - \frac{\beta(\beta-1)}{2} + 1.$$

Since $0 \in A$, it follows that $A = \{0, a_1, a_2\}$.

Now if $k = 4$, then it follows from Theorem 5 that

$$|(k-\beta)^{(\bar{\mathbf{r}})}A| = L(\bar{\mathbf{r}}, k-\beta) = \frac{k(k-1)}{2} - \frac{\alpha(\alpha-1)}{2} - \frac{\beta(\beta-1)}{2} + 1$$

if and only if $a_1 - a_0 = a_3 - a_2$, which implies that $a_3 = a_1 + a_2$. Since $0 \in A$, it follows that $A = \{0, a_1, a_2, a_1 + a_2\}$.

If $k \geq 5$, then it follows from Theorem 3 that

$$|(k-\beta)^{(\bar{\mathbf{r}})}A| = L(\bar{\mathbf{r}}, k-\beta) = \frac{k(k-1)}{2} - \frac{\alpha(\alpha-1)}{2} - \frac{\beta(\beta-1)}{2} + 1$$

if and only if A is an arithmetic progression. Hence

$$a_1 - a_0 = a_2 - a_1 = \cdots = a_{k-1} - a_{k-2},$$

which implies that $a_i = ia_1$ for $i = 1, \ldots, k-1$. Hence $A = a_1 * [0, k-1]$. This completes the proof. □

Remark 2. Let A be a finite set of $k \geq 3$ positive integers, and let α and β be nonnegative integers. The following remarks show that the equality in (3.1) may hold even if A is not an arithmetic progression.

(i) If $\alpha = k-1$ and $\beta = 0$, then $|\Sigma_{\alpha}^{\beta}(A)| = k+1$. Thus the equality in (3.1) holds.

(ii) If $\alpha = k-1$ and $\beta = 1$, then $|\Sigma_{\alpha}^{\beta}(A)| = k$. Thus the equality in (3.1) holds.

(iii) If $\alpha = k$ and $\beta = 0$, then $|\Sigma_{\alpha}^{\beta}(A)| = 1$. Thus the equality in (3.1) holds.

(iv) For the remaining values of α and β, one can draw the the conclusion using Facts 1.

Remark 3. Let A be a finite set of $k \geq 3$ nonnegative integers with $0 \in A$, and let α and β be nonnegative integers. The following remarks show that the equality in (3.2) may hold even if A is not an arithmetic progression.

(i) If $\alpha = k-1$ and $\beta = 0$, then $|\Sigma_{\alpha}^{\beta}(A)| = k$. Thus the equality in (3.2) holds.

(ii) If $\alpha = k-1$ and $\beta = 1$, then $|\Sigma_{\alpha}^{\beta}(A)| = k$. Thus the equality in (3.2) holds.

(iii) If $\alpha = k$ and $\beta = 0$, then $|\Sigma_{\alpha}^{\beta}(A)| = 1$. Thus the equality in (3.2) holds.

(iv) For the remaining values of α and β, one can draw the conclusion using Facts 1.

Theorem 10. *Let A be a finite set containing p positive integers and n negative integers, where $1 \le n \le p$. Let α and β be integers such that $1 \le \alpha \le k-2$, $0 \le \beta \le k-2$, and $\alpha+\beta \le k-1$, where $k = p+n$. Let $\mathcal{L}(\alpha,\beta,A)$ be defined as in Theorem 7. Then the following conclusions hold.*

(i) If $k = 3, \alpha = 1$, and $\beta = 0$, then $|\Sigma_\alpha^\beta(A)| = \mathcal{L}(\alpha,\beta,A)$ if and only if $A = d * \{-1,1,2\}$, where d is the smallest positive element of A.

(ii) If $k = 3, \alpha = 1$, and $\beta = 1$, then $|\Sigma_\alpha^\beta(A)| = \mathcal{L}(\alpha,\beta,A)$ if and only if $A = \{a_0, a_0+a_3, a_3\}$ with $a_0 < 0 < a_0+a_3 < a_3$.

(iii) If $k \ge 4$, then $|\Sigma_\alpha^\beta(A)| = \mathcal{L}(\alpha,\beta,A)$ if and only if $A = d * \{-n, -(n-1), \ldots, -1, 1, \ldots, p\}$, where d is the smallest positive element of A.

Proof. Let
$$A = \{a_0, a_1, \ldots, a_{n-1}, a_{n+1}, \ldots, a_{n+p}\}$$
and
$$A_0 = \{a_0, a_1, \ldots, a_{n-1}, a_n, a_{n+1}, \ldots, a_{n+p}\},$$
where $a_0 < a_1 < \cdots < a_{n-1} < 0 = a_n < a_{n+1} < \cdots < a_{n+p}$. Let
$$k = |A| = p+n,\ k_0 = |A_0| = p+n+1$$
and
$$\bar{\mathbf{r}} = (r_0, r_1, \ldots, r_{n-1}, r_n, r_{n+1}, \ldots, r_{n+p}),$$
where $r_0 = r_1 = \cdots = r_{n-1} = r_{n+1} = \cdots = r_{n+p} = 1$ and $r_n = k-\alpha-\beta$. Then it follows from Lemma 1 and Lemma 3 that
$$\Sigma_\alpha^\beta(A) = (k-\beta)^{(\bar{\mathbf{r}})}A_0.$$
Therefore,
$$|\Sigma_\alpha^\beta(A)| = |(k-\beta)^{(\bar{\mathbf{r}})}A_0|.$$
We can verify that $\mathcal{L}(\alpha,\beta,A) = L(\bar{\mathbf{r}}, k-\beta)$.

If $k = 3$, then clearly we have $n = 1$ and $p = 2$. Hence $A = \{a_0, a_2, a_3\}$ and $A_0 = \{a_0, a_1, a_2, a_3\}$ with $a_0 < 0 = a_1 < a_2 < a_3$, and
$$\bar{\mathbf{r}} = (r_0, r_1, r_2, r_3) = (1, 3-\alpha-\beta, 1, 1).$$
If $\alpha = 1$ and $\beta = 0$, then $r_1 = k-\alpha-\beta = 2$, and so it follows from Theorem 5 that
$$|(k-\beta)^{(\bar{\mathbf{r}})}A_0| = |\Sigma_\alpha^\beta(A)| = \mathcal{L}(\alpha,\beta,A) = L(\bar{\mathbf{r}}, k-\beta)$$

if and only if A_0 is an arithmetic progression. Hence

$$a_1 - a_0 = a_2 - a_1 = a_3 - a_2,$$

which implies that $a_0 = -a_2$ and $a_3 = 2a_2$, and so $A_0 = \{-a_2, 0, a_2, 2a_2\}$. Therefore, $A = \{-a_2, a_2, 2a_2\} = a_2 * \{-1, 1, 2\}$. Next, if $\alpha = 1$ and $\beta = 1$, then $r_1 = k - \alpha - \beta = 1$ and $r_2 = 1$, and so it follows from Theorem 5 that

$$|(k-\beta)^{(\bar{\mathbf{r}})} A_0| = |\Sigma_\alpha^\beta(A)| = \mathcal{L}(\alpha, \beta, A) = L(\bar{\mathbf{r}}, k - \beta)$$

if and only if $a_1 - a_0 = a_3 - a_2$, which implies that $a_2 = a_3 + a_0$. Therefore, $A = \{a_0, a_0 + a_3, a_3\}$, where $a_0 < 0 < a_0 + a_3 < a_3$.

Now, if $k \geq 4$, then, it follows from Theorem 3 that

$$|(k-\beta)^{(\bar{\mathbf{r}})} A_0| = |\Sigma_\alpha^\beta(A)| = \mathcal{L}(\alpha, \beta, A) = L(\bar{\mathbf{r}}, k)$$

if and only if A_0 is an arithmetic progression. Hence

$$\begin{aligned} a_1 - a_0 = a_2 - a_1 = \cdots = a_{n-1} - a_{n-2} = a_n - a_{n-1} \\ = a_{n+1} - a_n = a_{n+2} - a_{n+1} = \cdots = a_{n+p} - a_{n+p-1}, \end{aligned}$$

which implies that

$$a_{n-j} = -j a_{n+1} \text{ for } j = 1, \ldots, n$$

and

$$a_{n+j} = j a_{n+1} \text{ for } j = 2, \ldots, p.$$

Hence $A_0 = a_{n+1} * [-n, p]$. Therefore,

$$A = a_{n+1} * \{-n, -(n-1), \ldots, -1, 1, 2, \ldots, p\}.$$

This completes the proof. □

Remark 4. Let A be a set of $k \geq 3$ nonzero integers containing at least one positive integer and at least one negative integer. Let α and β be nonnegative integers.

(i) If $\alpha = k - 1$ and $\beta = 0$, then $|\Sigma_\alpha^\beta(A)| = k + 1$.

(ii) If $\alpha = k - 1$ and $\beta = 1$, then $|\Sigma_\alpha^\beta(A)| = k$.

(iii) If $\alpha = k$ and $\beta = 0$, then $|\Sigma_\alpha^\beta(A)| = 1$.

(iv) For the remaining values of α and β, one can draw the conclusion using Facts 1.

Theorem 11. *Let A be a finite set containing p positive integers, n negative integers, and zero, where $1 \leq n \leq p$. Let α and β be integers such that $1 \leq \alpha \leq k-2$, $0 \leq \beta \leq k-2$, and $\alpha + \beta \leq k-1$, where $k = p+n+1$. Let $\mathcal{L}_0(\alpha, \beta, A)$ be defined as in Theorem 8. Then*

$$|\Sigma_{\alpha}^{\beta}(A)| = \mathcal{L}_0(\alpha, \beta, A)$$

if and only if $A = d * [-n, p]$, where d is the smallest positive element of the set A.

Proof. Let

$$A = \{a_0, a_1, \ldots, a_{n-1}, a_n, a_{n+1}, \ldots, a_{n+p}\},$$

where $a_0 < a_1 < \cdots < a_n = 0 < a_{n+1} < \cdots < a_{n+p}$. Then $k = |A| = p+n+1$. Let

$$\bar{\mathbf{r}} = (r_0, r_1, \ldots, r_{n-1}, r_n, r_{n+1}, \ldots, r_{n+p}),$$

where $r_0 = r_1 = \cdots = r_{n-1} = r_{n+1} = \cdots = r_{n+p} = 1$ and $r_n = k - \alpha - \beta + 1$. It follows from Lemma 2 and Lemma 3 that

$$\Sigma_{\alpha}^{\beta}(A) = (k-\beta)^{(\bar{\mathbf{r}})}A.$$

We can verify that $\mathcal{L}_0(\alpha, \beta, A) = L(\bar{\mathbf{r}}, k)$. If $k = 3$, then clearly, $p = n = 1$. Hence

$$A = \{a_0, a_1, a_2\}$$

with $a_0 < 0 = a_1 < a_2$ and $\bar{\mathbf{r}} = (r_0, r_1, r_2) = (1, k-\alpha-\beta+1, 1)$. Since

$$r_1 = k - \alpha - \beta + 1 \geq 2,$$

it follows from Theorem 4 that

$$|(k-\beta)^{(\bar{\mathbf{r}})}A| = |\Sigma_{\alpha}^{\beta}(A)| = \mathcal{L}_0(\alpha, \beta, A) = L(\bar{\mathbf{r}}, k-\beta)$$

if and only if A is an arithmetic progression. Hence

$$a_1 - a_0 = a_2 - a_1,$$

which implies that $a_0 = -a_2$, and so $A = \{-a_2, 0, a_2\} = a_2 * [-1, 1]$.

If $k = 4$, then clearly we have $n = 1$ and $p = 2$. Hence $A = \{a_0, a_1, a_2, a_3\}$ with $a_0 < 0 = a_1 < a_2 < a_3$ and $\bar{\mathbf{r}} = (r_0, r_1, r_2, r_3) = (1, k-\alpha-\beta+1, 1, 1)$. Since $r_1 = k - \alpha - \beta + 1 \geq 2$, it follows from Theorem 5 that

$$|(k-\beta)^{(\bar{\mathbf{r}})}A| = |\Sigma_{\alpha}^{\beta}(A)| = \mathcal{L}_0(\alpha, \beta, A) = L(\bar{\mathbf{r}}, k-\beta)$$

if and only if A is an arithmetic progression. Hence

$$a_1 - a_0 = a_2 - a_1 = a_3 - a_2,$$

which implies that $a_0 = -a_2$ and $a_3 = 2a_2$. Therefore,

$$A = \{-a_2, 0, a_2, 2a_2\} = a_2 * [-1, 2].$$

If $k \geq 5$, then it follows from Theorem 3 that

$$|(k-\beta)^{(\bar{\mathbf{r}})}A| = |\Sigma_{\alpha}^{\beta}(A)| = \mathcal{L}_0(\alpha, \beta, A) = L(\bar{\mathbf{r}}, k-\beta)$$

if and only if A is an arithmetic progression. Hence

$$\begin{aligned} a_1 - a_0 = a_2 - a_1 = \cdots = a_{n-1} - a_{n-2} = a_n - a_{n-1} \\ = a_{n+1} - a_n = a_{n+2} - a_{n+1} = \cdots = a_{n+p} - a_{n+p-1}, \end{aligned}$$

which implies that

$$a_{n-j} = -ja_{n+1} \text{ for } j = 1, \ldots, n$$

and

$$a_{n+j} = ja_{n+1} \text{ for } j = 2, \ldots, p.$$

Hence $A = a_{n+1} * [-n, p]$. Thus in all cases, we have $|\Sigma_{\alpha}^{\beta}(A)| = \mathcal{L}_0(\alpha, \beta, A)$ if and only if $A = a_{n+1} * [-n, p]$. This completes the proof. □

Remark 5. Let A be a set of $k \geq 3$ integers containing zero, at least one positive integer, and at least one negative integer. Let α and β be nonnegative integers.

(i) If $\alpha = k-1$ and $\beta = 0$, then $|\Sigma_{\alpha}^{\beta}(A)| = k$.

(ii) If $\alpha = k-1$ and $\beta = 1$, then $|\Sigma_{\alpha}^{\beta}(A)| = k$.

(iii) If $\alpha = k$ and $\beta = 0$, then $|\Sigma_{\alpha}^{\beta}(A)| = 1$.

(iv) For the remaining values of α and β, one can draw the conclusion using Facts 1.

Remark 6. In Theorem 10 and Theorem 11, we have assumed that $n \leq p$. If $n > p$, then we can replace the set A by $-A$ and apply the above theorems to establish the corresponding inverse theorems.

4. Subsequence Sums

For convenience, we will use braces around the elements of a sequence whenever it is clear from the context that we are referring to a sequence (as opposed to a set). A finite sequence $\mathscr{A} = \{\underbrace{a_0, \ldots, a_0}_{t_0 \text{ times}}, \underbrace{a_1, \ldots, a_1}_{t_1 \text{ times}}, \ldots, \underbrace{a_{k-1}, \ldots, a_{k-1}}_{t_{k-1} \text{ times}}\}$ in G will be denoted by $(A, \bar{\mathbf{t}})$, where $A = \{a_0, a_1, \ldots, a_{k-1}\}$ is the set of distinct terms of the sequence $\mathscr{A}$ and $\bar{\mathbf{t}} = (t_0, t_1, \ldots, t_{k-1})$ is the k-tuple of repetitions of each element of the set A written in the order of the appearance of the elements in the set A. If $\mathscr{B}$ is a subsequence of $\mathscr{A}$, then we write $\mathscr{B} \subseteq \mathscr{A}$. The length of a sequence $\mathscr{A}$ is denoted by $|\mathscr{A}|$. Let α and β be nonnegative integers with $\alpha + \beta \leq |\mathscr{A}|$. Like subset sums, we define

$$\Sigma_{\alpha}^{\beta}(\mathscr{A}) = \{\sigma(\mathscr{B}) : \mathscr{B} \subseteq \mathscr{A} \text{ and } \alpha \leq |\mathscr{B}| \leq |\mathscr{A}| - \beta\},$$

where $\sigma(\mathscr{B})$ denotes the sum of all the terms of the subsequence $\mathscr{B} \subseteq \mathscr{A}$. The usual sets of subsequence sums $\Sigma(\mathscr{A})$ and $\Sigma_0(\mathscr{A})$ are special cases of $\Sigma_{\alpha}^{\beta}(\mathscr{A})$ for $(\alpha, \beta) = (1, 0)$ and $(\alpha, \beta) = (0, 0)$, respectively. If $\beta = 0$, then $\Sigma_{\alpha}^{0}(\mathscr{A})$ is simply denoted by $\Sigma_{\alpha}(\mathscr{A})$.

Bhanja and Pandey [5] proved some direct and inverse theorems for $\Sigma_{\alpha}(\mathscr{A})$ for arbitrary α in case $\mathscr{A}$ is a finite sequence of nonnegative integers including or excluding zero. The case $\alpha = 1$ has been studied by Mistri and Pandey [19], by Mistri, Pandey and Prakash [20], and by Jiang and Li [17]. In this section, we prove direct and inverse theorems for the subsequence sums $\Sigma_{\alpha}^{\beta}(\mathscr{A})$ in $\mathbb{Z}$ for an arbitrary finite sequence of integers (see Theorem 13, Theorem 14, Theorem 15, Theorem 16, Theorem 17, Theorem 18, Theorem 19 and Theorem 20). In case of $\beta = 0$, these results generalize and solve two problems of Bhanja and Pandey [6, Open Problems (1) and (2), Section 4] also.

Facts 12. The following facts allow us to consider the sumset $\Sigma_{\alpha}^{\beta}(\mathscr{A})$ only for the pairs (α, β) satisfying $1 \leq \alpha \leq |\mathscr{A}| - 1$ and $0 \leq \beta \leq |\mathscr{A}| - 1$.

(i) It is easy to see that $\Sigma_{\alpha}^{\beta}(\mathscr{A}) = \alpha^{(\bar{\mathbf{t}})} A$ if $\alpha + \beta = |\mathscr{A}|$. Since the direct and inverse theorems are well known for the restricted h-fold sumset in $\mathbb{Z}$ [23], we always assume that $\alpha + \beta \leq |\mathscr{A}| - 1$, and so $0 \leq \alpha \leq |\mathscr{A}| - 1$ and $0 \leq \beta \leq |\mathscr{A}| - 1$.

(ii) It is easy to verify that $\Sigma_{\alpha}^{\beta}(\mathscr{A}) = \sigma(\mathscr{A}) - \Sigma_{\beta}^{\alpha}(\mathscr{A})$, and thus

$$|\Sigma_{\alpha}^{\beta}(\mathscr{A})| = |\Sigma_{\beta}^{\alpha}(\mathscr{A})|.$$

(iii) Furthermore, $\Sigma_{0}^{\beta}(\mathscr{A}) = \Sigma_{1}^{\beta}(\mathscr{A})$ if $0 \in \Sigma_{1}^{\beta}(\mathscr{A})$, and $\Sigma_{0}^{\beta}(\mathscr{A}) = \Sigma_{1}^{\beta}(\mathscr{A}) \cup \{0\}$ if $0 \notin \Sigma_{1}^{\beta}(\mathscr{A})$. Therefore, we consider only positive values of α.

A simple set-theoretic argument yields the following lemmas.

Lemma 4. *Let $\mathscr{A} = (A, \bar{\mathbf{t}})$ be a finite sequence in G, where $A = \{a_1, \ldots, a_k\}$ is a nonempty finite subset of G with $0 \notin A$ and $\bar{\mathbf{t}} = (t_1, \ldots, t_k)$ is a k-tuple of positive integers. Let $h = t_1 + \cdots + t_k$. Let α and β be integers such that $1 \leq \alpha \leq h - 1$, $0 \leq \beta \leq h - 1$, and $\alpha + \beta \leq h - 1$. Let $A_0 = \{a_0, a_1, \ldots, a_k\}$ with $a_0 = 0$, and let $\bar{\mathbf{r}} = (h - \alpha - \beta, t_1, \ldots, t_k)$. Then*

$$\Sigma_{\alpha}^{\beta}(\mathscr{A}) = (h - \beta)^{(\bar{\mathbf{r}})} A_0.$$

Lemma 5. *Let $\mathscr{A} = (A, \bar{\mathbf{t}})$ be a finite sequence in G, where $A = \{a_0, a_1, \ldots, a_{k-1}\}$ is a nonempty finite subset of G with $a_0 = 0$ and $\bar{\mathbf{t}} = (t_0, t_1, \ldots, t_{k-1})$ is a k-tuple of positive integers. Let $h = t_0 + \cdots + t_{k-1}$. Let α and β be integers such that $1 \leq \alpha \leq h$, $0 \leq \beta \leq h-1$, and $\alpha + \beta \leq h$. Let $\bar{\mathbf{r}} = (h - \alpha - \beta + t_0, t_1, \ldots, t_{k-1})$. Then*

$$\Sigma_{\alpha}^{\beta}(\mathscr{A}) = (h-\beta)^{(\bar{\mathbf{r}})} A.$$

We prove the following direct theorems which give the optimal lower bound for the cardinality of $\Sigma_{\alpha}^{\beta}(\mathscr{A})$ in case of an arbitrary finite sequence $\mathscr{A}$ of integers containing positive integers, negative integers and (or) zero. In case of $\beta = 0$, Theorem 15 and Theorem 16 solve a problem of Bhanja and Pandey [6, Open problems (1), Section 4].

Theorem 13. *Let $k \geq 2$ be an integer. Let $\mathscr{A} = (A, \bar{\mathbf{t}})$ be a nonempty finite sequence of integers, where $A = \{a_1, \ldots, a_k\}$ with $0 < a_1 < \cdots < a_k$ and $\bar{\mathbf{t}} = (t_1, \ldots, t_k)$. Let $h = t_1 + \cdots + t_k$. Let α and β be integers such that $1 \leq \alpha \leq h-1$, $0 \leq \beta \leq h-1$, and $\alpha + \beta \leq h-1$. Let $\bar{\mathbf{r}} = (h - \alpha - \beta, t_1, \ldots, t_k)$. Then*

$$|\Sigma_{\alpha}^{\beta}(\mathscr{A})| \geq L(\bar{\mathbf{r}}, h-\beta). \tag{4.1}$$

The lower bound in (4.1) is best possible.

Proof. Let $A_0 = \{a_0, a_1, \ldots, a_k\}$ with $a_0 = 0$. Then it follows from Lemma 4 that

$$\Sigma_{\alpha}^{\beta}(\mathscr{A}) = (h-\beta)^{(\bar{\mathbf{r}})} A_0.$$

Now the lower bound in (4.1) easily follows from Theorem 2. We can see that the lower bound in (4.1) is best possible by taking the sequence $\mathscr{A} = (A, \bar{\mathbf{t}})$, where $A = [1, k]$ with $k \geq 2$. □

This theorem easily implies a theorem of Bhanja and Pandey [5, Theorem 3.1].

Theorem 14. *Let $k \geq 2$ be an integer. Let $\mathscr{A} = (A, \bar{\mathbf{t}})$ be a nonempty finite sequence of integers, where $A = \{a_0, a_1, \ldots, a_{k-1}\}$ with $0 = a_0 < a_1 < \cdots < a_{k-1}$ and $\bar{\mathbf{t}} = (t_0, t_1, \ldots, t_{k-1})$. Let $h = t_0 + \cdots + t_{k-1}$. Let α and β be integers such that $1 \leq \alpha \leq h$, $0 \leq \beta \leq h-1$, and $\alpha + \beta \leq h$. Let $\bar{\mathbf{r}} = (h - \alpha - \beta + t_0, t_1, \ldots, t_{k-1})$. Then*

$$|\Sigma_{\alpha}^{\beta}(\mathscr{A})| \geq L(\bar{\mathbf{r}}, h-\beta). \tag{4.2}$$

The lower bound in (4.2) is best possible.

Proof. It follows from Lemma 5 that

$$\Sigma_{\alpha}^{\beta}(\mathscr{A}) = (h-\beta)^{(\bar{\mathbf{r}})} A.$$

Now the lower bound in (4.2) easily follows from Theorem 2. We can see that the lower bound in (4.2) is best possible by taking the sequence $\mathscr{A} = (A, \bar{\mathbf{t}})$ of length at least 3, where $A = [0, k-1]$ with $k \geq 2$. □

This theorem easily implies a theorem of Bhanja and Pandey [5, Corollary 3.1].

Theorem 15. *Let n and p be positive integers such that $n \leq p$. Let $\mathscr{A} = (A, \bar{\mathbf{t}})$ be a finite sequence of integers, where*

$$A = \{a_0, a_1, \ldots, a_{n-1}, a_{n+1}, \ldots, a_{n+p}\}$$

with

$$a_0 < a_1 < \cdots < a_{n-1} < 0 < a_{n+1} < \cdots < a_{n+p}$$

and

$$\bar{\mathbf{t}} = (t_0, t_1, \ldots, t_{n-1}, t_{n+1}, \ldots, t_{n+p}).$$

Let $h = t_0 + \cdots + t_{n-1} + t_{n+1} + \cdots + t_{n+p}$. Let α and β be integers such that $1 \leq \alpha \leq h-1$, $0 \leq \beta \leq h-1$, and $\alpha + \beta \leq h-1$. Then

$$|\Sigma_{\alpha}^{\beta}(\mathscr{A})| \geq L(\bar{\mathbf{r}}, h-\beta), \tag{4.3}$$

where $\bar{\mathbf{r}} = (t_0, \ldots, t_{n-1}, h-\alpha-\beta, t_{n+1}, \ldots, t_{n+p})$. The lower bound in (4.3) is best possible.

Proof. Let $A_0 = \{a_0, a_1, \ldots, a_{n-1}, a_n, a_{n+1}, \ldots, a_{n+p}\}$ with $a_n = 0$. Then it follows from Theorem 4 and Lemma 3 that

$$\Sigma_{\alpha}^{\beta}(\mathscr{A}) = (h-\beta)^{(\bar{\mathbf{r}})} A_0,$$

and so

$$|\Sigma_{\alpha}^{\beta}(\mathscr{A})| = |(h-\beta)^{(\bar{\mathbf{r}})} A_0| \geq L(\bar{\mathbf{r}}, h-\beta).$$

We can see that the lower bound in (4.3) is best possible by taking the sequence $\mathscr{A} = (A, \bar{\mathbf{t}})$, where $A = [-n, p] \setminus \{0\}$ and $\bar{\mathbf{t}} = (t_0, t_1, \ldots, t_{n-1}, t_{n+1}, \ldots, t_{n+p})$. This completes the proof. □

Theorem 16. *Let n and p be positive integers with $n \leq p$. Let $\mathscr{A} = (A, \bar{\mathbf{t}})$ be a nonempty finite sequence of integers, where*

$$A = \{a_0, a_1, \ldots, a_{n-1}, a_n, a_{n+1}, \ldots, a_{n+p}\}$$

with

$$a_0 < a_1 < \cdots < a_{n-1} < 0 = a_n < a_{n+1} < \cdots < a_{n+p}$$

and

$$\bar{\mathbf{t}} = (t_0, t_1, \ldots, t_{n-1}, t_n, t_{n+1}, \ldots, t_{n+p}).$$

Let $h = t_0 + \cdots + t_{n+p}$. Let α and β be integers such that $1 \leq \alpha \leq h$, $0 \leq \beta \leq h-1$, and $\alpha + \beta \leq h-1$. Then

$$|\Sigma_{\alpha}^{\beta}(\mathscr{A})| \geq L(\bar{\mathbf{r}}, h-\beta), \tag{4.4}$$

where $\bar{\mathbf{r}} = (t_0, \ldots, t_{n-1}, h-\alpha-\beta+t_n, t_{n+1}, \ldots, t_{n+p})$. The lower bound in (4.4) is best possible.

Proof. It follows from Theorem 5 and Lemma 3 that

$$\Sigma_{\alpha}^{\beta}(\mathscr{A}) = (h-\beta)^{(\bar{\mathbf{r}})}A,$$

and so

$$|\Sigma_{\alpha}^{\beta}(\mathscr{A})| = |(h-\beta)^{(\bar{\mathbf{r}})}A| \geq L(\bar{\mathbf{r}}, h-\beta).$$

We can see that the lower bound in (4.4) is best possible by taking the sequence $\mathscr{A} = (A, \bar{\mathbf{t}})$, where $A = [-n, p]$ and $\bar{\mathbf{t}} = (t_0, t_1, \ldots, t_{n-1}, t_n, t_{n+1}, \ldots, t_{n+p})$. This completes the proof. □

The following inverse theorems for the subsequence sums describe the structure of the arbitrary finite sequences $\mathscr{A}$ of integers for which $|\Sigma_{\alpha}(\mathscr{A})|$ achieves the optimal lower bound. In case of $\beta = 0$, Theorem 19 and Theorem 20 solve another problem of Bhanja and Pandey [6, Open problems (2), Section 4].

Theorem 17. *Let $k \geq 2$ be an integer. Let $\mathscr{A} = (A, \bar{\mathbf{t}})$ be a finite sequence of integers, where $A = \{a_1, \ldots, a_k\}$ with $0 < a_1 < \cdots < a_k$ and $\bar{\mathbf{t}} = (t_1, \ldots, t_k)$. Let $h = t_1 + \cdots + t_k$. Let α and β be integers such that $1 \leq \alpha \leq h-2$, $0 \leq \beta \leq h-2$, and $\alpha + \beta \leq h-1$. Let $\bar{\mathbf{r}} = (h-\alpha-\beta, t_1, \ldots, t_k)$. Then the following conclusions hold.*

(a) If $k = 2$ and $t_1 = 1$, then $|\Sigma_{\alpha}^{\beta}(\mathscr{A})| = L(\bar{\mathbf{r}}, h-\beta)$. If $k = 2$ and $t_1 \geq 2$, then

$$|\Sigma_{\alpha}^{\beta}(\mathscr{A})| = L(\bar{\mathbf{r}}, h-\beta)$$

if and only if $\mathscr{A} = (A, \bar{\mathbf{t}})$, where $A = a_1 * [1, 2]$ and $\bar{\mathbf{t}} = (t_1, t_2)$.

(b) If $k = 3$ and $t_1 = t_2 = 1$, then

$$|\Sigma_{\alpha}^{\beta}(\mathscr{A})| = L(\bar{\mathbf{r}}, h-\beta)$$

if and only if $\mathscr{A} = (A, \bar{\mathbf{t}})$, where $A = \{a_1, a_2, a_1 + a_2\}$ with $0 < a_1 < a_2$ and $\bar{\mathbf{t}} = (1, 1, t_3)$. If $k = 3$ and either $t_1 \geq 2$ or $t_2 \geq 2$, then

$$|\Sigma_{\alpha}^{\beta}(\mathscr{A})| = L(\bar{\mathbf{r}}, h-\beta)$$

if and only if $\mathscr{A} = (A, \bar{\mathbf{t}})$, where $A = a_1 * [1, 3]$ and $\bar{\mathbf{t}} = (t_1, t_2, t_3)$.

(c) If $k \geq 4$, then $|\Sigma_{\alpha}^{\beta}(\mathscr{A})| = L(\bar{\mathbf{r}}, h-\beta)$ if and only if $\mathscr{A} = (A, \bar{\mathbf{t}})$, where $A = a_1 * [1, k]$ and $\bar{\mathbf{t}} = (t_1, \ldots, t_k)$.

Proof. Let $A_0 = \{a_0, a_1, \ldots, a_k\}$ with $a_0 = 0$. Then it follows from Lemma 4 that $\Sigma_{\alpha}^{\beta}(\mathscr{A}) = (h-\beta)^{(\bar{\mathbf{r}})}A_0$. Therefore,

$$|(h-\beta)^{(\bar{\mathbf{r}})}A_0| = |\Sigma_{\alpha}^{\beta}(\mathscr{A})| = L(\bar{\mathbf{r}}, h-\beta).$$

It is easy to see that $2 \leq h - \beta \leq r_0 + r_1 + \cdots + r_k - 2$.

Now if $k = 2$ and $t_1 = 1$, then it follows from Theorem 4 that

$$|(h-\beta)^{(\bar{\mathbf{r}})} A_0| = L(\bar{\mathbf{r}}, h-\beta),$$

which implies that

$$|\Sigma_{\alpha}^{\beta}(\mathscr{A})| = L(\bar{\mathbf{r}}, h-\beta).$$

If $k = 2$ and $t_1 \geq 2$, then again it follows from Theorem 4 that

$$|\Sigma_{\alpha}^{\beta}(\mathscr{A})| = |(h-\beta)^{(\bar{\mathbf{r}})} A_0| = L(\bar{\mathbf{r}}, h-\beta)$$

if and only if A_0 is a 3-term arithmetic progression, which implies that $A = a_1 * [1, 2]$. This proves part (a).

Now if $k = 3$ and $t_1 = t_2 = 1$, then it follows from Theorem 5 that

$$|\Sigma_{\alpha}^{\beta}(\mathscr{A})| = |(h-\beta)^{(\bar{\mathbf{r}})} A_0| = L(\bar{\mathbf{r}}, h-\beta)$$

if and only if $a_1 - a_0 = a_3 - a_2$. This implies that $A = \{a_1, a_2, a_1 + a_2\}$ with $0 < a_1 < a_2$. If $k = 3$ and either $t_1 \geq 2$ or $t_2 \geq 2$, then it follows from Theorem 5 that

$$|\Sigma_{\alpha}^{\beta}(\mathscr{A})| = |(h-\beta)^{(\bar{\mathbf{r}})} A_0| = L(\bar{\mathbf{r}}, h-\beta)$$

if and only if A is an arithmetic progression. Hence

$$a_1 - a_0 = a_2 - a_1 = a_3 - a_2,$$

which implies that $a_i = i a_1$ for $i = 1, 2, 3$. Hence $A = a_1 * [1, 3]$. This proves part (b).

If $k \geq 4$, then it follows from Theorem 3 that

$$|\Sigma_{\alpha}^{\beta}(\mathscr{A})| = |(h-\beta)^{(\bar{\mathbf{r}})} A_0| = L(\bar{\mathbf{r}}, h-\beta)$$

if and only if A is an arithmetic progression. Hence

$$a_1 - a_0 = a_2 - a_1 = \cdots = a_k - a_{k-1},$$

which implies that $a_i = i a_1$ for $i = 1, \ldots, k$. Hence $A = a_1 * [1, k]$. This proves part (c). □

Remark 7. Let $\mathscr{A} = (A, \bar{\mathbf{t}})$ be a finite sequence of integers, where $A = \{a_1, \ldots, a_k\}$ is a set of $k \geq 2$ positive integers with $a_1 < \cdots < a_k$ and $\bar{\mathbf{t}} = (t_1, \ldots, t_k)$. Let $h = t_1 + \cdots + t_k$. Let α and β be nonnegative integers, and let $\bar{\mathbf{r}} = (h - \alpha - \beta, t_1, \ldots, t_k)$.

(i) If $\alpha = h - 1$ and $\beta = 0$, then $|\Sigma_{\alpha}^{\beta}(\mathscr{A})| = k + 1$. It is easy to verify that $L(\bar{\mathbf{r}}, h-\beta) = L((1, t_1, \ldots, t_k), h) = k + 1$. Thus $|\Sigma_{\alpha}^{\beta}(\mathscr{A})|$ achieves the lower bound $L(\bar{\mathbf{r}}, h-\beta)$ in this case.

(ii) If $\alpha = h - 1$ and $\beta = 1$, then $|\Sigma_\alpha^\beta(\mathscr{A})| = k$.

(iii) If $\alpha = h$ and $\beta = 0$, then $|\Sigma_\alpha^\beta(\mathscr{A})| = 1$.

(iv) For the remaining values of α and β, one can draw the conclusion using Facts 12.

Theorem 18. *Let $k \geq 3$ be an integer. Let $\mathscr{A} = (A, \bar{\mathbf{t}})$ be a finite sequence of integers, where $A = \{a_0, a_1, \dots, a_{k-1}\}$ with $0 = a_0 < a_1 < \cdots < a_{k-1}$ and $\bar{\mathbf{t}} = (t_0, t_1, \dots, t_{k-1})$. Let $h = t_0 + t_1 + \cdots + t_{k-1}$. Let α and β be integers such that $1 \leq \alpha \leq h-2$, $0 \leq \beta \leq h-2$, and $\alpha+\beta \leq h-1$. Let $\bar{\mathbf{r}} = (h-\alpha-\beta+t_0, t_1, \dots, t_{k-1})$. Then the following conclusions hold.*

(a) If $k = 3$ and $t_1 = 1$, then $|\Sigma_\alpha^\beta(\mathscr{A})| = L(\bar{\mathbf{r}}, h-\beta)$. If $k = 3$ and $t_1 \geq 2$, then $|\Sigma_\alpha^\beta(\mathscr{A})| = L(\bar{\mathbf{r}}, h-\beta)$ if and only if $\mathscr{A} = (A, \bar{\mathbf{t}})$, where $A = a_1 * [0, 2]$ and $\bar{\mathbf{t}} = (t_0, t_1, t_2)$.

(b) If $k = 4$ and $t_1 = t_2 = 1$, then $|\Sigma_\alpha^\beta(\mathscr{A})| = L(\bar{\mathbf{r}}, h-\beta)$ if and only if $\mathscr{A} = (A, \bar{\mathbf{t}})$, where $A = \{0, a_1, a_2, a_1 + a_2\}$ with $0 < a_1 < a_2$ and $\bar{\mathbf{t}} = (t_0, 1, 1, t_3)$. If $k = 4$ and either $t_1 \geq 2$ or $t_2 \geq 2$, then $|\Sigma_\alpha^\beta(\mathscr{A})| = L(\bar{\mathbf{r}}, h-\beta)$ if and only if $\mathscr{A} = (A, \bar{\mathbf{t}})$, where $A = a_1 * [0, 3]$ and $\bar{\mathbf{t}} = (t_0, t_1, t_2, t_3)$.

(c) If $k \geq 5$, then $|\Sigma_\alpha^\beta(\mathscr{A})| = L(\bar{\mathbf{r}}, h-\beta)$ if and only if $\mathscr{A} = (A, \bar{\mathbf{t}})$, where $A = a_1 * [0, k-1]$ and $\bar{\mathbf{t}} = (t_0, t_1, \dots, t_{k-1})$.

Proof. The proof is similar to the proof of Theorem 18. □

Remark 8. Let $\mathscr{A} = (A, \bar{\mathbf{t}})$ be a finite sequence of integers, where

$$A = \{a_0, a_1, \dots, a_{k-1}\}$$

is a set of $k \geq 3$ nonnegative integers with $0 = a_0 < a_1 < \cdots < a_{k-1}$. Let $\bar{\mathbf{t}} = (t_0, t_1, \dots, t_k)$, and let $h = t_0 + t_1 + \cdots + t_k$. Let α and β be nonnegative integers, and let $\bar{\mathbf{r}} = (h - \alpha - \beta + t_0, t_1, \dots, t_{k-1})$.

(i) If $\alpha = h - 1$ and $\beta = 0$, then $|\Sigma_\alpha^\beta(\mathscr{A})| = k$. It is easy to verify that

$$L(\bar{\mathbf{r}}, h - \beta) = L((t_0 + 1, t_1, \dots, t_k), h) = k.$$

Thus $|\Sigma_\alpha^\beta(\mathscr{A})|$ achieves the lower bound $L(\bar{\mathbf{r}}, h - \beta)$ in this case.

(ii) If $\alpha = h - 1$ and $\beta = 1$, then $|\Sigma_\alpha^\beta(\mathscr{A})| = k$. It is easy to verify that

$$L(\bar{\mathbf{r}}, h - \beta) = L((t_0, t_1, \dots, t_k), h - 1) = k.$$

Thus $|\Sigma_\alpha^\beta(\mathscr{A})|$ achieves the lower bound $L(\bar{\mathbf{r}}, h - \beta)$ in this case also.

(iii) If $\alpha = h$ and $\beta = 0$, then $|\Sigma_\alpha^\beta(\mathscr{A})| = 1$. It is easy to verify that

$$L(\bar{\mathbf{r}}, h - \beta) = L((t_0, t_1, \dots, t_k), h) = 1.$$

Thus $|\Sigma_\alpha^\beta(\mathscr{A})|$ achieves the lower bound $L(\bar{\mathbf{r}}, h-\beta)$ in this case also.

(iv) For the remaining values of α and β, one can draw the conclusion using Facts 12.

Theorem 19. *Let n and p be integers such that $n \le p$. Let $\mathscr{A} = (A, \bar{\mathbf{t}})$ be a finite sequence of integers, where*

$$A = \{a_0, a_1, \dots, a_{n-1}, a_{n+1}, \dots, a_{n+p}\}$$

with

$$a_0 < a_1 < \cdots < a_{n-1} < 0 < a_{n+1} < \cdots < a_{n+p}$$

and

$$\bar{\mathbf{t}} = (t_0, t_1, \dots, t_{n-1}, t_{n+1}, \dots, t_{n+p}).$$

Let $h = t_0 + \cdots + t_{n-1} + t_{n+1} + \cdots + t_{n+p} \ge 3$. Let α and β be integers such that $1 \le \alpha \le h-2$, $0 \le \beta \le h-2$, and $\alpha + \beta \le h-1$. Let

$$\bar{\mathbf{r}} = (t_0, \dots, t_{n-1}, h - \alpha - \beta, t_{n+1}, \dots, t_{n+p}).$$

Then the following conclusions hold.

(a) If $k = 3$, $\alpha + \beta = h - 1$, and $t_2 = 1$, then $|\Sigma_\alpha^\beta(\mathscr{A})| = L(\bar{\mathbf{r}}, h-\beta)$ if and only if $\mathscr{A} = (A, \bar{\mathbf{t}})$, where $A = \{a_2 - a_3, a_2, a_3\}$ with $0 < a_2 < a_3$ and $\bar{\mathbf{t}} = (t_0, 1, t_3)$.

(b) In all other cases, $|\Sigma_\alpha^\beta(\mathscr{A})| = L(\bar{\mathbf{r}}, h-\beta)$ if and only if $\mathscr{A} = (A, \bar{\mathbf{t}})$, where $A = a_{n+1} * \{-n, \dots, -1, 1, \dots, p\}$.

Proof. Let $A_0 = \{a_0, a_1, \dots, a_{n-1}, a_n, a_{n+1}, \dots, a_{n+p}\}$ with $a_n = 0$. Then it follows from Lemma 4 and Lemma 3 that

$$\Sigma_\alpha^\beta(\mathscr{A}) = (h-\beta)^{(\bar{\mathbf{r}})} A_0.$$

Let $k = |A| = p + n$. If $k = 2$, then clearly, $p = n = 1$. Hence $A = \{a_0, a_2\}$ and $A_0 = \{a_0, a_1, a_2\}$ with $a_0 < 0 = a_1 < a_2$ and $\bar{\mathbf{r}} = (r_0, r_1, r_2) = (t_0, h - \alpha - \beta, t_2)$, where $t_0 + t_3 \ge 3$. Since $r_1 = h - \alpha \ge 2$, it follows from Theorem 4 that

$$|(h-\beta)^{(\bar{\mathbf{r}})} A_0| = |\Sigma_\alpha^\beta(\mathscr{A})| = L(\bar{\mathbf{r}}, h-\beta)$$

if and only if A_0 is an arithmetic progression. Hence

$$a_1 - a_0 = a_2 - a_1,$$

which implies that $a_0 = -a_2$, and so $A_0 = \{-a_2, 0, a_2\}$. Hence

$$A = \{-a_2, a_2\} = a_2 * \{-1, 1\}.$$

If $k = 3$, then clearly we have $n = 1$ and $p = 2$. Hence $A = \{a_0, a_2, a_3\}$ and $A_0 = \{a_0, a_1, a_2, a_3\}$ with $a_0 < 0 = a_1 < a_2 < a_3$, and

$$\bar{\mathbf{r}} = (r_0, r_1, r_2, r_3) = (t_0, h - \alpha - \beta, t_2, t_3).$$

If $\alpha + \beta = h - 1$ and $t_2 = 1$, then it follows from Theorem 5 that

$$|(h-\beta)^{(\bar{\mathbf{r}})} A_0| = |\Sigma_\alpha^\beta(\mathscr{A})| = L(\bar{\mathbf{r}}, h-\beta)$$

if and only if $a_1 - a_0 = a_3 - a_2$. This implies that $a_0 = a_2 - a_3$. Therefore, $A = \{a_2 - a_3, a_2, a_3\}$. If either $\alpha + \beta \leq h - 2$ or $t_2 \geq 2$, then it follows from Theorem 5 that $|(h-\beta)^{(\bar{\mathbf{r}})} A_0| = |\Sigma_\alpha^\beta(\mathscr{A})| = L(\bar{\mathbf{r}}, h-\beta)$ if and only if A_0 is an arithmetic progression. Hence

$$a_1 - a_0 = a_2 - a_1 = a_3 - a_2.$$

This implies that $a_0 = -a_2$ and $a_3 = 2a_2$, and so $A_0 = \{-a_2, 0, a_2, 2a_2\}$. Therefore,

$$A = \{-a_2, a_2, 2a_2\} = a_2 * \{-1, 1, 2\}.$$

If $k \geq 4$, then, it follows from Theorem 3 that

$$|(h-\beta)^{(\bar{\mathbf{r}})} A_0| = |\Sigma_\alpha^\beta(\mathscr{A})| = L(\bar{\mathbf{r}}, h-\beta)$$

if and only if A_0 is an arithmetic progression. Hence

$$\begin{aligned} a_1 - a_0 = a_2 - a_1 = \cdots = a_{n-1} - a_{n-2} = a_n - a_{n-1} \\ = a_{n+1} - a_n = a_{n+2} - a_{n+1} = \cdots = a_{n+p} - a_{n+p-1}, \end{aligned}$$

which implies that

$$a_{n-j} = -ja_{n+1} \text{ for } j = 1, \ldots, n$$

and

$$a_{n+j} = ja_{n+1} \text{ for } j = 2, \ldots, p.$$

Hence $A_0 = a_{n+1} * [-n, p]$. Therefore,

$$A = a_{n+1} * \{-n, -(n-1), \ldots, -1, 1, 2, \ldots, p\}.$$

This completes the proof. □

Remark 9. Let n and p be integers such that $n \le p$. Let $\mathscr{A} = (A, \bar{\mathbf{t}})$ be a finite sequence of integers, where

$$A = \{a_0, a_1, \ldots, a_{n-1}, a_{n+1}, \ldots, a_{n+p}\}$$

with

$$a_0 < a_1 < \cdots < a_{n-1} < 0 < a_{n+1} < \cdots < a_{n+p}$$

and

$$\bar{\mathbf{t}} = (t_0, t_1, \ldots, t_{n-1}, t_{n+1}, \ldots, t_{n+p}).$$

Let $h = t_0 + \cdots + t_{n-1} + t_{n+1} + \cdots + t_{n+p} \ge 3$. Let α and β be nonnegative integers, and let $\bar{\mathbf{r}} = (t_0, \ldots, t_{n-1}, h - \alpha - \beta, t_{n+1}, \ldots, t_{n+p})$.

(i) If $\alpha = h - 1$ and $\beta = 0$, then $|\Sigma_{\alpha}^{\beta}(\mathscr{A})| = k + 1$. It is easy to verify that $L(\bar{\mathbf{r}}, h - \beta) = k + 1$. Thus $|\Sigma_{\alpha}^{\beta}(\mathscr{A})|$ achieves the lower bound $L(\bar{\mathbf{r}}, h - \beta)$ in this case.

(ii) If $\alpha = h - 1$ and $\beta = 1$, then $|\Sigma_{\alpha}^{\beta}(\mathscr{A})| = k$.

(iii) If $\alpha = h$ and $\beta = 0$, then $|\Sigma_{\alpha}^{\beta}(\mathscr{A})| = 1$.

(iv) For the remaining values of α and β, one can draw the conclusion using Facts 12.

Theorem 20. *Let n and p be integers such that $n \le p$. Let $\mathscr{A} = (A, \bar{\mathbf{t}})$ be a finite sequence of integers, where*

$$A = \{a_0, a_1, \ldots, a_{n-1}, a_n, a_{n+1}, \ldots, a_{n+p}\}$$

with

$$a_0 < a_1 < \cdots < a_{n-1} < 0 = a_n < a_{n+1} < \cdots < a_{n+p}$$

and

$$\bar{\mathbf{t}} = (t_0, t_1, \ldots, t_{n-1}, t_n, t_{n+1}, \ldots, t_{n+p}).$$

Let $h = t_0 + \cdots + t_{n+p}$. Let α and β be integers such that $1 \le \alpha \le h - 2$, $0 \le \beta \le h - 2$, and $\alpha + \beta \le h$. Let $\bar{\mathbf{r}} = (t_0, \ldots, t_{n-1}, h - \alpha - \beta + t_n, t_{n+1}, \ldots, t_{n+p})$. Then the following conclusions hold.

(a) Suppose that $k = 3$ and $\alpha + \beta = h$. In this case, if $t_1 = 1$, then

$$|\Sigma_{\alpha}^{\beta}(\mathscr{A})| = L(\bar{\mathbf{r}}, h - \beta).$$

If $t_1 \ge 2$, then $|\Sigma_{\alpha}^{\beta}(\mathscr{A})| = L(\bar{\mathbf{r}}, h - \beta)$ if and only if $\mathscr{A} = (A, \bar{\mathbf{t}})$, where $A = a_2 * [-1, 1]$ and $\bar{\mathbf{t}} = (t_0, t_1, t_2)$.

(b) Suppose that $k = 3$ and $\alpha + \beta \le h - 1$. In this case, $|\Sigma_{\alpha}^{\beta}(\mathscr{A})| = L(\bar{\mathbf{r}}, h - \beta)$ if and only if $\mathscr{A} = (A, \bar{\mathbf{t}})$, where $A = a_2 * [-1, 1]$ and $\bar{\mathbf{t}} = (t_0, t_1, t_2)$.

(c) *Suppose that $k = 4$ and $\alpha + \beta = h$. In this case, if $t_1 = t_2 = 1$, then*

$$|\Sigma_{\alpha}^{\beta}(\mathscr{A})| = L(\bar{\mathbf{r}}, h - \beta)$$

if and only if $\mathscr{A} = (A, \bar{\mathbf{t}})$, where $A = \{a_2 - a_3, 0, a_2, a_3\}$ with $0 < a_2 < a_3$ and $\bar{\mathbf{t}} = (t_0, 1, 1, t_3)$. If either $t_1 \geq 2$ or $t_2 \geq 2$, then $|\Sigma_{\alpha}^{\beta}(\mathscr{A})| = L(\bar{\mathbf{r}}, h - \beta)$ if and only if $\mathscr{A} = (A, \bar{\mathbf{t}})$, where $A = a_2 * [-1, 2]$.

(d) *Suppose that $k = 4$ and $\alpha + \beta \leq h - 1$. In this case, $|\Sigma_{\alpha}^{\beta}(\mathscr{A})| = L(\bar{\mathbf{r}}, h - \beta)$ if and only if $\mathscr{A} = (A, \bar{\mathbf{t}})$, where $A = a_2 * [-1, 2]$ and $\bar{\mathbf{t}} = (t_0, t_1, t_2, t_3)$.*

(e) *In all other cases, $|\Sigma_{\alpha}^{\beta}(\mathscr{A})| = L(\bar{\mathbf{r}}, h - \beta)$ if and only if $\mathscr{A} = (A, \bar{\mathbf{t}})$, where $A = a_{n+1} * [-n, p]$.*

Proof. It follows from Lemma 5 and Lemma 3 that

$$\Sigma_{\alpha}^{\beta}(\mathscr{A}) = h^{(\bar{\mathbf{r}})}A.$$

Let $k = |A| = p + n + 1$. First assume that $k = 3$. Then clearly, $p = n = 1$. Hence $A = \{a_0, a_1, a_2\}$ with $a_0 < 0 = a_1 < a_2$ and

$$\bar{\mathbf{r}} = (r_0, r_1, r_2) = (t_0, h - \alpha + \beta + t_1, t_2).$$

If $t_1 = 1$ and $\alpha + \beta = h$, then it follows from Theorem 4 that

$$|(h - \beta)^{(\bar{\mathbf{r}})}A| = |\Sigma_{\alpha}^{\beta}(\mathscr{A})| = L(\bar{\mathbf{r}}, h - \beta).$$

If $t_1 \geq 2$ and $\alpha + \beta = h$, then it follows from Theorem 4 that

$$|(h - \beta)^{(\bar{\mathbf{r}})}A| = |\Sigma_{\alpha}^{\beta}(\mathscr{A})| = L(\bar{\mathbf{r}}, h - \beta)$$

if and only if A is an arithmetic progression. Hence $a_1 - a_0 = a_2 - a_1$, which implies that $a_0 = -a_2$, and so $A = \{-a_2, 0, a_2\} = a_2 * [-1, 1]$. This proves part (a). If $t_1 = 1$ and $\alpha + \beta \leq h - 1$, then it follows from Theorem 4 that

$$|(h - \beta)^{(\bar{\mathbf{r}})}A| = |\Sigma_{\alpha}^{\beta}(\mathscr{A})| = L(\bar{\mathbf{r}}, h - \beta)$$

if and only if A is an arithmetic progression. Hence $a_1 - a_0 = a_2 - a_1$, which implies that $a_0 = -a_2$, and so $A = \{-a_2, 0, a_2\} = a_2 * [-1, 1]$. This proves part (b).

Now assume that $k = 4$. Then clearly we have $n = 1$ and $p = 2$. Hence $A = \{a_0, a_1, a_2, a_3\}$ with $a_0 < 0 = a_1 < a_2 < a_3$ and

$$\bar{\mathbf{r}} = (r_0, r_1, r_2, r_3) = (t_0, h - \alpha + t_1, t_2, t_3).$$

If $t_1 = t_2 = 1$ and $\alpha + \beta = h$, then it follows from Theorem 5 that

$$|(h - \beta)^{(\bar{\mathbf{r}})}A| = |\Sigma_{\alpha}^{\beta}(\mathscr{A})| = L(\bar{\mathbf{r}}, h - \beta)$$

if and only if $a_1 - a_0 = a_3 - a_2$, which implies that $A = \{a_2 - a_3, 0, a_2, a_3\}$ with $0 < a_2 < a_3$. If $\alpha + \beta = h$ and either $t_1 \geq 2$ or $t_2 \geq 2$, then it follows from Theorem 5 that

$$|(h-\beta)^{(\bar{\mathbf{r}})}A| = |\Sigma_{\alpha}^{\beta}(\mathscr{A})| = L(\bar{\mathbf{r}}, h-\beta)$$

if and only if A is an arithmetic progression. Hence

$$a_1 - a_0 = a_2 - a_1 = a_3 - a_2,$$

which implies that $a_0 = -a_2$ and $a_3 = 2a_2$, and so $A = \{-a_2, 0, a_2, 2a_2\} = a_2 * [-1, 2]$. This proves part (c). If $\alpha+\beta \leq h-1$, then since $r_1 = h-\alpha+t_1 \geq 1+t_1 \geq 2$, it follows from Theorem 5 that

$$|(h-\beta)^{(\bar{\mathbf{r}})}A| = |\Sigma_{\alpha}^{\beta}(\mathscr{A})| = L(\bar{\mathbf{r}}, h-\beta)$$

if and only if A is an arithmetic progression. Hence

$$a_1 - a_0 = a_2 - a_1 = a_3 - a_2,$$

which implies that $a_0 = -a_2$ and $a_3 = 2a_2$. Therefore,

$$A = \{-a_2, 0, a_2, 2a_2\} = a_2 * [-1, 2].$$

This proves part (d).

If $k \geq 5$, then, it follows from Theorem 3 that

$$|(h-\beta)^{(\bar{\mathbf{r}})}A| = |\Sigma_{\alpha}^{\beta}(\mathscr{A})| = L(\bar{\mathbf{r}}, h-\beta)$$

if and only if A is an arithmetic progression. Hence

$$\begin{aligned} a_1 - a_0 = a_2 - a_1 = \cdots = a_{n-1} - a_{n-2} = a_n - a_{n-1} \\ = a_{n+1} - a_n = a_{n+2} - a_{n+1} = \cdots = a_{n+p} - a_{n+p-1}, \end{aligned}$$

which implies that

$$a_{n-j} = -ja_{n+1} \text{ for } j = 1, \ldots, n$$

and

$$a_{n+j} = ja_{n+1} \text{ for } j = 2, \ldots, p.$$

Hence $A = a_{n+1} * [-n, p]$. This proves part (e). This completes the proof. □

Remark 10. Let n and p be integers such that $n \leq p$. Let $\mathscr{A} = (A, \bar{\mathbf{t}})$ be a finite sequence of integers, where

$$A = \{a_0, a_1, \ldots, a_{n-1}, a_n, a_{n+1}, \ldots, a_{n+p}\}$$

with

$$a_0 < a_1 < \cdots < a_{n-1} < 0 = a_n < a_{n+1} < \cdots < a_{n+p}$$

and

$$\bar{\mathbf{t}} = (t_0, t_1, \ldots, t_{n-1}, t_n, t_{n+1}, \ldots, t_{n+p}).$$

Let $h = t_0 + \cdots + t_{n+p}$. Let α and β be nonnegative integers, and let

$$\bar{\mathbf{r}} = (t_0, \ldots, t_{n-1}, h - \alpha - \beta + t_n, t_{n+1}, \ldots, t_{n+p}).$$

(i) If $\alpha = h - 1$ and $\beta = 0$, then $|\Sigma_\alpha^\beta(\mathscr{A})| = k$. It is easy to verify that

$$L(\bar{\mathbf{r}}, h - \beta) = k.$$

Thus $|\Sigma_\alpha^\beta(\mathscr{A})|$ achieves the lower bound $L(\bar{\mathbf{r}}, h - \beta)$ in this case.

(ii) If $\alpha = h - 1$ and $\beta = 1$, then $|\Sigma_\alpha^\beta(\mathscr{A})| = k$. It is easy to verify that

$$L(\bar{\mathbf{r}}, h - \beta) = k.$$

Thus $|\Sigma_\alpha^\beta(\mathscr{A})|$ achieves the lower bound $L(\bar{\mathbf{r}}, h - \beta)$ in this case also.

(iii) If $\alpha = h$ and $\beta = 0$, then $|\Sigma_\alpha^\beta(\mathscr{A})| = 1$. It is easy to verify that

$$L(\bar{\mathbf{r}}, h - \beta) = 1.$$

Thus $|\Sigma_\alpha^\beta(\mathscr{A})|$ achieves the lower bound $L(\bar{\mathbf{r}}, h - \beta)$ in this case also.

(iv) For the remaining values of α and β, one can draw the conclusion using Facts 12.

Remark 11. In Theorem 15, Theorem 16, Theorem 19 and Theorem 20, we have assumed that $n \leq p$. If $n > p$, then we can replace the sequence $\mathscr{A}$ by $-\mathscr{A}$ and apply the corresponding theorems to establish the inverse theorems in this case. Here the sequence $-\mathscr{A}$ is obtained by replacing each term x of $\mathscr{A}$ by $-x$.

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