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A NOTE ON SYLVESTER'S SEQUENCE

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Abstract

Let (S_n) be the Sylvester sequence. Using Lebesgue-Nagell type equations of the form $x^2 + C = 4y^n$, we show that no Sylvester number can be a perfect power and no Sylvester number (except 3 and 7) can be expressed as $a^m \pm 1$, where a and m are integers greater than or equal to 2. Other information about this sequence is provided.

1. Introduction

The Sylvester sequence $(S_n)_{n\in\mathbb{N}}$, introduced by J. J. Sylvester [5] in 1880, is the integer sequence defined by

$$S_0 = 2$$

and the recursive relationship

$$S_n = 1 + \prod_{k=0}^{n-1} S_k. \tag{1}$$

The first few terms of (S_n) are

$$2, \quad 3, \quad 7, \quad 43, \quad 1807, \quad 3263443, \quad 10650056950807 \ldots.$$

This sequence is strictly increasing and also satisfies the recurrence relation

$$S_n = S_{n-1}^2 - S_{n-1} + 1. \tag{2}$$

Then

$$\frac{1}{S_n} = \frac{1}{S_n - 1} - \frac{1}{S_{n+1} - 1},$$

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which shows that

$$\sum_{n=0}^{+\infty} \frac{1}{S_n} = 1.$$

The Sylvester sequence is a well known sequence in number theory (Sequence A000058 in the OEIS), its reciprocals are used to develop, in an optimal way, finite Egyptian fraction representations of 1, and some of its properties are used in differential geometry and computer science.

It follows clearly from the definition that all terms (except S_0 and S_1) are of the form $6k+1$ and every two terms are coprime. Concerning the factorization of the terms in the Sylvester sequence, one can easily prove, through quadratic residues, that no prime of the form $6k-1$ divides a term of the sequence. Indeed, observe that Equation (2) can be rewritten as

$$4S_n = (2S_{n-1} - 1)^2 + 3, \tag{3}$$

so if a prime p divides a term S_n, then -3 is a quadratic residue modulo p and therefore p is necessarily of the form $6k+1$, according to the law of quadratic reciprocity.

Calculations [7] show that all known terms of this sequence are square-free, but the question of whether all terms are square-free is still unsolved.

In this note we show, by using a result established by F. Luca, SZ. Tengely and A. Togbé [2], on equations of Lebesgue-Nagell type $x^2 + C = 4y^n$, that no term of (S_n) is a perfect power and no term (except S_1 and S_2) can be expressed as $a^m \pm 1$, with $a \geq 2$ and $m \geq 2$. In addition, we determine the terms of (S_n) which are perfect numbers, triangular numbers or Mersenne numbers, and we make a remark on the digits of S_n.

2. Congruence Properties of Sylvester's Sequence

It follows from the definition that $S_n \equiv 1 \pmod{42}$, for all $n \geq 3$; $S_n \equiv 1 \pmod{1806}$, for all $n \geq 4$; and more generally, for any given positive integer k, $S_n \equiv 1 \pmod{S_1 S_2 ... S_k}$, for all $n \geq k+1$. Furthermore, we have the following congruence properties.

Proposition 1. *Let k be a positive integer. For all $m, n \geq k$ we have*

$$(S_k - 1)^2 \;\; \textit{divides} \;\; S_n - S_m. \tag{4}$$

In particular for all $n \geq 1$

$$S_n \equiv 3 \pmod{4}, \tag{5}$$

for all $n \geq 2$

$$S_n \equiv 7 \pmod{36}, \tag{6}$$

for all $n \geq 3$

$$S_n \equiv 43 \pmod{1764}. \tag{7}$$

Proof. We first show, by induction on the variable n, that $(S_k-1)^2$ divides $S_n - S_k$, for all $n \geq k$. The base case clearly holds, so assume that $(S_k-1)^2$ divides $S_n - S_k$. According to (2), we have

$$\begin{aligned} S_{n+1} - S_k &= S_n^2 - S_n + 1 - S_k \\ &\equiv S_k^2 - S_k + 1 - S_k \pmod{(S_k-1)^2} \\ &\equiv (S_k-1)^2 \pmod{(S_k-1)^2} \\ &\equiv 0 \pmod{(S_k-1)^2}, \end{aligned}$$

and hence the induction is complete. Now $S_n - S_k$ and $S_m - S_k$ are both divisible by $(S_k-1)^2$, so $S_n - S_m = S_n - S_k - (S_m - S_k)$ is divisible by $(S_k-1)^2$. □

Proposition 2. *Let k be a positive integer. For all $n \geq k$ we have*

$$(S_k^2+1) \text{ divides } S_n - (-1)^{n-k} S_k. \tag{8}$$

In particular for all $i \geq 1$

$$S_{2i} \equiv 7 \pmod{10}, \quad S_{2i+1} \equiv 3 \pmod{10}. \tag{9}$$

Proof. We proceed by induction on the variable n to show that $S_n \equiv (-1)^{n-k} S_k \pmod{(S_k^2+1)}$, for all $n \geq k$. The base case clearly holds, so assume that S_k^2+1 divides $S_n - (-1)^{n-k} S_k$. According to (2), we have

$$\begin{aligned} S_{n+1} &= S_n^2 - S_n + 1 \\ &\equiv S_k^2 - (-1)^{n-k} S_k + 1 \pmod{(S_k^2+1)} \\ &\equiv (-1)^{n+1-k} S_k \pmod{(S_k^2+1)}, \end{aligned}$$

and hence the induction is complete. □

Corollary 1. *The following statements hold.*
(*a*) *The numbers* 3 *and* 7 *are the only Mersenne Sylvester numbers.*
(*b*) *The number* 3 *is the only triangular Sylvester number.*
(*c*) *No Sylvester number can be a perfect number.*
(*d*) *Every Sylvester number, except* 2*, has a prime divisor of the form* $4k+3$.

Proof. (a) Mersenne numbers are numbers of the form $2^n - 1$, for some positive integer n. Clearly, S_1 and S_2 are Mersenne numbers but S_0 is not. Assume that there exists an integer $m \geq 3$ such that $S_m = 2^n - 1$ for some positive integer n.

Clearly, we can write $n = 3k + r$, with $r \in \{0, 1, 2\}$, so by (6) we have $8^k \cdot 2^r = 8 \pmod{36}$. One can easily prove, by induction, that for all $i \geq 1$,

$$8^i \equiv (-1)^{i+1} 8 \pmod{36},$$

and thus $(-1)^{k+1} 8 \cdot 2^r \equiv 8 \pmod{36}$. This is only possible if k is odd and $r = 0$, so $n = 6l + 3$, for some non-negative integer l. On the other hand, according to (7), we have

$$2^n \equiv 44 \pmod{1764},$$

and hence $(2^l)^6 \cdot 8 \equiv 44 \pmod{1764}$. But the prime number 7 is a divisor of 1764, so it follows from Fermat's little theorem that $1 \equiv 2 \pmod 7$, which is absurd.
(b) Triangular numbers are numbers of the form $\dfrac{m(m+1)}{2}$, for some positive integer m. It is clear that S_1 is a triangular number but S_0 and S_2 are not triangular numbers. Now assume that there exists $n \geq 3$ such that S_n is a triangular number. So, there exists a positive integer m such that $S_n = \frac{m(m+1)}{2}$. Relation (3) yields

$$(2m + 1)^2 - 2(2S_{n-1} - 1)^2 = 7,$$

thus, according to (6), we have $(2m + 1)^2 \equiv 21 \pmod{36}$. But calculation shows that 21 is not a square modulo 36, and hence we have a contradiction.
(c) Clearly S_0 and S_1 are not perfect numbers. For $n \geq 2$, the term S_n is odd and, according to (6), it is equal to $7 \pmod{36}$, while an odd perfect number must be equal to $1 \pmod{12}$ or $9 \pmod{36}$, according to [6].
(d) Let $n \geq 1$. If all the prime divisors of S_n are of the form $4k + 1$, then S_n would be of the form $4k + 1$, which would contradict Relation (5). □

Remark 1 (On the Digits of the Sylvester Sequence)**.** With the exception of S_0, the last digit of the Sylvester numbers is 3 or 7, according to (9). More precisely, it is 3 for odd indices and 7 for even indices. In other words, starting from the index 1, the Sylvester sequence is periodic modulo 10 and its period is equal to 2. More generally, for any integer $b > 1$, the Sylvester sequence is ultimately periodic modulo b. Indeed, according to the pigeonhole principle, there exist two positive integers N and T such that $S_N = S_{N+T}$ modulo b. So using Relation (2) we obtain $S_n = S_{n+T}$ modulo b, for all $n \geq N$. On the other hand, let us also note that the digital root of the Sylvester numbers (except S_0 and S_1) equals 7. Recall that the digital root d of a non-negative number N is one of the digits $0, 1, 2, ..., 9$. To calculate d, let N_1 be the sum of the digits of N. Then let N_2 be the sum of the digits of N_1, and we repeat this process until we get one-digit result. It is easy to see that if 9 is not a divisor of N, then $d \equiv N \pmod 9$. Here, according to (6), for all $n \geq 2$ we have

$$S_n \equiv 7 \pmod 9,$$

so the digital root of S_n is 7.

Remark 2. Proposition 1 also makes it possible to show that the Sylvester version (see [1] and [3] for other versions) of the Brocard-Ramanujan Diophantine equation

$$n! + 1 = S_m^2$$

has no solution (n, m) in positive integers. Indeed, a quick computation shows that there is no solution for $n < 6$. If we assume that $n! + 1 = S_m^2$, for some non-negative integers n, m, with $n \geq 6$, then, according to (6), we have $S_m^2 \equiv 13 \pmod{36}$, and hence $S_m^2 \equiv 4 \pmod 9$. On the other hand, since $n \geq 6$, we have $n! + 1 \equiv 1 \pmod 9$, and hence $1 \equiv 4 \pmod 9$, which is absurd.

3. The Main Result

Now by using Proposition 1 and a theorem on the equations of Lebesgue-Nagell type

$$x^2 + C = 4y^n, \quad (n, x, y) \in \mathbb{N}^3, \quad C \in \mathbb{Z}, \tag{10}$$

we prove the following result.

Proposition 3. *The following statements hold.*
(*a*) *No Sylvester number can be a perfect power.*
(*b*) *No Sylvester number (except S_1 and S_2) can be expressed as $a^m \pm 1$, where a and m are integers greater than or equal to 2.*

Before giving the proof, let us recall our main tool concerning Equation (10).

Theorem 1 ([2], Theorem 1.1)**.** *The only integer solutions (n, x, y) of the Diophantine equation*

$$x^2 + 3 = 4y^n, \quad x, y \geq 1, \quad \gcd(x, y) = 1, \quad n \geq 3,$$

are the triples

$$(n, 1, 1), \ \textit{for all} \ \ n \geq 3 \ \textit{and} \ (3, 37, 7).$$

The only integer solutions (n, x, y) of the Diophantine equation

$$x^2 + 7 = 4y^n, \quad x, y \geq 1, \quad \gcd(x, y) = 1, \quad n \geq 3,$$

are the triples

$$(3, 5, 2), \quad (5, 11, 2), \ \textit{and} \ (13, 181, 2).$$

Proof of Proposition 3. (a) Clearly, the terms S_0, S_1, and S_2 are not perfect powers. Assume, on the contrary, that

$$S_m = y^n, \tag{11}$$

for some $m \geq 3,\ y > 1$ and $n \geq 2$.
`Case 1:` $n = 2$. According to (6) we have

$$y^2 \equiv 7 \pmod{36},$$

but 7 is not a square modulo 36, so we have a contradiction.
`Case 2:` $n \geq 3$. Relations (3) and (11) yield

$$(2S_{m-1} - 1)^2 + 3 = 4y^n, \tag{12}$$

so the triple $(n, 2S_{m-1} - 1, y)$ is a solution of Equation (10), with $C = 3$. To use Theorem 1, we first need to check that $2S_{m-1} - 1$ and y are coprime. Indeed, Relation (6) implies that $S_{n-1} \equiv 1 \pmod 3$, so $2S_{m-1} - 1 \equiv 1 \pmod 3$, and hence the number 3 does not divide $2S_{m-1} - 1$. Therefore $\gcd(2S_{m-1} - 1, y) = 1$, according to (12). Now, according to Theorem 1, we have

$$2S_{m-1} - 1 = 1 \textit{ or } 2S_{m-1} - 1 = 37,$$

so $S_{m-1} = 1$ or $S_{m-1} = 19$. But 1 and 19 are not terms of the Sylvester sequence, hence we have a contradiction.
(b) Assume that $S_m = a^n \pm 1$, for some integers n, m, a, with $m \geq 3, a \geq 2$, and $n \geq 2$. The case where $S_m = a^n + 1$ is easily dismissed. In fact, if $S_m = a^n + 1$, then, by Equation (1), we have $S_0 S_1 ... S_{m-1} = a^n$. But every two terms of the Sylvester sequence are relatively prime, so n cannot be greater than 1, hence we have a contradiction.

Now assume that $S_m = a^n - 1$. By (3) we have

$$(2S_{m-1} - 1)^2 + 7 = 4a^n, \tag{13}$$

so the triple $(n, 2S_{m-1} - 1, a)$ is a solution of Equation (10), with $C = 7$.
`Case 1:` $n = 2$. According to (6), we have $S_{m-1} \equiv 7 \pmod{36}$, thus

$$(2S_{m-1} - 1)^2 + 7 \equiv -4 \pmod{36},$$

which implies, according to (13), that the equation $-4 = 4y^2$ has a solution modulo 36, but a simple calculation shows the opposite.
`Case 2:` $n \geq 3$. In this case we shall use Theorem 1, so let us check that $2S_{m-1} - 1$ and a are coprime. Relation (7) leads to $2S_{m-1} - 1 \equiv 85 \pmod{1764}$, but 7 divides 1764, so $2S_{m-1} - 1 \equiv 1 \pmod 7$, which clearly shows that 7 does not divide $2S_{m-1} - 1$ and therefore $\gcd(2S_{m-1} - 1, a) = 1$ according to (13). Now, according to Theorem 1, we have

$$2S_{m-1} - 1 = 5, \text{ or } 2S_{m-1} - 1 = 11, \text{ or } 2S_{m-1} - 1 = 181,$$

and hence $S_{m-1} = 3$, or $S_{m-1} = 6$, or $S_{m-1} = 91$. But 6 and 91 are not terms of the Sylvester sequence, and S_{m-1} cannot be equal to 3 because m is assumed to be greater than 2, so we have a contradiction. $\square$

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