A REMARK ON SINGULAR DUALS OF MOEBIUS MAPS

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Abstract

Let (B, T) be a fibred map. A standard method for the determination of the density of an invariant measure is provided by the theory of dual maps, a generalization of backward continued fractions. A dual map $(B^{\#}, T^{\#})$ is called a natural dual if there is a differentiable map M with the property $M \circ T = T^{\#} \circ M$. In this paper we present the surprising result of a family of fibred maps (B, T) such that the set $B^{\#}$ of every natural dual is a one-point set.

1. Introduction

The search for invariant measures has seen a lot of publications, starting with [3] and through the years that followed (see, for example, [1] and [2]).

Let $T : B \to B$ be a map on the interval $B = [a, b]$ subject to the following conditions. There is a partition $a = x_0 < x_1 < ... < x_N = b$ such that the map T is injective on every interval $[x_j, x_{j+1}]$ and $T[x_j, x_{j+1}] = B$, , $0 \leq j < N$ (a special case of a *fibred system*; see [7]). The inverse function $V_j : B \to [x_j, x_{j+1}]$ is called an *inverse branch* of T.

If there is a matrix

$$V_k = \begin{pmatrix} a_{00}^k & a_{01}^k \\ a_{10}^k & a_{11}^k \end{pmatrix}$$

which corresponds to a map

$$V_k x = \frac{a_{10}^k + a_{11}^k x}{a_{00}^k + a_{01}^k x},$$

then we call it a *Moebius map* (see [6]).

A Moebius map $T^{\#} : B^{\#} \to B^{\#}$, $B^{\#} := [a^{\#}, b^{\#}]$, is called a *dual map* if its inverse branches are given by the transposed matrices $V_k^{\#}$, i.e.,

$$V_k^{\#} y = \frac{a_{01}^k + a_{11}^k y}{a_{00}^k + a_{10}^k y}.$$

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Then

$$h(x) = \int_{B^\#} \frac{dy}{(1+xy)^2}$$

is the density of an invariant measure for (B, T) (see [7], [6]). If there exists a symmetric matrix M with the associated map

$$M(t) = \frac{B + Dt}{A + Bt}$$

such that

$$M : B \to B^\# \text{ and } MV_k = V_k^\# M$$

for all k, then we call it a *natural dual*. If $B^\#$ shrinks to one point then the dual can be seen as a natural dual with a singular matrix M. We call it a *singular dual* (see [6] and [5]). If no suitable non-singular matrix M exists we call the dual an *exceptional dual*.

In this paper we consider Moebius maps which are constructed in the following way. Let $T : [0,1] \to [0,1]$ be a Moebius map with inverse branches defined by the matrices

$$V_\alpha = \begin{pmatrix} 1 & 2\alpha - 1 \\ 0 & \alpha \end{pmatrix},\ 0 < \alpha \leq 1,\ V_\beta = \begin{pmatrix} 1 & 1 - 2\beta \\ 1 & -\beta \end{pmatrix},\ \beta < 1$$

and $S; [0,1] \to [0,1]$ the Moebius maps with inverse branches defined by the matrices

$$V_\gamma = \begin{pmatrix} 2 & \gamma - 1 \\ 1 & -1 \end{pmatrix},\ -1 < \gamma,\ V_\delta = \begin{pmatrix} 2 & \delta - 1 \\ 1 & -\delta \end{pmatrix},\ 0 \leq \delta.$$

Then we investigate the map defined by $(S \circ T)x = S(Tx)$. Its inverse branches are given by the following matrices:

$$V_{\alpha\gamma} = V_\alpha V_\gamma = \begin{pmatrix} 2\alpha + 1 & -2\alpha + \gamma \\ \alpha & -\alpha \end{pmatrix};\ V_{\alpha\delta} = V_\alpha V_\delta = \begin{pmatrix} 2\alpha + 1 & 2\alpha\delta - 1 \\ \alpha & \alpha\delta \end{pmatrix};$$

$$V_{\beta\gamma} = V_\beta V_\gamma = \begin{pmatrix} 3 - 2\beta & 2\beta + \gamma - 2 \\ 2 - \beta & \beta + \gamma - 1 \end{pmatrix};\ V_{\beta\delta} = V_\beta V_\delta = \begin{pmatrix} 3 - 2\beta & 2\delta - 2\beta\delta - 1 \\ 2 - \beta & \delta - \beta\delta - 1 \end{pmatrix}.$$

If we suppose that our given system $(B, S \circ T)$ has a natural dual $(B^\#, (S \circ T)^\#)$, then the four branches $V_{\alpha\gamma}^\#$, $V_{\alpha\delta}^\#$, $V_{\beta\gamma}^\#$, and $V_{\beta,\delta}^\#$ follow the same order as the four branches of $S \circ T$ if the map $M : B \to B^\#$ is increasing. They follow the reverse order if the map $M : B \to B^\#$ is decreasing.

We denote the fixed point of $V_{\beta\gamma}^\#$, which is one endpoint of $B^\#$, by η. Let $\xi = V_{\alpha\gamma}^\# \eta$ be the other endpoint of $B^\#$. The map $(S \circ T)^\#$ further satisfies the following equations:

$$V_{\beta\gamma}^\# \xi = V_{\beta\delta}^\# \xi, \quad V_{\alpha\delta}^\# \xi = V_{\alpha\gamma}^\# \xi, \quad \text{and } V_{\beta\delta}^\# \eta = V_{\alpha\delta}^\# \eta.$$

The main result of this paper is the following result: if the given system $(B, (S \circ T))$ has a natural dual then it is a singular dual.

2. The Main Result

Lemma 1. *The equation* $\frac{-2+2\beta+\gamma+(-1+\beta+\gamma)\eta}{3-2\beta+(2-\beta)} = \eta$ *has the solutions* $\eta = -1$ *and* $\eta = \frac{-2+2\beta+\gamma}{2-\beta}$.

Proof. It is easy to see that $\eta = -1$ and $\eta = \frac{-2+2\beta+\gamma}{2-\beta}$ are the solutions of the corresponding quadratic equation. □

Lemma 2. *The case* $\eta = -1$ *cannot occur.*

Proof. The equation $V^{\#}_{\beta\delta}\eta = V^{\#}_{\alpha\delta}\eta$ together with $\eta = -1$ leads to

$$\delta = \frac{-1+\alpha\delta}{1+\alpha}.$$

Then $\delta = -1$, which is not an allowed value for δ. □

Lemma 3. *The central equations*

$$\alpha + \alpha\delta + \delta = \beta + \gamma + \beta\delta$$

and

$$\alpha^2\gamma + \alpha^2\gamma\delta + 2\alpha + \beta + \beta\gamma + 2\alpha^2 + 2\alpha^2\delta + 4\alpha\delta + 2\alpha\gamma\delta \\ = 2 + \alpha\gamma^2 + 2\gamma + 2\alpha\beta + \alpha\gamma + \alpha\beta\gamma + \alpha\beta\gamma\delta + 2\alpha\beta\delta$$

hold.

Proof. We start with the proof of the first central equation. The equation $V^{\#}_{\beta\gamma}\xi = V^{\#}_{\beta\delta}\xi$ shows that

$$\xi = \frac{1-2\beta+\gamma+2\delta-\beta\delta}{\beta+\gamma-\delta+\beta\delta}.$$

The equation $V^{\#}_{\alpha\delta}\xi = V^{\#}_{\alpha\gamma}\xi$ gives the representation

$$\xi = \frac{1-2\alpha+\gamma-2\alpha\delta}{\alpha+\alpha\delta}.$$

If we compare the two representations of ξ then there is value λ such that

$$1 - 2\beta + \gamma + 2\delta - \beta\delta = \lambda(1 - 2\alpha + \gamma - 2\alpha\delta)$$

and

$$\beta + \gamma - \delta + \beta\delta = \lambda(\alpha + \alpha\delta).$$

We multiply the second equation by 2 and add it to the first equation. Then

$$1 + \gamma = \lambda(1 + \gamma).$$

Hence $\lambda = 1$ and the central equation can be deduced.

The proof of the second central equation follows. From $\eta = \frac{-2+2\beta+\gamma}{2-\beta}$ and the equation $\xi = V^{\#}_{\alpha\gamma}\eta$ we calculate

$$\xi = \frac{-2\alpha + 2\gamma + \alpha\gamma - \beta\gamma}{2\alpha + 2 - \beta + \alpha\gamma}.$$

Then we have

$$\xi = \frac{1 - 2\alpha + \gamma - 2\alpha\delta}{\alpha + \alpha\delta} = \frac{-2\alpha + 2\gamma - \alpha\gamma - \beta\gamma}{2\alpha + 2 - \beta + \alpha\gamma}.$$

A tedious calculation leads to the second central equation. □

It easy to write down solutions for the central equations. However, in all cases we find $\eta = \xi$. This leads to our main theorem.

Theorem 1. *If $(B^{\#}, (S \circ T)^{\#})$ is a natural dual of $(B, S \circ T)$ then the dual map $(B^{\#}, (S \circ T)^{\#})$ is a singular dual.*

Proof. We will reduce the second central equation to a shorter form by using the first central equation. From

$$\begin{aligned}
&\alpha^2\gamma + \alpha^2\gamma\delta + 2\alpha + \beta + \beta\gamma + 2\alpha^2 + 2\alpha^2\delta + 4\alpha\delta + 2\alpha\gamma\delta \\
&\quad = \alpha^2\gamma + \alpha^2\gamma\delta + 2\alpha + \beta + \beta\gamma + 2\alpha(\alpha + \delta + \alpha\delta) + 2\alpha\delta + 2\alpha\gamma\delta \\
&\quad = \alpha^2\gamma + \alpha^2\gamma\delta + 2\alpha + \beta + \beta\gamma + 2\alpha(\beta + \gamma + \beta\delta) + 2\alpha\delta + 2\alpha\gamma\delta \\
&\quad = 2 + \alpha\gamma^2 + 2\gamma + 2\alpha\beta + \alpha\gamma + \alpha\beta\gamma + \alpha\beta\gamma\delta + 2\alpha\beta\delta
\end{aligned}$$

we deduce

$$\alpha^2\gamma + \alpha^2\gamma\delta + 2\alpha\gamma\delta + 2\alpha + \beta + \beta\gamma + \alpha\gamma + 2\alpha\delta = 2 + \alpha^2\gamma + 2\gamma + \alpha\beta\gamma\delta + \alpha\beta\gamma.$$

Then

$$\begin{aligned}
&\alpha^2\gamma + \alpha^2\gamma\delta + 2\alpha\gamma\delta + 2\alpha + \beta + \beta\gamma + \alpha\gamma + 2\alpha\delta \\
&\quad = \alpha\gamma(\alpha + \alpha\delta + \delta) + \alpha\gamma\delta + +2\alpha + \beta + \beta\gamma + \alpha\gamma + 2\alpha\delta \\
&\quad = \alpha\gamma(\beta + \gamma + \beta\delta) + \alpha\gamma\delta + +2\alpha + \beta + \beta\gamma + \alpha\gamma + 2\alpha\delta \\
&\quad = 2 + \alpha^2\gamma + 2\gamma + 2\alpha\beta + \alpha\gamma + \alpha\beta\gamma\delta + \alpha\beta\gamma + 2\alpha\beta\delta.
\end{aligned}$$

Then we obtain our final result:

$$\beta + \alpha\gamma\delta + 2\alpha + \beta\gamma + \alpha\gamma + 2\alpha\delta = 2 + 2\gamma.$$

Since

$$\eta = \frac{-2 + 2\beta + \gamma}{2 - \beta} = -1 + \frac{\beta + \gamma}{2 - \beta}$$

and
$$\xi = \frac{1-2\alpha+\gamma-2\alpha\delta}{\alpha+\alpha\delta} = -1 + \frac{1-\alpha+\gamma-\alpha\delta}{\alpha+\alpha\delta},$$
we see that the equation $\xi = \eta$ is equivalent to
$$\beta + \alpha\gamma\delta + 2\alpha + \beta\gamma + \alpha\gamma + 2\alpha\delta = 2 + 2\gamma.$$
□

Remark 1. The same result clearly applies to $T \circ S$. If μ denotes the invariant measure for $S \circ T$, then ν defined as $\nu(E) = \mu(T^{-1}E)$ is invariant for $T \circ S$.

Example 1. Let $\alpha = 1$, $\beta = 0$, $\gamma = 2$, $\delta = \frac{1}{2}$. Then we have
$$Tx = \begin{cases} \dfrac{x}{1-x}, & 0 \le x < \frac{1}{2} \\ \dfrac{1-x}{x}, & \frac{1}{2} \le x \le 1, \end{cases} \quad \text{and} \quad Sx = \begin{cases} \dfrac{1-2x}{1+x}, & 0 \le x < \frac{1}{2} \\ \dfrac{-2+4x}{1+x}, & \frac{1}{2} \le x \le 1. \end{cases}$$
Then $S \circ T$ has a singular dual on $B^{\#} = \{0\}$. The density of the invariant measure for $S \circ T$ is $h(x) = 1$. The map $T \circ S$ also has a singular dual on $B^{\#} = \{1\}$ and the density of the invariant measure is $h(x) = \frac{1}{(1+x)^2}$.

Remark 2. Exceptional duals can be found for the configuration $V^{\#}_{\beta\gamma}$, $V^{\#}_{\alpha\gamma}$, $V^{\#}_{\alpha\delta}$, and $V^{\#}_{\beta\delta}$ or its reverse order. Again η is given by $V^{\#}_{\beta\gamma}\eta = \eta$, but $\xi = V^{\#}_{\beta\delta}\eta$.

Example 2. (1) If $\eta = -1$, an example is given by $\alpha = 1$, $\beta = \gamma = \delta = 0$ and $B^{\#} = [-1, 0]$. (2) If $\eta = \frac{-2+2\beta+\gamma}{2-\beta}$, an example is $\alpha = \frac{1}{2}$, $\beta = 0$, $\gamma = \delta = 2$ and $B^{\#} = [0, 1]$.

3. A Further Result

One can try the following approach which is similar to the *backward conditions* in [4]. Let the matrix
$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$
satisfy the conditions
$$MV_{\alpha\gamma} = V^{\#}_{\gamma\alpha}M,\ MV_{\alpha\delta} = V^{\#}_{\delta\alpha}M, MV_{\beta\gamma} = V^{\#}_{\gamma\beta}M,\ MV_{\beta\delta} = V^{\#}_{\delta\beta}M.$$
Then we obtain the following result.

Theorem 2. *If $B^{\#} = MB$, then $h(x) = \int_{B^{\#}} \frac{dy}{(1+xy)^2}$ is the density of the invariant measure for $S \circ T$. Furthermore $h = h(x)$ is the density for the map T and for the map S.*

Proof. The first assertion is more or less obvious. The second assertion is more interesting. The matrix

$$M_1 = \begin{pmatrix} a_1 & b_1 \\ b_1 & d_1 \end{pmatrix},$$

which satisfies

$$M_1 V_\alpha = V_\alpha^\# M_1,\ M_1 V_\beta = V_\beta^\# M_1,$$

has the entries $a_1 = 1 - \alpha$, $b_1 = 2\alpha - 1$, and $d_1 = 2 - 3\alpha - \beta$.

From the equation $MV_{\alpha\gamma} = V_{\gamma\alpha}^\# M$, we find $a = 1 - \alpha$, $b = 2\alpha - 1$, and $d = (-2\alpha + \gamma)a - (\alpha + 2)b$. From $MV_{\beta\gamma} = V_{\gamma\beta}^\# M$, we find $a(2 - 2\beta - \gamma) = b(\beta - 2)$ and hence $\gamma - \alpha\gamma = 2\alpha - \beta$. This gives $d = 2 - 3\alpha - \beta$ and $M = M_1$.

To prove $M = M_2$, we apply the map $N(x) = 1 - x$ which exchanges the maps T and S. □

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