ON HIGHER MOMENTS OF DIRICHLET COEFFICIENTS ATTACHED TO SYMMETRIC POWER L-FUNCTIONS OVER CERTAIN SEQUENCES OF POSITIVE INTEGERS

Guodong Hua
School of Math. and Stats., Weinan Normal University, Shaanxi, Weinan, China
and
Qindong Mathematical Research Institute, Weinan Normal University, Shaanxi, Weinan, China
huasdu@163.com

Received: 3/15/23, Accepted: 2/23/24, Published: 3/15/24

Abstract

Let j be a fixed integer such that $2 \leq j \leq 8$. Let f be a normalized primitive holomorphic cusp form of even integral weight for the full modular group $\Gamma = SL(2,\mathbb{Z})$. Denote by $\lambda_{\mathrm{sym}^2 f}(n)$ the n-th normalized coefficient of the Dirichlet expansion of the symmetric square L-function $L(\mathrm{sym}^2 f, s)$ attached to f. In this paper, we are interested in the average behavior of the summatory function

$$\sum_{\substack{a_1^2+a_2^2+a_3^2+a_4^2+a_5^2+a_6^2 \leq x \\ (a_1,a_2,a_3,a_4,a_5,a_6)\in\mathbb{Z}^6}} \lambda^j_{\mathrm{sym}^2 f}(a_1^2+a_2^2+a_3^2+a_4^2+a_5^2+a_6^2)$$

for x sufficiently large. In a similar manner, we also consider the mean square of coefficients of the Dirichlet expansions of two symmetric power L-functions attached to two distinct primitive holomorphic cusp forms over the same sequence.

1. Introduction

The Fourier coefficients of modular forms are important and interesting objects in number theory. Let H_k^* be the set of all normalized primitive holomorphic cusp forms of even integral weight $k \geq 2$ for the full modular group $\Gamma = SL(2,\mathbb{Z})$. Then the Hecke eigenform $f(z) \in H_k^*$ has the Fourier expansion at the cusp ∞:

$$f(z) = \sum_{n=1}^{\infty} \lambda_f(n) n^{\frac{k-1}{2}} e(nz), \quad \Im(z) > 0,$$

DOI: 10.5281/zenodo.10821702

where $e(z) = e^{2\pi iz}$, and $\lambda_f(n)$ is the n-th normalized Fourier coefficient (Hecke eigenvalue) such that $\lambda_f(1) = 1$. Then $\lambda_f(n)$ is real and satisfies the multiplicative property

$$\lambda_f(m)\lambda_f(n) = \sum_{d|(m,n)} \lambda_f\left(\frac{mn}{d^2}\right),$$

where $m \geq 1$ and $n \geq 1$ are positive integers. In 1974, P. Deligne [5] proved the Ramanujan-Petersson conjecture

$$|\lambda_f(n)| \leq d(n), \tag{1}$$

where $d(n)$ is the classical divisor function. By Equation (1), Deligne's bound is equivalent to the fact that there exist $\alpha_f(p), \beta_f(p) \in \mathbb{C}$ satisfying

$$\alpha_f(p) + \beta_f(p) = \lambda_f(p) \quad \text{and} \quad \alpha_f(p)\beta_f(p) = |\alpha_f(p)| = |\beta_f(p)| = 1. \tag{2}$$

More generally, for all positive integers $l \geq 1$, one has

$$\lambda_f(p^l) = \alpha_f(p)^l + \alpha_f(p)^{l-1}\beta_f(p) + \cdots + \alpha_f(p)\beta_f(p)^{l-1} + \beta_f(p)^l.$$

It is an important topic to consider the average behavior of Hecke eigenvalues of cusp forms in various aspects (see, e.g., [13, 16, 36, 38, 52]). In 2013, Zhai [54] considered the average behavior of the power sum

$$U_j(f;x) := \sum_{a^2+b^2 \leq x} \lambda_f(a^2+b^2)^j$$

for $x \geq 1, 2 \leq j \leq 8$ and $a, b, j \in \mathbb{Z}$. Indeed, he successfully proved that

$$U_j(f;x) = x\tilde{P}_j(\log x) + O\big(x^{\alpha_j+\varepsilon}\big),$$

where $\tilde{P}_j$ with $j = 2, \ldots, 8$ are polynomials with degrees $\deg \tilde{P}_2 = 0, \deg \tilde{P}_4 = 1, \deg \tilde{P}_6 = 4, \deg \tilde{P}_8 = 13$, and $\deg \tilde{P}_j \equiv 0$ for $j = 3, 5, 7$. The exponents α_j are given by

$$\alpha_2 = \frac{8}{11}, \qquad \alpha_3 = \frac{17}{20}, \qquad \alpha_4 = \frac{43}{46}, \qquad \alpha_5 = \frac{83}{86},$$

$$\alpha_6 = \frac{184}{187}, \qquad \alpha_7 = \frac{355}{357}, \qquad \alpha_8 = \frac{752}{755}.$$

Very recently, the results of Zhai were refined and generalized for all $j \geq 2$ by Xu [53], by using the recent breakthrough of Newton and Thorne [33, 34], along with some nice analytic properties of the associated L-functions.

Let $\lambda_{\mathrm{sym}^j f}(n)$ denote the n-th normalized coefficient of the Dirichlet expansion of the j-th symmetric power L-function $L(\mathrm{sym}^j f, s)$ (see Section 2 for more details). Fomenko [6] proved that

$$\sum_{n\le x}\lambda_{\mathrm{sym}^2 f}(n) \ll x^{\frac{1}{2}}(\log x)^2.$$

Later, this sum were studied by many authors (see, e.g., [25, 29, 44]). The analogous cases for symmetric power lifting $\mathrm{sym}^j\pi_f$ for large j were considered by Lau and Lü [30], and Tang and Wu [51].

On the other hand, Fomenko [7] studied the sum of $\lambda^2_{\mathrm{sym}^2 f}(n)$. Later, this result was improved and generalized by a number of authors (see, e.g., [50, 11, 31, 45]). Recently, Sankaranarayanan, Singh, and Srinivas [45] proved that

$$\sum_{n\le x}\lambda^2_{\mathrm{sym}^3 f}(n) = c_1 x + O\big(x^{\frac{15}{17}+\varepsilon}\big),$$

and

$$\sum_{n\le x}\lambda^2_{\mathrm{sym}^4 f}(n) = c_2 x + O\big(x^{\frac{12}{13}+\varepsilon}\big),$$

where $c_1, c_2 > 0$ are some suitable constants. Very recently, Luo et al. [31] established the asymptotic formulas

$$\sum_{n\le x}\lambda^2_{\mathrm{sym}^j f}(n) = \tilde{c}_j x + O\big(x^{\tilde{\theta}_j+\varepsilon}\big), \quad 3\le j\le 6,$$

and

$$\sum_{n\le x}\lambda^2_{\mathrm{sym}^j f}(n) = \tilde{c}_j x + O\big(x^{\tilde{\theta}_j}\big), \qquad j = 7, 8,$$

where $\tilde{c}_j (3\le j\le 8)$ is a suitable constant, and $\tilde{\theta}_3 = \frac{551}{635}, \tilde{\theta}_4 = \frac{929}{1013}, \tilde{\theta}_5 = \frac{1391}{1475}, \tilde{\theta}_6 = \frac{979}{1021}, \tilde{\theta}_7 = \frac{63}{65}, \tilde{\theta}_8 = \frac{40}{41}$, respectively. In the same paper, the authors also proved that

$$\sum_{n\le x}\lambda^j_{\mathrm{sym}^2 f}(n) = xP_j(\log x) + O\big(x^{\theta_j+\varepsilon}\big),$$

where P_j is a polynomial with $\deg P_3 = 0, \deg P_4 = 2, \deg P_5 = 5, \deg P_6 = 14, \deg P_7 = 35, \deg P_8 = 90$, and $\theta_3 = \frac{971}{1055}, \theta_4 = \frac{262}{269}, \theta_5 = \frac{3237}{3265}, \theta_6 = \frac{4923}{4937}, \theta_7 = \frac{7442}{7449}, \theta_8 = \frac{89771}{89799}$, respectively.

In [46], Sharma and Sankaranarayanan considered the asymptotic behavior of the sum

$$U_{f,j}(x) := \sum_{\substack{n=a^2+b^2+c^2+d^2\le x\\ (a,b,c,d)\in\mathbb{Z}^4}} \lambda^j_{\mathrm{sym}^2 f}(n)$$

for $j = 2$ for $x \geq x_0$, where x_0 is sufficiently large. In fact, the authors established the formula

$$U_{f,2}(x) = c_f x^2 + O_f\big(x^{\frac{9}{5}+\varepsilon}\big)$$

for any $\varepsilon > 0$, where $c_f > 0$ is some suitable constant depending on f. Very recently, Sharma and Sankaranarayanan [47] established the asymptotic formulae for $U_{f,j}(x)$ with $j = 3, 4$. In fact, they proved that

$$U_{f,3}(x) = c_1 x^2 + O_f\big(x^{\frac{27}{14}+\varepsilon}\big),$$

and

$$U_{f,4}(x) = c_2 x^2 \log x + O_f\big(x^{\frac{160}{81}+\varepsilon}\big),$$

where c_1, c_2 are suitable effective constants depending on f. Afterwards, the author and his collaborators gave some refinements and generalizations concerning the above results of Sharma and Sankaranarayanan, the interested readers can refer to [14, 15, 17].

In [48], Sharma and Sankaranarayanan investigated another type of summatory function related to the coefficients of the symmetric power L-function

$$S_{f,j}(x) = \sum_{\substack{a_1^2+a_2^2+a_3^2+a_4^2+a_5^2+a_6^2 \leq x \\ (a_1,a_2,a_3,a_4,a_5,a_6)\in\mathbb{Z}^6}} \lambda^2_{\mathrm{sym}^j f}(a_1^2+a_2^2+a_3^2+a_4^2+a_5^2+a_6^2),$$

with $j = 2$. In fact, they proved the asymptotic formula

$$S_{f,2}(x) = c'_{f,2} x^3 + O\big(x^{\frac{14}{5}+\varepsilon}\big),$$

where $c'_{f,2}$ is an effective constant. Very recently, Sharma and Sankaranarayanan [49] considered the asymptotic formulae for $S_{f,j}(x)$ for all $j \geq 2$, by using the celebrated work of Newton and Thorne [33, 34], along with some individual and averaged subconvexity bounds of associated L-functions. More precisely, for $j \geq 2$, they established that

$$S_{f,j}(x) = c'_{f,j} x^3 + O\big(x^{3-\frac{6}{3(j+1)^2+1}+\varepsilon}\big),$$

where $c'_{f,j}$ is some effective constant depending on f and associated L-functions.

Let $f \in H_k^*$ be a Hecke eigenform, and let $\lambda_{\mathrm{sym}^j f}(n)$ be the coefficients of the Dirichlet expansion of the j-th symmetric power L-function associated to f. Inspired by the above results, the aim in this paper is to consider the summatory function

$$S^*_{f,j}(x) := \sum_{\substack{a_1^2+a_2^2+a_3^2+a_4^2+a_5^2+a_6^2 \leq x \\ (a_1,a_2,a_3,a_4,a_5,a_6)\in\mathbb{Z}^6}} \lambda^j_{\mathrm{sym}^2 f}(a_1^2+a_2^2+a_3^2+a_4^2+a_5^2+a_6^2), \tag{3}$$

with $3 \leq j \leq 8$. More precisely, we are able to establish the following result.

Theorem 1. *Let $S^*_{f,j}(x)$ be defined by Equation* (3)*. For $3 \le j \le 8$ and any $\varepsilon > 0$, we have*

$$S^*_{f,j}(x) = x^3 P^*_j(\log x) + O\big(x^{\theta^*_j+\varepsilon}\big)$$

where P^*_j is a polynomial with $\deg P^*_3 \equiv 0, \deg P^*_4 = 2, \deg P^*_5 = 5, \deg P^*_6 = 14, \deg P^*_7 = 35, \deg P^*_8 = 90,$ *and*

$$\begin{aligned} \theta^*_3 &= \frac{79}{27}, \qquad \theta^*_4 = \frac{241}{81}, \qquad \theta^*_5 = \frac{727}{243}, \\ \theta^*_6 &= \frac{2185}{729}, \qquad \theta^*_7 = \frac{6559}{2187}, \qquad \theta^*_8 = \frac{19681}{6561}. \end{aligned}$$

Let $f \in H^*_{k_1}$ and $g \in H^*_{k_2}$ be two distinct Hecke eigenforms. Define

$$S_{f,g,i,j}(x) := \sum_{\substack{a_1^2+a_2^2+a_3^2+a_4^2+a_5^2+a_6^2 \le x \\ (a_1,a_2,a_3,a_4,a_5,a_6)\in\mathbb{Z}^6}} \lambda^2_{\mathrm{sym}^i f}\Big(\sum_{r=1}^6 a_r^2\Big)\lambda^2_{\mathrm{sym}^j g}\Big(\sum_{r=1}^6 a_r^2\Big), \tag{4}$$

where $i, j \ge 2$ are two fixed positive integers. In a similar manner as that of Theorem 1, we are also able to prove the following theorem.

Theorem 2. *Let $S_{f,g,i,j}(x)$ be defined by Equation* (4)*. For $i, j \ge 2$ any two fixed integers and any $\varepsilon > 0$, we have*

$$S_{f,g,i,j}(x) = c_{f,g,i,j}x^3 + O\big(x^{3-\frac{2}{(i+1)^2(j+1)^2}+\varepsilon}\big),$$

where $c_{f,g,i,j}$ is an effective constant given by

$$\begin{aligned} c_{f,g,i,j} \;=\; & \frac{16}{3}L(3,\chi)\prod_{i_1=1}^{i}\prod_{j_1=1}^{j} L(sym^{2i_1}f,1)L(sym^{2j_1}g,1)L(sym^{2i_1}f\otimes sym^{2j_1}g,1) \\ & \times\prod_{i_1=1}^{i}\prod_{j_1=1}^{j} L(sym^{2i_1}f\otimes\chi,3)L(sym^{2j_1}g\otimes\chi,3) \\ & \times L(sym^{2i_1}f\otimes sym^{2j_1}g\otimes\chi,3)H_{i,j}(3), \end{aligned}$$

and χ is the non-principal Dirichlet character modulo 4*, and $H_{i,j}(3) \ne 0$.*

Throughout the paper, we always assume that $f \in H^*_{k_1}$ and $g \in H^*_{k_2}$ be two distinct Hecke eigenforms. Also, we denote by $\varepsilon > 0$ an arbitrarily small positive constant that may vary in different occurrences. The symbol p always denotes a prime number.

2. Preliminaries

In this section,we introduce some background on the analytic properties of automorphic L-functions and give some useful lemmas which play important roles in the proof of the main results in this paper.

Let $f \in H_{k_1}^*$ be a Hecke eigenform of even integral weight k for the full modular group $\Gamma = SL(2,\mathbb{Z})$, and let $\lambda_f(n)$ denote its n-th normalized Fourier coefficient. The Hecke L-function $L(f,s)$ associated to f is defined by

$$\begin{aligned} L(f,s) &= \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^s} = \prod_p (1-\lambda_f(p)p^{-s}+p^{-2s})^{-1} \\ &= \prod_p \left(1-\frac{\alpha_f(p)}{p^s}\right)^{-1}\left(1-\frac{\beta_f(p)}{p^s}\right)^{-1},\ \Re(s)>1, \end{aligned}$$

where $\alpha_f(p)$ and $\beta_f(p)$ are the local parameters satisfying Equation (2). The j-th symmetric power L-function associated with f is defined by

$$L(\mathrm{sym}^j f,s) = \prod_p \prod_{m=0}^{j} (1-\alpha_f(p)^{j-m}\beta_f(p)^m p^{-s})^{-1},\ \Re(s)>1.$$

We may expand it into a Dirichlet series

$$\begin{aligned} L(\mathrm{sym}^j f,s) &= \sum_{n=1}^{\infty} \frac{\lambda_{\mathrm{sym}^j f}(n)}{n^s}, \\ &= \prod_p \left(1+\frac{\lambda_{\mathrm{sym}^j f}(p)}{p^s}+\cdots+\frac{\lambda_{\mathrm{sym}^j f}(p^k)}{p^{ks}}+\cdots\right),\ \Re(s)>1. \end{aligned} \tag{5}$$

Obviously, $\lambda_{\mathrm{sym}^j f}(n)$ is a real multiplicative function. In particular, for $j=1$, we have $L(\mathrm{sym}^1 f,s) = L(f,s)$. Let $g \in H_{k_2}^*$ be a Hecke eigenform. The Rankin-Selberg L-function $L(\mathrm{sym}^i f\otimes \mathrm{sym}^j g,s)$ attached to $\mathrm{sym}^i f$ and $\mathrm{sym}^j g$ is defined as

$$\begin{aligned} L(\mathrm{sym}^i f\otimes \mathrm{sym}^j g,s) &= \prod_p \prod_{m=0}^{i} \prod_{m'=0}^{j} (1-\alpha_f(p)^{i-m}\beta_f(p)^m \\ &\quad \times \alpha_g(p)^{j-m'}\beta_g(p)^{m'}p^{-s})^{-1} \\ &= \sum_{n=1}^{\infty} \frac{\lambda_{\mathrm{sym}^i f\otimes \mathrm{sym}^j g}(n)}{n^s},\quad \Re(s)>1, \end{aligned} \tag{6}$$

where $\alpha_g(p)$ and $\beta_g(p)$ are the local parameters of g defined in a manner similar to that of f in Equation (2), and where f and g are not necessarily different. Similarly, $\lambda_{\mathrm{sym}^i f\otimes \mathrm{sym}^j g}(n)$ is also a real multiplicative function. From Equation (2), it is not hard to find that

$$|\lambda_{\mathrm{sym}^j f}(n)| \le d_{j+1}(n) \quad \text{and} \quad |\lambda_{\mathrm{sym}^i f\otimes \mathrm{sym}^j g}(n)| \le d_{(i+1)(j+1)}(n) \tag{7}$$

for all $i, j \geq 1$, where $d_\nu(n)$ denotes the ν-dimensional divisor function, which is defined as the number of ordered representations $n = n_1 \dots n_\nu$ with integers $n_1, \dots, n_\nu \geq 1$.

Let χ be a Dirichlet character modulo q. Then we can define the *twisted* j-th symmetric power L-function by the Euler product representation with degree $j+1$

$$\begin{aligned} L(\mathrm{sym}^j f \otimes \chi, s) &= \prod_p \prod_{m=0}^{j} (1 - \alpha_f(p)^{j-m} \beta_f(p)^m \chi(p) p^{-s})^{-1} \\ &= \sum_{n=1}^{\infty} \frac{\lambda_{\mathrm{sym}^j f}(n) \chi(n)}{n^s} \end{aligned}$$

for $\Re(s) > 1$. In the analogous manner, we can also define the Rankin-Selberg convolution L-function attached to $\mathrm{sym}^i f$ and $\mathrm{sym}^j g \otimes \chi$ by the Euler product representation with degree $(i+1)(j+1)$

$$\begin{aligned} L(\mathrm{sym}^i f \otimes \mathrm{sym}^j g \otimes \chi, s) &= \prod_p \prod_{m=0}^{i} \prod_{m'=0}^{j} (1 - \alpha_f(p)^{i-m} \beta_f(p)^m \\ &\quad \times \alpha_g(p)^{j-m'} \beta_g(p)^{m'} \chi(p) p^{-s})^{-1} \\ &= \sum_{n=1}^{\infty} \frac{\lambda_{\mathrm{sym}^i f \otimes \mathrm{sym}^j g}(n) \chi(n)}{n^s}, \quad \Re(s) > 1. \end{aligned}$$

We may expand it into a Dirichlet series

$$\begin{aligned} L(\mathrm{sym}^i f \otimes \mathrm{sym}^j g \otimes \chi, s) &= \prod_p \left(1 + \sum_{k \geq 1} \frac{\lambda_{\mathrm{sym}^i f \otimes \mathrm{sym}^j g}(p^k) \chi(p^k)}{p^{ks}} \right) \\ &= \sum_{n \geq 1} \frac{\lambda_{\mathrm{sym}^i f \otimes \mathrm{sym}^j g}(n) \chi(n)}{n^s}. \end{aligned}$$

It is standard that

$$\lambda_f(p^j) = \lambda_{\mathrm{sym}^j f}(p) = \frac{\alpha_f(p)^{j+1} - \beta_f(p)^{j+1}}{\alpha_f(p) - \beta_f(p)} = \sum_{m=0}^{j} \alpha_f(p)^{j-m} \beta_f(p)^m,$$

which can be rewritten as

$$\lambda_f(p^j) = \lambda_{\mathrm{sym}^j f}(p) = \widetilde{U}_j(\lambda_f(p)/2),$$

where $\widetilde{U}_j(x)$ is the j-th Chebyshev polynomial of the second kind. For any prime number p, we also have

$$\lambda_{\mathrm{sym}^i f \otimes \mathrm{sym}^j g}(p) = \lambda_{\mathrm{sym}^i f}(p) \lambda_{\mathrm{sym}^j g}(p) = \lambda_f(p^i) \lambda_g(p^j). \tag{8}$$

As is well-known, to a primitive form f is associated an automorphic cuspidal representation π_f of $GL_2(\mathbb{A}_{\mathbb{Q}})$, and hence an automorphic L-function $L(\pi_f, s)$ which coincides with $L(f, s)$. It is predicted that π_f gives rise to a symmetric power lift– an automorphic representation whose L-function is the symmetric power L-function attached to f.

For $1 \le j \le 8$, the Langlands functoriality conjecture, which states that $\mathrm{sym}^j f$ is automorphic cuspidal, was established in a series of important work by Gelbart and Jacquet [8], Kim [28], Kim and Shahidi [27, 26], Shahidi [43], and Clozel and Thorne [2, 3, 4]. Very recently, Newton and Thorne [33, 34] proved that $\mathrm{sym}^j f$ corresponds with a cuspidal automorphic representation of $GL_{j+1}(\mathbb{A}_{\mathbb{Q}})$ for all $j \ge 1$ (with f being a holomorphic cusp form). From the work of about the Rankin-Selberg theory developed by Jacquet, Piatetski-Shapiro, Shalika [23], Jacquet and Shalika [21, 22], Shahidi [39, 40, 41, 42], and the reformulation of Rudnick and Sarnak [37], we know that $L(\mathrm{sym}^j f, s), L(\mathrm{sym}^i f \otimes \mathrm{sym}^j g, s), (1 \le i \le j)$ and its twisted L-functions have analytic continuations to the whole complex plane (except possibly for simple poles at $s = 0, 1$ if $\mathrm{sym}^j \pi_f \cong \mathrm{sym}^j \pi_g$) and satisfy certain Riemann-type functional equations. We refer the interested readers to [20, Chapter 5] for a more comprehensive treatment.

Let

$$r_k(n) := \#\big\{(n_1, n_2, \ldots, n_k) \in \mathbb{Z}^k \ : \ n_1^2 + n_2^2 + \ldots + n_k^2 = n\big\}.$$

In this paper, we are concerned with the function $r_6(n)$. From [49, Lemma 2.1], we learn that for any positive integer,

$$r_6(n) = 16 \sum_{d|n} \chi(d')d^2 - 4 \sum_{d|n} \chi(d)d^2,$$

where $n = dd'$, and χ is the non-principal Dirichlet character modulo 4, i.e.,

$$\chi(n) = \begin{cases} 1, & \text{if } n \equiv 1 \pmod 4, \\ -1, & \text{if } n \equiv -1 \pmod 4, \\ 0, & \text{if } n \equiv 0 \pmod 2. \end{cases}$$

We can also rewrite $r_6(n)$ as

$$\begin{aligned} r_6(n) &= 16 \sum_{d|n} \chi(d) \frac{n^2}{d^2} - 4 \sum_{d|n} \chi(d)d^2 \\ &:= 16l(n) - 4v(n) \\ &:= l_1(n) - v_1(n). \end{aligned} \tag{9}$$

It is not hard to find that $l(n)$ and $v(n)$ are multiplicative since the non-principal character $\chi(n)$ is multiplicative. Note that

$$l(p) = p^2 + \chi(p) \quad \text{and} \quad l(p^2) = p^4 + p^2\chi(p) + \chi(p^2),$$

and

$$v(p) = 1 + p^2\chi(p) \quad \text{and} \quad v(p^2) = 1 + p^2\chi(p) + p^4\chi(p^2).$$

Let $3 \le j \le 8$ be any fixed positive integer. From the definition of $S^*_{f,j}(x)$, along with Equations (3) and (9), we have

$$\begin{aligned} S^*_{f,j}(x) &= \sum_{n\le x} \lambda^j_{\mathrm{sym}^2 f}(n) \sum_{\substack{n=a_1^2+a_2^2+a_3^2+a_4^2+a_5^2+a_6^2 \\ (a_1,a_2,a_3,a_4,a_5,a_6)\in\mathbb{Z}^6}} 1 \\ &= \sum_{n\le x} \lambda^j_{\mathrm{sym}^2 f}(n) r_6(n) \\ &= 16\sum_{n\le x} \lambda^j_{\mathrm{sym}^2 f}(n) l(n) - 4\sum_{n\le x} \lambda^j_{\mathrm{sym}^2 f}(n) v(n). \end{aligned} \tag{10}$$

For the sake of simplicity, for $l \ge 1$, let

$$\prod_{\chi}{}' L(\mathrm{sym}^l f, s) := L(\mathrm{sym}^l f, s-2) L(\mathrm{sym}^l f \otimes \chi, s),$$

which means that $L(\mathrm{sym}^l f, s-2)$ and $L(\mathrm{sym}^l f \otimes \chi, s)$ occur in pairs.

Lemma 1. *Let j be an integer such that $3 \le j \le 8$. Let $f \in H^*_{k_1}$ be a Hecke eigenform. Define*

$$\mathcal{F}_j(s) := \sum_{n=1}^{\infty} \frac{\lambda^j_{sym^2 f}(n) l(n)}{n^s}.$$

Then

$$\mathcal{F}_j(s) = G_j(s) H_j(s),$$

where

$$\begin{aligned} G_3(s) &= \zeta(s-2)L(s,\chi)\prod_{\chi}{}' L(sym^2 f, s)^3 L(sym^4 f, s)^2 L(sym^6 f, s), \\ G_4(s) &= \zeta(s-2)^3 L(s,\chi)^3 \prod_{\chi}{}' L(sym^2 f, s)^6 L(sym^4 f, s)^6 L(sym^6 f, s)^3 \\ &\quad \times L(sym^8 f, s), \\ G_5(s) &= \zeta(s-2)^6 L(s,\chi)^6 \prod_{\chi}{}' L(sym^2 f, s)^{15} L(sym^4 f, s)^{15} L(sym^6 f, s)^{10} \\ &\quad \times L(sym^8 f, s)^4 L(sym^{10} f, s), \end{aligned}$$

$$\begin{aligned} G_6(s) &= \zeta(s-2)^{15}L(s,\chi)^{15}\prod_{\chi}{}' L(sym^2 f,s)^{36}L(sym^4 f,s)^{40}L(sym^6 f,s)^{29} \\ &\quad \times L(sym^8 f,s)^{15}L(sym^{10} f,s)^{5}L(sym^{12} f,s), \\ G_7(s) &= \zeta(s-2)^{36}L(s,\chi)^{36}\prod_{\chi}{}' L(sym^2 f,s)^{91}L(sym^4 f,s)^{105}L(sym^6 f,s)^{84} \\ &\quad \times L(sym^8 f,s)^{39}L(sym^{10} f,s)^{21}L(sym^{12} f,s)^{6}L(sym^{14} f,s), \\ G_8(s) &= \zeta(s-2)^{91}L(s,\chi)^{91}\prod_{\chi}{}' L(sym^2 f,s)^{232}L(sym^4 f,s)^{280}L(sym^6 f,s)^{238} \\ &\quad \times L(sym^8 f,s)^{154}L(sym^{10} f,s)^{76}L(sym^{12} f,s)^{28}L(sym^{14} f,s)^{7} \\ &\quad \times L(sym^{16} f,s), \end{aligned}$$

and χ is a non-principal Dirichlet character modulo 4. The function $H_j(s)$ admits a Dirichlet series which converges uniformly and absolutely in the half-plane $\Re(s) \geq \frac{5}{2}+\varepsilon$, and $H_j(s) \neq 0$ for $\Re(s) = 3$.

Proof. Since $\lambda^j_{\mathrm{sym}^2 f}(n)l(n)$ is a multiplicative function, and also satisfies the bound $O(n^{2+\varepsilon})$ for any $\varepsilon > 0$, for $\Re(s) > 3$, we have the Euler product

$$\mathcal{F}_j(s) = \prod_p \left(1 + \sum_{k\geq 1} \frac{\lambda^j_{\mathrm{sym}^2 f}(p^k)l(p^k)}{p^{ks}}\right). \tag{11}$$

We only give the proof of the case $j = 8$, since other cases can be handled by a similar argument. For $j = 8$, from [30, (13)] and [31, Lemma 2.1], we learn that

$$\begin{aligned} \lambda^8_{\mathrm{sym}^2 f}(p) &= 91 + 232\lambda_{\mathrm{sym}^2 f}(p) + 280\lambda_{\mathrm{sym}^4 f}(p) + 238\lambda_{\mathrm{sym}^6 f}(p) + 154\lambda_{\mathrm{sym}^8 f}(p) \\ &\quad + 76\lambda_{\mathrm{sym}^{10} f}(p) + 28\lambda_{\mathrm{sym}^{12} f}(p) + 7\lambda_{\mathrm{sym}^{14} f}(p) + \lambda_{\mathrm{sym}^{16} f}(p). \end{aligned} \tag{12}$$

For $\Re(s) > 3$, the L-function

$$\begin{aligned} G_8(s) &:= \zeta(s-2)^{91}L(s,\chi)^{91}\prod_{\chi}{}' L(\mathrm{sym}^2 f,s)^{232}L(\mathrm{sym}^4 f,s)^{280}L(\mathrm{sym}^6 f,s)^{238} \\ &\quad \times L(\mathrm{sym}^8 f,s)^{154}L(\mathrm{sym}^{10} f,s)^{76}L(\mathrm{sym}^{12} f,s)^{28}L(\mathrm{sym}^{14} f,s)^{7} \\ &\quad \times L(\mathrm{sym}^{16} f,s) \end{aligned} \tag{13}$$

can be represented as

$$G_8(s) := \prod_p \left(1 + \sum_{k\geq 1} \frac{b(p^k)}{p^{ks}}\right). \tag{14}$$

It is not hard to find that

$$\begin{aligned}\lambda_{\mathrm{sym}^2 f}^8(p)l(p) &= \lambda_{\mathrm{sym}^2 f}^8(p)(p^2+\chi(p)) \\ &= \big(91+232\lambda_{\mathrm{sym}^2 f}(p)+280\lambda_{\mathrm{sym}^4 f}(p)+238\lambda_{\mathrm{sym}^6 f}(p) \\ &\quad +154\lambda_{\mathrm{sym}^8 f}(p)+76\lambda_{\mathrm{sym}^{10} f}(p)+28\lambda_{\mathrm{sym}^{12} f}(p) \\ &\quad +7\lambda_{\mathrm{sym}^{14} f}(p)+\lambda_{\mathrm{sym}^{16} f}(p)\big)(p^2+\chi(p)) \\ &= b(p). \end{aligned} \tag{15}$$

Putting Equations (11)−(15) together, for $\Re(s) > 3$, we obtain

$$\begin{aligned}\mathcal{F}_8(s) &= G_8(s) \times \prod_p \left(1 + \frac{\lambda_{\mathrm{sym}^2 f}^8(p^2)l(p^2) - b(p^2)}{p^{2s}} + \cdots\right) \\ &:= \zeta(s-2)^{91} L(s,\chi)^{91} \prod_{\chi}{}' L(\mathrm{sym}^2 f, s)^{232} L(\mathrm{sym}^4 f, s)^{280} L(\mathrm{sym}^6 f, s)^{238} \\ &\quad \times L(\mathrm{sym}^8 f, s)^{154} L(\mathrm{sym}^{10} f, s)^{76} L(\mathrm{sym}^{12} f, s)^{28} L(\mathrm{sym}^{14} f, s)^{7} \\ &\quad \times L(\mathrm{sym}^{16} f, s) H_8(s).\end{aligned}$$

By Equation (7) and the bound $l(n) \ll n^{2+\varepsilon}$ for any $\varepsilon > 0$, the function $H_8(s)$ converges uniformly and absolutely in the half-plane $\Re(s) \geq \frac{5}{2} + \varepsilon$ for any $\varepsilon > 0$. □

In a similar manner, for $l \geq 1$, we define

$$\prod_{\chi}{}^{*} L(\mathrm{sym}^l f, s) := L(\mathrm{sym}^l f, s) L(\mathrm{sym}^l f \otimes \chi, s-2),$$

which means that $L(\mathrm{sym}^l f, s)$ and $L(\mathrm{sym}^l f \otimes \chi, s-2)$ occur in pairs.

Lemma 2. *Let j be an integer such that $3 \leq j \leq 8$. Let $f \in H_{k_1}^*$ be a Hecke eigenform. Define*

$$\widetilde{\mathcal{F}}_j(s) := \sum_{n=1}^{\infty} \frac{\lambda_{sym^2 f}^j(n) v(n)}{n^s}.$$

Then

$$\widetilde{\mathcal{F}}_j(s) = \widetilde{G}_j(s) \widetilde{H}_j(s),$$

where

$$\begin{aligned}\widetilde{G}_3(s) &= \zeta(s) L(s-2,\chi) \prod_{\chi}{}^{*} L(sym^2 f, s)^3 L(sym^4 f, s)^2 L(sym^6 f, s), \\ \widetilde{G}_4(s) &= \zeta(s)^3 L(s-2,\chi)^3 \prod_{\chi}{}^{*} L(sym^2 f, s)^6 L(sym^4 f, s)^6 L(sym^6 f, s)^3 \\ &\quad \times L(sym^8 f, s),\end{aligned}$$

$$\begin{aligned}
\widetilde{G}_5(s) &= \zeta(s)^6 L(s-2,\chi)^6 \prod_{\chi}^{*} L(sym^2 f,s)^{15} L(sym^4 f,s)^{15} L(sym^6 f,s)^{10} \\
&\quad \times L(sym^8 f,s)^4 L(sym^{10} f,s), \\
\widetilde{G}_6(s) &= \zeta(s)^{15} L(s-2,\chi)^{15} \prod_{\chi}^{*} L(sym^2 f,s)^{36} L(sym^4 f,s)^{40} L(sym^6 f,s)^{29} \\
&\quad \times L(sym^8 f,s)^{15} L(sym^{10} f,s)^5 L(sym^{12} f,s), \\
\widetilde{G}_7(s) &= \zeta(s)^{36} L(s-2,\chi)^{36} \prod_{\chi}^{*} L(sym^2 f,s)^{91} L(sym^4 f,s)^{105} L(sym^6 f,s)^{84} \\
&\quad \times L(sym^8 f,s)^{39} L(sym^{10} f,s)^{21} L(sym^{12} f,s)^6 L(sym^{14} f,s), \\
\widetilde{G}_8(s) &= \zeta(s)^{91} L(s-2,\chi)^{91} \prod_{\chi}^{*} L(sym^2 f,s)^{232} L(sym^4 f,s)^{280} L(sym^6 f,s)^{238} \\
&\quad \times L(sym^8 f,s)^{154} L(sym^{10} f,s)^{76} L(sym^{12} f,s)^{28} L(sym^{14} f,s)^7 \\
&\quad \times L(sym^{16} f,s),
\end{aligned}$$

and χ is a non-principal Dirichlet character modulo 4. The function $\widetilde{H}_j(s)$ admits a Dirichlet series which converges uniformly and absolutely in the half-plane $\Re(s) \geq \frac{5}{2}+\varepsilon$, and $\widetilde{H}_j(s) \neq 0$ for $\Re(s) = 3$.

Proof. Since $\lambda^j_{\mathrm{sym}^2 f}(n)v(n)$ is a multiplicative function, and also satisfies the bound $O(n^{2+\varepsilon})$ for any $\varepsilon > 0$, for $\Re(s) > 3$, we have the Euler product

$$\widetilde{\mathcal{F}}_j(s) = \prod_p \left(1 + \sum_{k\geq 1} \frac{\lambda^j_{\mathrm{sym}^2 f}(p^k)v(p^k)}{p^{ks}}\right). \tag{16}$$

We only give the proof of the case $j = 8$, since other cases can be handled by a similar argument. For $j = 8$, from [30, (13)] and [31, Lemma 2.1], we learn that

$$\begin{aligned}
\lambda^8_{\mathrm{sym}^2 f}(p) &= 91 + 232\lambda_{\mathrm{sym}^2 f}(p) + 280\lambda_{\mathrm{sym}^4 f}(p) + 238\lambda_{\mathrm{sym}^6 f}(p) + 154\lambda_{\mathrm{sym}^8 f}(p) \\
&\quad +76\lambda_{\mathrm{sym}^{10} f}(p) + 28\lambda_{\mathrm{sym}^{12} f}(p) + 7\lambda_{\mathrm{sym}^{14} f}(p) + \lambda_{\mathrm{sym}^{16} f}(p).
\end{aligned} \tag{17}$$

For $\Re(s) > 3$, the L-function

$$\begin{aligned}
\widetilde{G}_8(s) &:= \zeta(s)^{91} L(s-2,\chi)^{91} \prod_{\chi}^{*} L(\mathrm{sym}^2 f,s)^{232} L(\mathrm{sym}^4 f,s)^{280} L(\mathrm{sym}^6 f,s)^{238} \\
&\quad \times L(\mathrm{sym}^8 f,s)^{154} L(\mathrm{sym}^{10} f,s)^{76} L(\mathrm{sym}^{12} f,s)^{28} L(\mathrm{sym}^{14} f,s)^7 \\
&\quad \times L(\mathrm{sym}^{16} f,s)
\end{aligned} \tag{18}$$

can be represented as

$$\widetilde{G}_8(s) := \prod_p \left(1 + \sum_{k\geq 1} \frac{h(p^k)}{p^{ks}}\right). \tag{19}$$

It is not hard to find that

$$\begin{aligned}\lambda^8_{\mathrm{sym}^2 f}(p)v(p) &= \lambda^8_{\mathrm{sym}^2 f}(p)(1+p^2\chi(p)) \\ &= \big(91+232\lambda_{\mathrm{sym}^2 f}(p)+280\lambda_{\mathrm{sym}^4 f}(p)+238\lambda_{\mathrm{sym}^6 f}(p) \\ &\quad +154\lambda_{\mathrm{sym}^8 f}(p)+76\lambda_{\mathrm{sym}^{10} f}(p)+28\lambda_{\mathrm{sym}^{12} f}(p) \\ &\quad +7\lambda_{\mathrm{sym}^{14} f}(p)+\lambda_{\mathrm{sym}^{16} f}(p)\big)(1+p^2\chi(p)) \\ &= h(p). \qquad (20)\end{aligned}$$

Putting Equations (16)-(20) together, for $\Re(s) > 3$ we obtain

$$\begin{aligned}\widetilde{\mathcal{F}}_8(s) &= \widetilde{G}_8(s)\times\prod_p\left(1+\frac{\lambda^8_{\mathrm{sym}^2 f}(p^2)v(p^2)-h(p^2)}{p^{2s}}+\cdots\right) \\ &:= \zeta(s)^{91}L(s-2,\chi)^{91}\prod_\chi^{*}L(\mathrm{sym}^2 f,s)^{232}L(\mathrm{sym}^4 f,s)^{280}L(\mathrm{sym}^6 f,s)^{238} \\ &\quad \times L(\mathrm{sym}^8 f,s)^{154}L(\mathrm{sym}^{10} f,s)^{76}L(\mathrm{sym}^{12} f,s)^{28}L(\mathrm{sym}^{14} f,s)^{7} \\ &\quad \times L(\mathrm{sym}^{16} f,s)H_8(s).\end{aligned}$$

By Equation (7) and the bound $v(n) \ll n^{2+\varepsilon}$ for any $\varepsilon > 0$, it follows that $\widetilde{H}_8(s)$ converges uniformly and absolutely in the half-plane $\Re(s) \geq \frac{5}{2}+\varepsilon$ for any $\varepsilon > 0$. □

Lemma 3. *Let $i, j \geq 2$ be any two fixed integers. Let $f \in H^*_{k_1}$ and $g \in H^*_{k_2}$ be two distinct Hecke eigenforms. Define*

$$\mathcal{F}_{f,g,i,j}(s) := \sum_{n=1}^{\infty}\frac{\lambda^2_{sym^i f}(n)\lambda^2_{sym^j g}(n)l(n)}{n^s}.$$

Then

$$\mathcal{F}_{f,g,i,j}(s) = G_{i,j}(s)H_{i,j}(s),$$

where

$$\begin{aligned}G_{i,j}(s) &= \zeta(s-2)L(s,\chi)\prod_\chi{}' \Bigg\{\prod_{i_1=1}^{i}\prod_{j_1=1}^{j}L(sym^{2i_1}f,s)L(sym^{2j_1}g,s) \\ &\quad \times L(sym^{2i_1}f\otimes sym^{2j_1}g,s)\Bigg\},\end{aligned}$$

and χ is a non-principal Dirichlet character modulo 4. The function $H_{i,j}(s)$ admits a Dirichlet series which converges uniformly and absolutely in the half-plane $\Re(s) \geq \frac{5}{2}+\varepsilon$, and $H_{i,j}(s) \neq 0$ for $\Re(s) = 3$.

Proof. This can be proved by an argument similar to of Lemma 2.1 by noting that

$$\begin{aligned}
\lambda^2_{\mathrm{sym}^i f}(p)\lambda^2_{\mathrm{sym}^j g}(p)l(p) &= \lambda_f^2(p^i)\lambda_g^2(p^j)l(p)\\
&= \Big(1+\sum_{i_1=1}^{i}\lambda_f(p^{2i_1})\Big)\Big(1+\sum_{j_1=1}^{j}\lambda_g(p^{2j_1})\Big)(p^2+\chi(p))\\
&= \Big(1+\sum_{i_1=1}^{i}\lambda_{\mathrm{sym}^{2i_1}f}(p)\Big)\Big(1+\sum_{j_1=1}^{j}\lambda_{\mathrm{sym}^{2j_1}g}(p)\Big)\\
&\quad \times(p^2+\chi(p)).
\end{aligned}$$

This completes the proof of the lemma. ☐

Lemma 4. *Let $3 \le j \le 8$ be any given integer. Let $f \in H^*_{k_1}$ be a Hecke eigenform. Define*

$$\widetilde{\mathcal{F}}_{f,g,i,j}(s) := \sum_{n=1}^{\infty}\frac{\lambda^2_{sym^i f}(n)\lambda^2_{sym^j g}(n)v(n)}{n^s}.$$

Then

$$\widetilde{\mathcal{F}}_{f,g,i,j}(s) = \widetilde{G}_{i,j}(s)\widetilde{H}_{i,j}(s),$$

where

$$\begin{aligned}
\widetilde{G}_{i,j}(s) &= \zeta(s)L(s-2,\chi)\prod_{\chi}^{*}\Big\{\prod_{i_1=1}^{i}\prod_{j_1=1}^{j}L(sym^{2i_1}f,s)L(sym^{2j_1}g,s)\\
&\quad \times L(sym^{2i_1}f\otimes sym^{2j_1}g,s)\Big\},
\end{aligned}$$

and χ is a non-principal Dirichlet character modulo 4. The function $\widetilde{H}_{i,j}(s)$ admits a Dirichlet series which converges uniformly and absolutely in the half-plane $\Re(s) \ge \frac{5}{2}+\varepsilon$, and $\widetilde{H}_{i,j}(s) \ne 0$ for $\Re(s) = 3$.

Proof. This can be proved using an approach similar to Lemma 3. ☐

Lemma 5. *For any $\varepsilon > 0$, we have*

$$\int_1^T \left|\zeta\Big(\frac{1}{2}+it\Big)\right|^{12} dt \ll T^{2+\varepsilon}, \tag{21}$$

uniformly for $T \ge 1$, and

$$\zeta(\sigma+it) \ll \big(1+|t|\big)^{\max\{\frac{13}{42}(1-\sigma),0\}+\varepsilon}, \tag{22}$$

$$L(\sigma+it,\chi) \ll \big(1+|t|\big)^{\max\{\frac{1}{3}(1-\sigma),0\}+\varepsilon}, \tag{23}$$

uniformly for $\frac{1}{2} \le \sigma \le 1+\varepsilon$ and $|t| \ge 1$.

Proof. This first result is given by Heath-Brown [9], the second result is the recent breakthrough due to Bourgain [1, Theorem 5], and the third result follows from Heath-Brown [10] and the Phragmén-Lindelöf principle for a strip [20, Theorem 5.53]. □

Lemma 6. *For any $\varepsilon > 0$, we have*

$$L(sym^2 f, \sigma + it) \ll \big(1 + |t|\big)^{\max\{\frac{6}{5}(1-\sigma), 0\}+\varepsilon}$$

uniformly for $\frac{1}{2} \leq \sigma \leq 2$ and $|t| \geq 1$.

Proof. The result follows from the recent work of Lin, Nunes, and Qi [32, Corollary 1.2] and the Phragmén-Lindelöf convexity principle for a strip. □

We state some basic definitions and analytic properties of *general* L-functions. Let $L(\phi, s)$ be a Dirichlet series (associated with the object ϕ) that admits an Euler product of degree $m \geq 1$, namely

$$L(\phi, s) = \sum_{n=1}^{\infty} \frac{\lambda_\phi(n)}{n^s} = \prod_{p<\infty} \prod_{j=1}^{m} \left(1 - \frac{\alpha_\phi(p, j)}{p^s}\right)^{-1},$$

where $\alpha_\phi(p, j), j = 1, 2, \cdots, m$ are the local parameters of $L(\phi, s)$ at a finite prime p. Suppose that this series and its Euler product are absolutely convergent for $\Re(s) > 1$. We denote the gamma factor by

$$L_\infty(\phi, s) = \prod_{j=1}^{m} \pi^{-\frac{s+\mu_\phi(j)}{2}} \Gamma\left(\frac{s + \mu_\phi(j)}{2}\right)$$

with local parameters $\mu_\phi(j), j = 1, 2, \cdots, m$, of $L(\phi, s)$ at ∞. The complete L-function $\Lambda(\phi, s)$ is defined by

$$\Lambda(\phi, s) = q(\phi)^{\frac{s}{2}} L_\infty(\phi, s) L(\phi, s),$$

where $q(\phi)$ is the conductor of $L(\phi, s)$. We assume that $\Lambda(\phi, s)$ admits an analytic continuation to the whole complex plane $\mathbb{C}$ and is holomorphic everywhere except for possible poles of finite order at $s = 0, 1$. Furthermore, we assume that it satisfies a functional equation of the Riemann-type

$$\Lambda(\phi, s) = \epsilon_\phi \Lambda(\tilde{\phi}, 1 - s)$$

where ϵ_ϕ is the root number with $|\epsilon_\phi| = 1$ and $\tilde{\phi}$ is the dual of ϕ such that $\lambda_{\tilde{\phi}}(n) = \overline{\lambda_\phi(n)}, L_\infty(\tilde{\phi}, s) = L_\infty(\phi, s)$ and $q(\tilde{\phi}) = q(\phi)$. We write $\phi \in S_e^{\#}$ if it is endowed with the above conditions. We say the L-function $L(\phi, s)$ satisfies the Ramanujan conjecture if $\lambda_\phi(n) \ll n^\varepsilon$ for any ε.

From above, we note that the L-functions $L(\mathrm{sym}^j f, s)$ and $L(\mathrm{sym}^i f \otimes \mathrm{sym}^j g, s)$, and their twisted L-functions are the general L-functions in the sense of Perelli [35]. For these L-functions, we have the following individual or averaged convexity bounds.

Lemma 7. *Let χ be a primitive character modulo q. For the general L-functions $\mathfrak{L}^{\mathbf{d}}_{\mathbf{m},\mathbf{n}}(s,\chi)$ of degree $2A$ indicated above, we have*

$$\int_T^{2T} |\mathfrak{L}^{\mathbf{d}}_{\mathbf{m},\mathbf{n}}(\sigma+it,\chi)|^2 dt \ll (qT)^{2A(1-\sigma)+\varepsilon} \tag{24}$$

uniformly for $\frac{1}{2} \leq \sigma \leq 1$ and $T \geq 1$. Furthermore,

$$\mathfrak{L}^{\mathbf{d}}_{\mathbf{m},\mathbf{n}}(\sigma+it,\chi) \ll (q(|t|+1))^{\max\{A(1-\sigma),0\}+\varepsilon} \tag{25}$$

uniformly for $-\varepsilon \leq \sigma \leq 1+\varepsilon$.

Proof. This can be derived by following an argument similar to that of Zou et al. [55], which was originally deduced from Jiang and Lü [24]. □

3. Proofs of Theorems 1 and 2

We only give the proof of Theorem 1, since Theorem 2 can be handled by a similar approach. In this section, we only give the proof of the case $j=8$ in Theorem 1 in detail, since other cases can be handled by a similar approach.

Proof of Theorem 1*, Case $j=8$.* From Equation (10), we know that

$$S^*_{f,j}(x) = 16\sum_{n\leq x} \lambda^j_{\mathrm{sym}^2 f}(n) l(n) - 4\sum_{n\leq x} \lambda^j_{\mathrm{sym}^2 f}(n) v(n). \tag{26}$$

Firstly, we consider the sum $16\sum_{n\leq x}\lambda^j_{\mathrm{sym}^2 f}(n)l(n)$. For $j=8$, by applying Perron's formula [20, Proposition 5.54] for the generating function $\mathcal{F}_8(s)$ in Lemma 1, we get

$$16\sum_{n\leq x} \lambda^8_{\mathrm{sym}^2 f}(n) l(n) = \frac{16}{2\pi i}\int_{\eta-iT}^{\eta+iT} \mathcal{F}_8(s)\frac{x^s}{s}ds + O\left(\frac{x^{3+\varepsilon}}{T}\right), \tag{27}$$

where $\eta = 3+\varepsilon$, and $6 \leq T \leq x$ is some parameter to be chosen later.

By shifting the line of integration in Equation (27) to the parallel line with $\Re(s) = \kappa := \frac{5}{2}+\varepsilon$, and using Cauchy's residue theorem, we obtain

$$\begin{aligned} 16\sum_{n\leq x} \lambda^8_{\mathrm{sym}^2 f}(n) l(n) &= 16\mathrm{Res}_{s=2}\left\{\mathcal{F}_8(s)\frac{x^s}{s}\right\} + O\left(\frac{x^{3+\varepsilon}}{T}\right) \\ &\quad + \frac{16}{2\pi i}\left\{\int_{\kappa-iT}^{\kappa+iT} + \int_{\kappa+iT}^{\eta+iT} + \int_{\eta-iT}^{\kappa-iT}\right\}\mathcal{F}_8(s)\frac{x^s}{s}ds \end{aligned}$$

$$:= x^3 P_8^*(\log x) + J_1 + J_2 + J_3 + O\left(\frac{x^{3+\varepsilon}}{T}\right), \tag{28}$$

where $P_8^*(t)$ is a polynomial in t of degree 90. In fact, the residue of the integrand coming from the pole at $s = 3$ with order 91, which is derived from the factor $\zeta(s-2)$.

Next we evaluate the integrals J_1, J_2 and J_3. Let

$$G_8^*(s) \quad = \quad \zeta(s)^{91} L(\mathrm{sym}^2 f, s)^{232} L_8(s),$$

where

$$\begin{aligned} L_8(s) \quad = \quad & L(\mathrm{sym}^4 f, s)^{280} L(\mathrm{sym}^6 f, s)^{238} L(\mathrm{sym}^8 f, s)^{154} L(\mathrm{sym}^{10} f, s)^{76} \\ & \times L(\mathrm{sym}^{12} f, s)^{28} L(\mathrm{sym}^{14} f, s)^{7} L(\mathrm{sym}^{16} f, s) \end{aligned}$$

is an L-function of degree $3^8 - 787 = 5774$.

For J_1, by Lemmas 5 and 6 and Equation (24), along with Hölder's inequality, we have

$$\begin{aligned} J_1 \quad \ll \quad & x^{\frac{5}{2}+\varepsilon} \log T \max_{1 \le T_1 \le T} \left\{ T_1^{-1} \int_{T_1/2}^{T_1} \left| G_8^*\left(\frac{1}{2}+it\right)\right| dt \right\} + x^{\frac{5}{2}+\varepsilon} \\ \ll \quad & x^{\frac{5}{2}+\varepsilon} \log T \max_{1 \le T_1 \le T} \left\{ \frac{1}{T_1} \left(\int_{T_1/2}^{T_1} \left| \zeta\left(\frac{1}{2}+it\right)^{91} \right|^{12} dt \right)^{\frac{1}{12}} \right. \\ & \times \left(\int_{T_1/2}^{T_1} \left| L\left(\mathrm{sym}^2 f, \frac{1}{2}+it\right)^{232} \right|^{\frac{12}{5}} dt \right)^{\frac{5}{12}} \\ & \left. \times \left(\int_{T_1/2}^{T_1} \left| L_8\left(\frac{1}{2}+it\right) \right|^2 dt \right)^{\frac{1}{2}} \right\} + x^{\frac{5}{2}+\varepsilon} \\ \ll \quad & x^{\frac{5}{2}+\varepsilon} \log T \max_{1 \le T_1 \le T} \left\{ \frac{1}{T_1} \left(\max_{T_1/2 \le t \le T_1} \left| \zeta\left(\frac{1}{2}+it\right) \right|^{1080} \right. \right. \\ & \times \left. \int_{T_1/2}^{T_1} \left| \zeta\left(\frac{1}{2}+it\right) \right|^{12} dt \right)^{\frac{1}{12}} \left(\max_{T_1/2 \le t \le T_1} \left| L\left(\mathrm{sym}^2 f, \frac{1}{2}+it\right) \right|^{2774/5} \right. \\ & \times \left. \left. \int_{T_1/2}^{T_1} \left| L\left(\mathrm{sym}^2 f, \frac{1}{2}+it\right) \right|^2 dt \right)^{\frac{5}{12}} \left(\int_{T_1/2}^{T_1} \left| L_8\left(\frac{1}{2}+it\right) \right|^2 dt \right)^{\frac{1}{2}} \right\} + x^{\frac{5}{2}+\varepsilon} \\ \ll \quad & x^{\frac{5}{2}+\varepsilon} T^{-1+(\frac{13}{42}\times\frac{1}{2}\times 1080+2)\times\frac{1}{12}+(\frac{6}{5}\times\frac{1}{2}\times\frac{2774}{5}+\frac{1}{2}\times 3)\times\frac{5}{12}+\frac{1}{2}\times\frac{1}{2}\times 5774+\varepsilon} \\ \ll \quad & x^{\frac{5}{2}+\varepsilon} T^{\frac{1340573}{840}+\varepsilon}. \end{aligned} \tag{29}$$

For the integrals over the horizontal segments J_2 and J_3, by Equations (22) and

(25), along with Lemma 3, we have

$$
\begin{aligned}
J_2+J_3 &\ll \int_{\frac{1}{2}+\varepsilon}^{1+\varepsilon} x^{\sigma+2}\left|\zeta(\sigma+it)^{91}L(\mathrm{sym}^2 f,\sigma+it)^{232}L_8(\sigma+it)\right|T^{-1}d\sigma \\
&\ll \max_{\frac{1}{2}+\varepsilon\le\sigma\le 1+\varepsilon} x^{\sigma+2}T^{(\frac{13}{42}\times 91+\frac{6}{5}\times 232+\frac{1}{2}\times 5774)(1-\sigma)+\varepsilon}T^{-1} \\
&\ll \frac{x^{3+\varepsilon}}{T}+x^{\frac{5}{2}+\varepsilon}T^{\frac{95747}{60}+\varepsilon}. \qquad (30)
\end{aligned}
$$

Combining Equations (27)-(30), we obtain

$$
16\sum_{n\le x}\lambda^8_{\mathrm{sym}^2 f}(n)l(n)=x^3P_8^*(\log x)+O\left(\frac{x^{3+\varepsilon}}{T}\right)+O\big(x^{\frac{5}{2}+\varepsilon}T^{\frac{1340573}{840}+\varepsilon}\big).
$$

On taking $\frac{x^3}{T}=x^{\frac{5}{2}}T^{\frac{1340573}{840}}$, i.e., $T=x^{\frac{420}{1341413}}$, we get

$$
16\sum_{n\le x}\lambda^8_{\mathrm{sym}^2 f}(n)l(n)=x^3P_8^*(\log x)+O\big(x^{\frac{4023819}{1341413}+\varepsilon}\big). \qquad (31)
$$

Now we compute the explicit form of the coefficients of the polynomial $P_8^*(\log x)$. From [19, (1.11)] we learn that $\zeta(s)$ has the Laurent expansion at the simple pole $s=1$:

$$
\zeta(s)=\frac{1}{s-1}+\gamma_0+\sum_{n=1}^{\infty}\gamma_j(s-1)^j,
$$

where $\gamma_j, j=0,1,\ldots$ are suitable constants. In particular, $\gamma:=\gamma_0$ is Euler's constant.

By the Leibniz rule and the method for the computation of residue at the pole $s=3$ for an integrand function, we have

$$
\begin{aligned}
x^3P_8^*(\log x) &= 16\mathrm{Res}_{s=3}\left\{\mathcal{F}_8(s)\frac{x^s}{s}\right\} \\
&= \frac{16}{3}\cdot\frac{1}{90!}L(3,\chi)^{91}L(\mathrm{sym}^2 f,1)^{232}L(\mathrm{sym}^4 f,1)^{280}L(\mathrm{sym}^6 f,1)^{238} \\
&\quad\times L(\mathrm{sym}^8 f,1)^{154}L(\mathrm{sym}^{10} f,1)^{76}L(\mathrm{sym}^{12} f,1)^{28}L(\mathrm{sym}^{14} f,1)^{7} \\
&\quad\times L(\mathrm{sym}^{16} f,1)L(\mathrm{sym}^2 f\otimes\chi,3)^{232}L(\mathrm{sym}^4 f\otimes\chi,3)^{280} \\
&\quad\times L(\mathrm{sym}^6 f\otimes\chi,3)^{238}L(\mathrm{sym}^8 f\otimes\chi,3)^{154} \\
&\quad\times L(\mathrm{sym}^{10} f\otimes\chi,3)^{76}L(\mathrm{sym}^{12} f\otimes\chi,3)^{28}L(\mathrm{sym}^{14} f\otimes\chi,3)^{7} \\
&\quad\times L(\mathrm{sym}^{16} f\otimes\chi,3)H_8(3)x^3(\log x)^{90}+\ldots+c_f^*x^3,
\end{aligned}
$$

where c_f^* is some suitable constant depending on f and various associated L-functions.

Similarly, for $j = 8$, by applying Perron's formula [20, Proposition 5.54] for the generating function $\widetilde{\mathcal{F}}_8(s)$ in Lemma 2, we get

$$4\sum_{n\leq x}\lambda^8_{\mathrm{sym}^2 f}(n)v(n) = \frac{4}{2\pi i}\int_{\eta-iT}^{\eta+iT}\widetilde{\mathcal{F}}_8(s)\frac{x^s}{s}ds + O\bigg(\frac{x^{3+\varepsilon}}{T}\bigg), \tag{32}$$

where $\eta = 3+\varepsilon$, and $6 \leq T \leq x$ is some parameter to be chosen later.

By shifting the line of integration in Equation (32) to the parallel line with $\Re(s) = \kappa := \frac{5}{2}+\varepsilon$ and using Cauchy's residue theorem, we obtain

$$\begin{aligned}4\sum_{n\leq x}\lambda^8_{\mathrm{sym}^2 f}(n)v(n) &= \frac{4}{2\pi i}\bigg\{\int_{\kappa-iT}^{\kappa+iT}+\int_{\kappa+iT}^{\eta+iT}+\int_{\eta-iT}^{\kappa-iT}\bigg\}\mathcal{F}_8(s)\frac{x^s}{s}ds\\ &\quad +O\bigg(\frac{x^{3+\varepsilon}}{T}\bigg),\\ &:= I_1+I_2+I_3+O\bigg(\frac{x^{3+\varepsilon}}{T}\bigg),\end{aligned} \tag{33}$$

since in this case there is no singularity in the rectangle obtained, and the integrand $\mathcal{F}_8(s)\frac{x^s}{s}$ is analytic in this region.

$$\widetilde{G}^*_8(s) = L(s,\chi)^{91}\widetilde{L}_8(s),$$

where

$$\begin{aligned}\widetilde{L}_8(s) &= L(\mathrm{sym}^2 f\otimes\chi,s)^{232}L(\mathrm{sym}^4 f\otimes\chi,s)^{280}L(\mathrm{sym}^6 f\otimes\chi,s)^{238}\\ &\quad\times L(\mathrm{sym}^8 f\otimes\chi,s)^{154}L(\mathrm{sym}^{10} f\otimes\chi,s)^{76}\\ &\quad\times L(\mathrm{sym}^{12} f\otimes\chi,s)^{28}L(\mathrm{sym}^{14} f\otimes\chi,s)^{7}L(\mathrm{sym}^{16} f\otimes\chi,s)\end{aligned}$$

is an L-function of degree $3^8 - 91 = 6470$.

For the integrals I_2 and I_3 over the horizontal segments, by Equations (23) and (25), we have

$$\begin{aligned}I_2+I_3 &\ll \int_{\frac{1}{2}+\varepsilon}^{1+\varepsilon}T^{-1}|\widetilde{G}^*_8(\sigma+it)|x^{\sigma+2}d\sigma\\ &\ll \max_{\frac{1}{2}+\varepsilon\leq\sigma\leq 1+\varepsilon}x^{\sigma+2}T^{(\frac{1}{3}\times 91+\frac{1}{2}\times 6470)(1-\sigma)+\varepsilon}T^{-1}\\ &\ll \frac{x^{3+\varepsilon}}{T}+x^{\frac{5}{2}+\varepsilon}T^{\frac{9790}{6}+\varepsilon}.\end{aligned} \tag{34}$$

For the integral I_1 over the vertical segment, by Equation (24) and the Cauchy-

Schwarz inequality, we get

$$\begin{aligned} I_1 &\ll x^{\frac{5}{2}+\varepsilon} \max_{1\le T_1\le T/2}\left\{T^{-1}\left(\int_{T_1}^{2T_1}\left|L\left(\frac{1}{2}+it,\chi\right)^{91}\right|^2 dt\right)^{\frac{1}{2}}\right. \\ &\quad \left.\times\left(\int_{T_1}^{2T_1}\left|\widetilde{L}_8\left(\frac{1}{2}+it,\chi\right)\right|^2 dt\right)^{\frac{1}{2}}\right\} + x^{\frac{5}{2}+\varepsilon} \\ &\ll x^{\frac{5}{2}+\varepsilon}T^{\frac{6557}{4}+\varepsilon}. \end{aligned} \tag{35}$$

Putting Equations (33)-(35) together, we obtain

$$4\sum_{n\le x}\lambda^8_{\mathrm{sym}^2 f}(n)v(n) = O\left(\frac{x^{3+\varepsilon}}{T}\right) + O\big(x^{\frac{5}{2}+\varepsilon}T^{\frac{6557}{4}+\varepsilon}\big). \tag{36}$$

Now we choose $x^{\frac{5}{2}+\varepsilon}T^{\frac{6557}{4}} = \frac{x^3}{T}$, i.e., $T = x^{\frac{2}{6561}}$, we get

$$4\sum_{n\le x}\lambda^8_{\mathrm{sym}^2 f}(n)v(n) = O\big(x^{\frac{19681}{6561}+\varepsilon}\big). \tag{37}$$

Combining Equations (26), (31) and (37), we get

$$S^*_{f,j}(x) = x^3 P^*_8(\log x) + O\big(x^{\frac{19681}{6561}+\varepsilon}\big).$$

This completes the proof of Theorem 1. □

Acknowledgements. The author would like to extend his gratitude to Professors Guangshi Lü, Bin Chen, Bingrong Huang, Yujiao Jiang, Research fellow Zhiwei Wang, and Dr. Wei Zhang, for their constant encouragement and valuable suggestions. The author is extremely grateful to the anonymous referees for their meticulous checking, for thoroughly reporting countless typos and inaccuracies as well as for their valuable comments. These corrections and additions have made the manuscript clearer and more readable. This work was financially supported in part by The National Key Research and Development Program of China (Grant No. 2021YFA1000700), Natural Science Basic Research Program of Shaanxi (Program Nos. 2023-JC-QN-0024, 2023-JC-YB-077), Foundation of Shaanxi Educational Committee (2023-JC-YB-013) and Shaanxi Fundamental Science Research Project for Mathematics and Physics (Grant No. 22JSQ010).

References

[1] J. Bourgain, Decoupling, exponential sums and the Riemann zeta function, *J. Amer. Math. Soc.* **30** (2017), 205-224.

[2] L. Clozel and J. A. Thorne, Level-raising and symmetric power functoriality. I, *Compos. Math.* **150** (2014), 729-748.

[3] L. Clozel and J. A. Thorne, Level-raising and symmetric power functoriality. II, *Ann. of Math.* **181** (2015), 303-359.

[4] L. Clozel and J. A. Thorne, Level-raising and symmetric power functoriality. III, *Duke Math. J.* **166** (2017), 325-402.

[5] P. Deligne, La conjecture de Weil. I, *Inst. Hautes Études Sci. Publ. Math.* **43** (1974), 273-307.

[6] O. M. Fomenko, Identities involving the coefficients of automorphic L-functions, *J. Math. Sci.* **133** (2006), 1749-1755.

[7] O. M. Fomenko, Mean value theorems for automorphic L-functions, *St. Petersburg Math. J.* **19** (2008), 853-866.

[8] S. Gelbart and H. Jacquet, A relation between automorphic representations of $GL(2)$ and $GL(3)$, *Ann. Sci. École Norm. Sup.* **11** (1978), 471-542.

[9] D. R. Heath-Brown, The twelfth power moment of Riemann-function, *Quart. J. Math. (Oxford)* **29** (1978), 443-462.

[10] D. R. Heath-Brown, The growth rate of the Dedekind zeta-function on the critical line, *Acta Arith* **49** (4) (1988), 323-339.

[11] X. G. He, Integral power sums of Fourier coefficients of symmetric square L-functions, *Proc. Amer. Math. Soc.* **147** (2019), 2847-2856.

[12] G. W. Hu, Y. J. Jiang, and G. S. Lü, The Fourier coefficients of Θ-series in arithmetic progressions, *Mathematika* **66** (1) (2020), 39-55.

[13] G. D. Hua, Sums of Fourier coefficients associated with Hecke cusp forms, *Lith. Math. J.* **62** (3) (2022), 349-359.

[14] G. D. Hua, The average behaviour of Hecke eigenvalues over certain sparse sequence of positive integers, *Res. Number Theory* **8** (4) (2022), Paper No. 95, 20 pp.

[15] G. D. Hua, Discrete mean square of the coefficients of triple product L-functions over certain sparse sequence, *Bull. Sci. Math.* **180** (2022), 12 pp.

[16] G. D. Hua, On general divisor problems of Hecke eigenvalues of cusp forms, *Period. Math. Hung.* **87** (2023), 340-350.

[17] G. D. Hua, Bin Chen, L. J. Pan, and X. F. Chen, On higher moments of Dirichlet coefficients attached to symmetric square L-functions over certain sparse sequence, *Rend. Circ. Mat. Palermo, II. Ser.* **72** (2) (2023), 4195-4208.

[18] A. Ivić, On zeta-functions associated with Fourier coefficients of cusp forms, in: *Proc. Amal. Conf. Analytic Number Theory (Maiori, 1989)*, Univ. Saleron, Salerno, 1992, pp. 231-246.

[19] A. Ivić, *The Riemann zeta-function. Theory and applications*, Dover Publications, New York, 2003.

[20] H. Iwaniec and E. Kowalski, *Analytic Number Theory*, Amer. Math. Soc. Colloquium Publ., No. **53**, Amer. Math. Soc. Providence, RI, 2004.

[21] H. Jacquet and J. A. Shalika, On Euler products and the classification of automorphic representations. I, *Amer. J. Math.* **103** (3) (1981), 499-558.

[22] H. Jacquet and J. A. Shalika, On Euler products and the classification of automorphic representations. II, *Amer. J. Math.* **103** (4) (1981), 777-815.

[23] H. Jacquet, I. I. Piatetski, and J. A. Shalika, Rankin-Selberg convolutions, *Amer. J. Math.* **105** (2) (1983), 367-464.

[24] Y. J. Jiang and G. S. Lü, On the higher mean over arithmetic progressions of Fourier coefficients of cusp form, *Acta Arith.* **166** (3) (2014), 231-252.

[25] Y. J. Jiang and G. S. Lü, Uniform estimates for sums of coefficients of symmetric square L-function, *J. Number Theory* **148** (2015), 220-234.

[26] H. Kim and F. Shahidi, Cuspidality of symmetric power with applications, *Duke Math. J.* **112** (2002), 177-197.

[27] H. Kim and F. Shahidi, Functorial products for $GL_2 \times GL_3$ and functorial symmetric cube for GL_2, with an appendix by C. J. Bushnell and G. Heniart, *Ann. of Math.* **155** (2002), 837-893.

[28] H. Kim, Functoriality for the exterior square of GL_4 and symmetric fourth of GL_2, Appendix 1 by D. Ramakrishan, Appendix 2 by H. Kim and P. Sarnak, *J. Amer. Math. Soc.* **16** (2003), 139-183.

[29] G. S. Lü, Uniform estimates for sums of Fourier coefficients of cusp forms, *Acta Math. Hugar.* **124** (2009), 83-97.

[30] Y.-K. Lau and G. S. Lü, Sums of Fourier coefficients of cusp forms, *Quart. J. Math.* **62** (2011), 687-716.

[31] S. Luo, H. X. Lao, and A. Y. Zou, Asymptotics for Dirichlet coefficients of symmetric power L-functions, *Acta Arith.* **199** (2021), 253-268.

[32] Y. X. Lin, R. Nunes, and Z. Qi, Strong subconvexity for self-dual $GL(3)$ L-functions, *Int. Math. Res. Not.* **13** (2023), 11453-11470.

[33] J. Newton and A. Thorne, Symmetric power functoriality for holomorphic modular forms, *Publ. Math. Inst. Hautes Études Sci.* **134** (2021), 1-116.

[34] J. Newton and A. Thorne, Symmetric power functoriality for holomorphic modular forms. II, *Publ. Math. Inst. Hautes Études Sci.* **134** (2021), 117-152.

[35] A. Perelli, General L-functions, *Ann. Mat. Pura Appl* **130** (4) (1982), 287-306.

[36] R. A. Rankin, Contributions to the theory of Ramanujan's function $\tau(n)$ and similar arithmetical functions II. The order of the Fourier coefficients of the integral modular forms, *Proc. Cambridge Philos. Soc.* **35** (1939), 357-372.

[37] Z. Rudnick and P. Sarnak, Zeros of principal L-functions and random matrix theory, *Duke Math. J.* **81** (2) (1996), 269-322.

[38] A. Selberg, Bemerkungen über eine Dirichletsche, die mit der Theorie der Modulformen nahe verbunden ist, *Arch. Math. Naturvid.* **43** (1940), 47-50.

[39] F. Shahidi, On certain L-functions, *Amer. J. Math.* **103** (2) (1981), 297-355.

[40] F. Shahidi, Fourier transforms of intertwining operators and Plancherel measure for $GL(n)$, *Amer. J. Math.* **106** (1) (1984), 67-111.

[41] F. Shahidi, Local coefficients as Artin factors for real groups, *Duke Math. J.* **52** (4) (1984), 973-1007.

[42] F. Shahidi, A proof of Langland's conjecture on Plancherel measures; complementary series for p-adic groups, *Ann. of Math.* **132** (2) (1990), 273-330.

[43] F. Shahidi, Third symmetric power L-functions for $GL(2)$, *Compos. Math.* **70** (1989), 245-273.

[44] A. Sankaranarayanan, On a sum involving Fourier coefficients of cusp forms, *Lith. Math. J.* **46** (2006), 459-474.

[45] A. Sankaranarayanan, S. K. Singh, and K. Srinivas, Discrete mean square estimates for coefficients of symmetric power L-functions, *Acta Arith.* **190** (2019), 193-208.

[46] A. Sharma and A. Sankaranarayana, Discrete mean square of the coefficients of symmetric square L-functions on certain sequence of positive numbers, *Res. Number Theory* **8** (1) (2022), Paper No. 19, 13 pp.

[47] A. Sharma and A. Sankaranarayana, Higher moments of the Fourier coefficients of symmetric square L-functions on certain sequence, *Rend. Circ. Mat. Palermo, II. Ser.* **72** (2) (2022), 1399-1416.

[48] A. Sharma and A. Sankaranarayana, Average behavior of the Fourier coefficients of symmtric square L-function over some sequence of integers, *Integers* **22** (2022), #A74, 17 pp.

[49] A. Sharma and A. Sankaranarayana, On the average behavior of the Fourier coefficients of jth symmetric power L-function over a certain sequences of positive integers, *Czech. Math. J.* **73** (3) (2023), 885-901.

[50] H. C. Tang, Estimates for the Fourier coefficients of symmetric square L-functions, *Arch. Math. (Basel)* **100** (2013), 123-130.

[51] H. C. Tang and J. Wu, Fourier coefficients of symmetric power L-functions, *J. Number Theory* **167** (2016), 147-160.

[52] J. Wu, Power sums of Hecke eigenvalues and applications, *Acta Arith.* **137** (2009), 333-344.

[53] C. R. Xu, General asymptotic formula of Fourier coefficients of cusp forms over sum of two squares, *J. Number Theory* **236** (2022), 214-229.

[54] S. Zhai, Average behavior of Fourier coefficients of cusp forms over sum of two squares, *J. Number Theory* **133** (2013), 3862-3876.

[55] A. Y. Zou, H. X. Lao, and L. L. Wei, Distributions of Fourier coefficients of cusp forms over arithmetic progressions, *Integers* **22** (2022), #A10, 16pp.