# INTEGER GROUP DETERMINANTS FOR $\mathrm{C}_4^2$

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**Abstract**

We give a complete description of the integer group determinant for $\mathrm{C}_4^2$, where $\mathrm{C}_4$ is the cyclic group of order 4.

## 1. Introduction

At the meeting of the American Mathematical Society in Hayward, California, in April 1977, Olga Taussky-Todd [11] asked if one could characterize the values that can be obtained as the determinant of an integer circulant matrix [3, 4]. This is the same as asking for the values obtained by the group determinant for a cyclic group when it is evaluated on integers. One may ask the same question for any finite group, not just for the cyclic groups.

Recall that for a finite group $G$, assigning a variable $x_g$ for each $g \in G$, the group determinant of $G$ is defined as $\det \left(x_{gh^{-1}}\right)_{g,h \in G}$. The group determinant is called an integer group determinant of $G$ when the all variables $x_g$ are integers. We denote the set of all integer group determinants of $G$ by $S(G)$:

$$S(G) := \left\{ \det \left(x_{gh^{-1}}\right)_{g,h \in G} \mid x_g \in \mathbb{Z} \right\}.$$

For every group $G$ of order at most 15, $S(G)$ was determined (see [6, 8]). Let $\mathrm{C}_n$ be the cyclic group of order $n$ and $\mathrm{D}_n$ be the dihedral group of order $n$. For the groups of order 16, the complete descriptions of $S(G)$ were obtained for $\mathrm{D}_{16}$ [1, Theorem 5.3], $\mathrm{C}_{16}$ [17], $\mathrm{C}_2^4$ [18] and $\mathrm{C}_8 \times \mathrm{C}_2$ [13, Theorem 1.5]. In this paper, we determine $S\left(\mathrm{C}_4^2\right)$.

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**Theorem 1.** *Let* $P := \{p \mid p \equiv -3 \pmod 8$ *is a prime number*$\}$ *and*

$$\begin{aligned} A := \{&(8k-3)(8l-3)(8m-3)(8n-3) \mid \\ &\quad k \in \mathbb{Z},\ 8l-3, 8m-3, 8n-3 \in P,\ k+l \not\equiv m+n \pmod 2\} \\ \subsetneq &\{16m-7 \mid m \in \mathbb{Z}\}. \end{aligned}$$

*Then we have*

$$S\left(\mathrm{C}_4^2\right) = \left\{16m+1,\ 2^{15}p(2m+1),\ 2^{16}m \mid m \in \mathbb{Z},\ p \in P\right\} \cup A.$$

There are fourteen groups of order 16 up to isomorphism [2, 19], and five of them are abelian. Our result leaves $\mathrm{C}_4 \times \mathrm{C}_2^2$ as the only abelian group of order 16 for which $S(G)$ has not been determined (this group and all of the non-abelian groups of order 16 have recently been resolved as well [5, 9, 10, 12, 15, 16] and [7, Theorems 3.1 and 4.1]).

## 2. Preliminaries

For any $\overline{r} \in \mathrm{C}_n$ with $r \in \{0, 1, \ldots, n-1\}$, we denote the variable $x_{\overline{r}}$ by $x_r$, and let

$$D_n(x_0, x_1, \ldots, x_{n-1}) := \det\left(x_{gh^{-1}}\right)_{g,h \in \mathrm{C}_n}.$$

For any $(\overline{r}, \overline{s}) \in \mathrm{C}_4^2$ with $r, s \in \{0, 1, 2, 3\}$, we denote the variable $y_{(\overline{r},\overline{s})}$ by $y_j$, where $j := r + 4s$, and let

$$D_{4\times 4}(y_0, y_1, \ldots, y_{15}) := \det\left(y_{gh^{-1}}\right)_{g,h \in \mathrm{C}_4^2}.$$

From the $G = \mathrm{C}_4$ and $H = \{\overline{0}, \overline{2}\}$ case of [13, Theorem 1.1], we have the following corollary.

**Corollary 1.** *We have*

$$\begin{aligned} D_4(x_0, x_1, x_2, x_3) &= D_2(x_0 + x_2, x_1 + x_3) D_2\left(x_0 - x_2, \sqrt{-1}(x_1 - x_3)\right) \\ &= \left\{(x_0 + x_2)^2 - (x_1 + x_3)^2\right\}\left\{(x_0 - x_2)^2 + (x_1 - x_3)^2\right\}. \end{aligned}$$

**Remark 1.** From Corollary 1, we have $D_4(x_0, x_1, x_2, x_3) = -D_4(x_1, x_2, x_3, x_0)$.

From the $H = K = \mathrm{C}_4$ case of [14, Theorem 1.1], we have the following corollary.

**Corollary 2.** *Let* $D := D_{4\times 4}(y_0, y_1, \ldots, y_{15})$. *Then we have*

$$D = \prod_{k=0}^{3} D_4\left(\sum_{s=0}^{3} \sqrt{-1}^{ks} y_{4s}, \sum_{s=0}^{3} \sqrt{-1}^{ks} y_{1+4s}, \sum_{s=0}^{3} \sqrt{-1}^{ks} y_{2+4s}, \sum_{s=0}^{3} \sqrt{-1}^{ks} y_{3+4s}\right).$$

Throughout this paper, we assume that $a_0, a_1, \dots, a_{15} \in \mathbb{Z}$, and let

$$\begin{aligned} b_i &:= (a_i + a_{i+8}) + (a_{i+4} + a_{i+12}), && 0 \le i \le 3, \\ c_i &:= (a_i + a_{i+8}) - (a_{i+4} + a_{i+12}), && 0 \le i \le 3, \\ d_i &:= a_i - a_{i+8}, && 0 \le i \le 7, \\ \alpha_i &:= d_i + \sqrt{-1} d_{i+4}, && 0 \le i \le 3. \end{aligned}$$

**Remark 2.** For any $0 \le i \le 3$, the following hold:

(1) $b_i \equiv c_i \equiv d_i + d_{i+4} \pmod 2$;

(2) $b_i + c_i \equiv 2d_i \pmod 4$;

(3) $b_i - c_i \equiv 2d_{i+4} \pmod 4$.

Let

$$\boldsymbol{a} := (a_0, a_1, \dots, a_{15}), \qquad \boldsymbol{b} := (b_0, b_1, b_2, b_3), \qquad \boldsymbol{c} := (c_0, c_1, c_2, c_3),$$
$$\beta := (\alpha_0 + \alpha_2)^2 - (\alpha_1 + \alpha_3)^2, \qquad \gamma := (\alpha_0 - \alpha_2)^2 + (\alpha_1 - \alpha_3)^2.$$

From Corollaries 1 and 2, we have the following relation which will be frequently used in this paper:

$$D_{4\times 4}(\boldsymbol{a}) = D_4(\boldsymbol{b}) D_4(\boldsymbol{c}) \beta\overline{\beta}\gamma\overline{\gamma},$$

where $\overline{\alpha}$ denotes the complex conjugate of $\alpha \in \mathbb{C}$. From

$$\begin{aligned} \beta &= (\alpha_0 + \alpha_2 + \alpha_1 + \alpha_3)(\alpha_0 + \alpha_2 - \alpha_1 - \alpha_3) \\ &= \left\{(d_0 + d_2 + d_1 + d_3) + \sqrt{-1}(d_4 + d_6 + d_5 + d_7)\right\} \\ &\quad \times \left\{(d_0 + d_2 - d_1 - d_3) + \sqrt{-1}(d_4 + d_6 - d_5 - d_7)\right\}, \\ \gamma &= \left\{(\alpha_0 - \alpha_2) + \sqrt{-1}(\alpha_1 - \alpha_3)\right\}\left\{(\alpha_0 - \alpha_2) - \sqrt{-1}(\alpha_1 - \alpha_3)\right\} \\ &= \left\{(d_0 - d_2 - d_5 + d_7) + \sqrt{-1}(d_4 - d_6 + d_1 - d_3)\right\} \\ &\quad \times \left\{(d_0 - d_2 + d_5 - d_7) + \sqrt{-1}(d_4 - d_6 - d_1 + d_3)\right\}, \end{aligned}$$

we have the following lemma.

**Lemma 1.** *The following hold:*

$$\begin{aligned} \beta\overline{\beta} &= \left\{(d_0 + d_2 + d_1 + d_3)^2 + (d_4 + d_6 + d_5 + d_7)^2\right\} \\ &\quad \times \left\{(d_0 + d_2 - d_1 - d_3)^2 + (d_4 + d_6 - d_5 - d_7)^2\right\}, \\ \gamma\overline{\gamma} &= \left\{(d_0 - d_2 - d_5 + d_7)^2 + (d_4 - d_6 + d_1 - d_3)^2\right\} \\ &\quad \times \left\{(d_0 - d_2 + d_5 - d_7)^2 + (d_4 - d_6 - d_1 + d_3)^2\right\}. \end{aligned}$$

**Remark 3.** From Lemma 1, each $\beta\overline{\beta}$ and $\gamma\overline{\gamma}$ is invariant under the replacing of $(d_0, \dots, d_7)$ with $(d_4, d_5, d_6, d_7, d_0, d_1, d_2, d_3)$.

**Lemma 2.** *We have* $D_{4\times 4}(\boldsymbol{a}) \equiv D_4(\boldsymbol{b}) \equiv D_4(\boldsymbol{c}) \equiv \beta\overline{\beta} \equiv \gamma\overline{\gamma} \pmod 2$.

*Proof.* From Corollary 1, it follows that for any $x_0, x_1, x_2, x_3 \in \mathbb{Z}$,

$$D_4(x_0, x_1, x_2, x_3) \equiv x_0 + x_1 + x_2 + x_3 \pmod 2.$$

Also, from Lemma 1, we have

$$\beta\overline{\beta} \equiv d_0 + d_1 + \cdots + d_7 \equiv \gamma\overline{\gamma} \pmod 2.$$

Therefore, from Remark 2 (1), we have $D_4(\boldsymbol{b}) \equiv D_4(\boldsymbol{c}) \equiv \beta\overline{\beta} \equiv \gamma\overline{\gamma} \pmod 2$. □

The following lemma is immediately obtained from Lemma 1.

**Lemma 3.** *We have*

$$\begin{aligned}\beta\overline{\beta} &= \left\{(d_0+d_2)^2+(d_4+d_6)^2+(d_1+d_3)^2+(d_5+d_7)^2\right\}^2\\ &\quad -4\left\{(d_0+d_2)(d_1+d_3)+(d_4+d_6)(d_5+d_7)\right\}^2,\\ \gamma\overline{\gamma} &= \left\{(d_0-d_2)^2+(d_4-d_6)^2+(d_1-d_3)^2+(d_5-d_7)^2\right\}^2\\ &\quad -4\left\{(d_0-d_2)(d_5-d_7)-(d_4-d_6)(d_1-d_3)\right\}^2.\end{aligned}$$

By direct calculation, we have the following lemma.

**Lemma 4.** *We have*

$$\begin{aligned}&\left\{(d_0+d_2)^2+(d_4+d_6)^2+(d_1+d_3)^2+(d_5+d_7)^2\right\}^2\\ &\quad -\left\{(d_0-d_2)^2+(d_4-d_6)^2+(d_1-d_3)^2+(d_5-d_7)^2\right\}^2\\ &= 8\left(d_0^2+d_2^2+d_4^2+d_6^2+d_1^2+d_3^2+d_5^2+d_7^2\right)(d_0d_2+d_4d_6+d_1d_3+d_5d_7).\end{aligned}$$

**Lemma 5.** *The following hold:*

(1) $2(d_0d_2+d_4d_6+d_1d_3+d_5d_7) \equiv b_0b_2+b_1b_3+c_0c_2+c_1c_3 \pmod 4$*;*

(2) $2(d_0d_7+d_2d_5+d_4d_3+d_6d_1) \equiv b_0b_3+b_2b_1-c_0c_3-c_2c_1 \pmod 4$*;*

(3) $2(d_0d_3+d_2d_1+d_4d_7+d_6d_5) \equiv b_0b_3+b_2b_1+c_0c_3+c_2c_1 \pmod 4$*;*

(4) $2(d_0d_5+d_2d_7+d_4d_1+d_6d_3) \equiv b_0b_1+b_2b_3-c_0c_1-c_2c_3 \pmod 4$*;*

(5) $2(d_0d_1+d_2d_3+d_4d_5+d_6d_7) \equiv b_0b_1+b_2b_3+c_0c_1+c_2c_3 \pmod 4$.

*Proof.* We obtain (1) from

$$\begin{aligned}
b_0b_2 + b_1b_3 + c_0c_2 + c_1c_3 &= (a_0 + a_8 + a_4 + a_{12})(a_2 + a_{10} + a_6 + a_{14}) \\
&\quad + (a_1 + a_9 + a_5 + a_{13})(a_3 + a_{11} + a_7 + a_{15}) \\
&\quad + (a_0 + a_8 - a_4 - a_{12})(a_2 + a_{10} - a_6 - a_{14}) \\
&\quad + (a_1 + a_9 - a_5 - a_{13})(a_3 + a_{11} - a_7 - a_{15}) \\
&= 2(a_0 + a_8)(a_2 + a_{10}) + 2(a_4 + a_{12})(a_6 + a_{14}) \\
&\quad + 2(a_1 + a_9)(a_3 + a_{11}) + 2(a_5 + a_{13})(a_7 + a_{15}) \\
&\equiv 2(d_0d_2 + d_4d_6 + d_1d_3 + d_5d_7) \pmod 4.
\end{aligned}$$

In the same way, we can prove (2)–(5). □

Let $P := \{p \mid p \equiv -3 \pmod 8 \text{ is a prime number}\}$. It is well known that a positive integer $n$ is expressible as a sum of two squares if and only if in the prime factorization of $n$, every prime of the form $4k + 3$ occurs an even number of times. From this, we have the following corollary.

**Corollary 3.** *If $a^2 + b^2 \equiv -3 \pmod 8$, then there exist $k \in \mathbb{Z}$ and $8l - 3 \in P$ satisfying $a^2 + b^2 = (8k + 1)(8l - 3)$.*

## 3. Integer Group Determinant of $C_4$

**Lemma 6** ([13, Lemmas 4.6 and 4.7])**.** *For any $k, l, m, n \in \mathbb{Z}$, the following hold:*

(1) $D_4(2k + 1, 2l, 2m, 2n) \equiv 8m + 1 \pmod{16}$;

(2) $D_4(2k, 2l + 1, 2m + 1, 2n + 1) \equiv 8(k + l + n) - 3 \pmod{16}$.

Let $\mathbb{Z}_{\text{odd}}$ be the set of all odd numbers.

**Lemma 7.** *For any $k, l, m, n \in \mathbb{Z}$, the following hold:*

(1) $D_4(2k, 2l, 2m, 2n) \in \begin{cases} 2^4\mathbb{Z}_{\text{odd}}, & k + m \not\equiv l + n \pmod 2, \\ 2^8\mathbb{Z}, & k + m \equiv l + n \pmod 2; \end{cases}$

(2) $D_4(2k+1, 2l+1, 2m+1, 2n+1) \in \begin{cases} 2^4\mathbb{Z}_{\text{odd}}, & k + m \not\equiv l + n \pmod 2, \\ 2^7\mathbb{Z}_{\text{odd}}, & (k + m)(l + n) \equiv -1 \pmod 4, \\ 2^9\mathbb{Z}, & \textit{otherwise}; \end{cases}$

(3) $D_4(2k, 2l + 1, 2m, 2n + 1)$

$$\in \begin{cases} 2^5\mathbb{Z}_{\text{odd}}, & k - m \equiv l - n \equiv 1 \pmod 2, \\ 2^6\mathbb{Z}_{\text{odd}}, & k \equiv m \pmod 2,\ (2k + 2l + 1)(2m + 2n + 1) \equiv \pm 3 \pmod 8, \\ 2^7\mathbb{Z}, & \textit{otherwise}; \end{cases}$$

(4) $D_4(2k, 2l, 2m+1, 2n+1)$

$$\in \begin{cases} 2^4\mathbb{Z}_{\mathrm{odd}}, & (2k+2m+1)(2l+2n+1) \equiv \pm 3 \pmod 8, \\ 2^5\mathbb{Z}, & (2k+2m+1)(2l+2n+1) \equiv \pm 1 \pmod 8. \end{cases}$$

To prove Lemma 7, we remark the following.

**Remark 4** ([18, Remark 3.5])**.** For any $k, l, m, n \in \mathbb{Z}$, the following hold:

(1) $(2k+2l+1)(2m+2n+1) \equiv 1 \pmod 8$ if and only if $k-m \equiv -l+n \pmod 4$;

(2) $(2k+2l+1)(2m+2n+1) \equiv -1 \pmod 8$ if and only if $k+m \equiv -l-n-1 \pmod 4$;

(3) $(2k+2l+1)(2m+2n+1) \equiv 3 \pmod 8$ if and only if $k+m \equiv 1-l-n \pmod 4$;

(4) $(2k+2l+1)(2m+2n+1) \equiv -3 \pmod 8$ if and only if $k-m \equiv 2-l+n \pmod 4$.

*Proof of Lemma* 7. We obtain (1) from

$$\begin{aligned} D_4(2k, 2l, 2m, 2n) &= \left\{(2k+2m)^2 - (2l+2n)^2\right\}\left\{(2k-2m)^2 + (2l-2n)^2\right\} \\ &= 2^4\left\{(k+m)^2 - (l+n)^2\right\}\left\{(k-m)^2 + (l-n)^2\right\}. \end{aligned}$$

We obtain (2) from

$$\begin{aligned} &D_4(2k+1, 2l+1, 2m+1, 2n+1) \\ &\quad = \left\{(2k+2m+2)^2 - (2l+2n+2)^2\right\}\left\{(2k-2m)^2 + (2l-2n)^2\right\} \\ &\quad = 2^4\left\{(k+m+1)^2 - (l+n+1)^2\right\}\left\{(k-m)^2 + (l-n)^2\right\} \end{aligned}$$

and

$$(k+m+1)^2 - (l+n+1)^2 \in \begin{cases} \mathbb{Z}_{\mathrm{odd}}, & k+m \not\equiv l+n \pmod 2, \\ 2^2\mathbb{Z}_{\mathrm{odd}}, & (k+m)(l+n) \equiv -1 \pmod 4, \\ 2^4\mathbb{Z}, & (k+m)(l+n) \equiv 1 \pmod 4, \\ 2^3\mathbb{Z}, & k+m \equiv l+n \equiv 0 \pmod 2, \end{cases}$$

$$(k-m)^2 + (l-n)^2 \in \begin{cases} \mathbb{Z}_{\mathrm{odd}}, & k+m \not\equiv l+n \pmod 2, \\ 2\mathbb{Z}_{\mathrm{odd}}, & k+m \equiv l+n \equiv 1 \pmod 2, \\ 2^2\mathbb{Z}, & k+m \equiv l+n \equiv 0 \pmod 2. \end{cases}$$

We prove (3). From Remark 4, we have

$$\left\{(k+m)^2-(l+n+1)^2\right\}\left\{(k-m)^2+(l-n)^2\right\}$$
$$\in \begin{cases} 2\mathbb{Z}_{\text{odd}}, & k-m\equiv l-n\equiv 1 \pmod 2,\\ 2^3\mathbb{Z}, & k\not\equiv m,\ l\equiv n \pmod 2,\\ 2^3\mathbb{Z}, & \text{when (I)},\\ 2^2\mathbb{Z}_{\text{odd}}, & \text{when (II)},\end{cases}$$

where

(I) $k\equiv m \pmod 2$ and $(2k+2l+1)(2m+2n+1)\equiv \pm 1 \pmod 8$;

(II) $k\equiv m \pmod 2$ and $(2k+2l+1)(2m+2n+1)\equiv \pm 3 \pmod 8$.

Therefore, (3) is obtained from

$$\begin{aligned} &D_4(2k,2l+1,2m,2n+1)\\ &\quad=\left\{(2k+2m)^2-(2l+2n+2)^2\right\}\left\{(2k-2m)^2+(2l-2n)^2\right\}\\ &\quad=2^4\left\{(k+m)^2-(l+n+1)^2\right\}\left\{(k-m)^2+(l-n)^2\right\}.\end{aligned}$$

We prove (4). From Remark 4, we have

$$k+m-l-n\in\begin{cases} 2^2\mathbb{Z}, & (2k+2m+1)(2l+2n+1)\equiv 1 \pmod 8,\\ \mathbb{Z}_{\text{odd}}, & (2k+2m+1)(2l+2n+1)\equiv -1 \text{ or } 3 \pmod 8,\\ 2\mathbb{Z}_{\text{odd}}, & (2k+2m+1)(2l+2n+1)\equiv -3 \pmod 8,\end{cases}$$

$$k+m+l+n+1\in\begin{cases} \mathbb{Z}_{\text{odd}}, & (2k+2m+1)(2l+2n+1)\equiv 1 \text{ or } -3 \pmod 8,\\ 2^2\mathbb{Z}, & (2k+2m+1)(2l+2n+1)\equiv -1 \pmod 8,\\ 2\mathbb{Z}_{\text{odd}}, & (2k+2m+1)(2l+2n+1)\equiv 3 \pmod 8.\end{cases}$$

Therefore, (4) is obtained from

$$\begin{aligned} &D_4(2k,2l,2m+1,2n+1)\\ &\quad=\left\{(2k+2m+1)^2-(2l+2n+1)^2\right\}\left\{(2k-2m-1)^2+(2l-2n-1)^2\right\}\\ &\quad=2^3(k+m+l+n+1)(k+m-l-n)\\ &\qquad\times\left\{2(k-m)(k-m-1)+2(l-n)(l-n-1)+1\right\}. \qquad\square\end{aligned}$$

## 4. Odd Values Must Be of the Stated Form

In this section, we prove that the odd values must be of the stated form. Let

$$\begin{aligned} A:=\{&(8k-3)(8l-3)(8m-3)(8n-3)\mid\\ &k\in\mathbb{Z},\ 8l-3,8m-3,8n-3\in P,\ k+l\not\equiv m+n \pmod 2\}.\end{aligned}$$

**Lemma 8.** *We have* $S(\mathrm{C}_4^2) \cap \mathbb{Z}_{\mathrm{odd}} \subset \{16m+1 \mid m \in \mathbb{Z}\} \cup A$.

To prove Lemma 8, we use the following five lemmas.

**Lemma 9.** *Let* $b_0 + b_2 \not\equiv b_1 + b_3 \pmod 2$. *Then we have the following:*

(1) *If* $(b_0b_2 + b_1b_3, c_0c_2 + c_1c_3) \equiv (0,0)$ *or* $(2,2) \pmod 4$, *then*

$$D_4(\boldsymbol{b})D_4(\boldsymbol{c}) \in \{(8k+1)(8l+1) \mid k,l \in \mathbb{Z},\ k \equiv l \pmod 2\};$$

(2) *If* $(b_0b_2 + b_1b_3, c_0c_2 + c_1c_3) \equiv (0,2)$ *or* $(2,0) \pmod 4$, *then*

$$D_4(\boldsymbol{b})D_4(\boldsymbol{c}) \in \{(8k+1)(8l+1) \mid k,l \in \mathbb{Z},\ k \not\equiv l \pmod 2\};$$

(3) *If* $(b_0b_2 + b_1b_3, c_0c_2 + c_1c_3) \equiv (1,1)$ *or* $(-1,-1) \pmod 4$, *then*

$$D_4(\boldsymbol{b})D_4(\boldsymbol{c}) \in \{(8k-3)(8l-3) \mid k,l \in \mathbb{Z},\ k \equiv l \pmod 2\};$$

(4) *If* $(b_0b_2 + b_1b_3, c_0c_2 + c_1c_3) \equiv (1,-1)$ *or* $(-1,1) \pmod 4$, *then*

$$D_4(\boldsymbol{b})D_4(\boldsymbol{c}) \in \{(8k-3)(8l-3) \mid k,l \in \mathbb{Z},\ k \not\equiv l \pmod 2\}.$$

*Proof.* First, we prove (1) and (2). If $b_0b_2 + b_1b_3 \equiv 0 \pmod 2$, then exactly three of $b_0, b_1, b_2, b_3$ are even. On the other hand, for any $k, l, m, n \in \mathbb{Z}$,

$$m \equiv 0 \pmod 2 \text{ if and only if } (2k+1)(2m) + (2l)(2n) \equiv 0 \pmod 4.$$

Therefore, from Remarks 1 and 2 (1) and Lemma 6 (1), we have (1) and (2). Next, we prove (3) and (4). If $b_0b_2 + b_1b_3 \equiv 1 \pmod 2$, then exactly one of $b_0, b_1, b_2, b_3$ is even. On the other hand, for any $k, l, m, n \in \mathbb{Z}$,

$$k+l+n \equiv 0 \pmod 2 \text{ if and only if } (2k)(2m+1) + (2l+1)(2n+1) \equiv 1 \pmod 4.$$

Therefore, from Remarks 1 and 2 (1) and Lemma 6 (2), we have (3) and (4). □

The following lemma is immediately obtained from Lemma 9.

**Lemma 10.** *If* $b_0 + b_2 \not\equiv b_1 + b_3 \pmod 2$, *then*

$$D_4(\boldsymbol{b})D_4(\boldsymbol{c}) \equiv 1 - 4(b_0b_2 + b_1b_3 + c_0c_2 + c_1c_3) \pmod{16}.$$

**Lemma 11.** *Let* $b_0 + b_2 \not\equiv b_1 + b_3 \pmod 2$ *and* $\beta\overline{\beta} \equiv -3 \pmod 8$. *Then the following hold:*

(1) *If* $b_0b_2 + b_1b_3 + c_0c_2 + c_1c_3 \equiv 0 \pmod 4$, *then*

$$D_4(\boldsymbol{b})D_4(\boldsymbol{c}) \in \{(8k-3)(8l-3) \mid k \in \mathbb{Z},\ 8l-3 \in P,\ k \not\equiv l \pmod 2\};$$

(2) *If* $b_0b_2 + b_1b_3 + c_0c_2 + c_1c_3 \equiv 2 \pmod 4$, *then*

$$D_4(\boldsymbol{b})D_4(\boldsymbol{c}) \in \{(8k-3)(8l-3) \mid k \in \mathbb{Z},\ 8l-3 \in P,\ k \equiv l \pmod 2\}.$$

*Proof.* From $b_0 + b_2 \not\equiv b_1 + b_3 \pmod 2$ and Remark 2 (1), $d_0 + d_2 + d_4 + d_6 \not\equiv d_1 + d_3 + d_5 + d_7 \pmod 2$ holds. Moreover, from $\beta\overline{\beta} \equiv -3 \pmod 8$ and Lemma 3, exactly one of $d_0 + d_2$, $d_1 + d_3$, $d_4 + d_6$, $d_5 + d_7$ is even. For any $k, l \in \mathbb{Z}$ satisfying $D_4(\boldsymbol{b})D_4(\boldsymbol{c}) = (8k-3)(8l-3)$, it follows from Lemma 10 that $k \not\equiv l \pmod 2$ if $b_0b_2 + b_1b_3 + c_0c_2 + c_1c_3 \equiv 0 \pmod 4$; $k \equiv l \pmod 2$ if $b_0b_2 + b_1b_3 + c_0c_2 + c_1c_3 \equiv 2 \pmod 4$. Below, we prove that there exist $k \in \mathbb{Z}$ and $8l-3 \in P$ satisfying $D_4(\boldsymbol{b})D_4(\boldsymbol{c}) = (8k-3)(8l-3)$. First, suppose that $b_0b_2 + b_1b_3 \equiv 0 \pmod 2$. Then, exactly three of $b_0, b_1, b_2, b_3$ are even. From Remark 1, we have

$$\begin{aligned} D_4(\boldsymbol{b})D_4(\boldsymbol{c}) &= D_4(b_1, b_2, b_3, b_0)D_4(c_1, c_2, c_3, c_0) \\ &= D_4(b_2, b_3, b_0, b_1)D_4(c_2, c_3, c_0, c_1) \\ &= D_4(b_3, b_0, b_1, b_2)D_4(c_3, c_0, c_1, c_2). \end{aligned}$$

Therefore, from Remark 2 (1), we may assume without loss of generality that $\boldsymbol{b} \equiv \boldsymbol{c} \equiv (1, 0, 0, 0) \pmod 2$. From Remark 2, we have

$$2(d_1 + d_3) \equiv b_1 + b_3 + c_1 + c_3 \equiv b_1 + b_3 - c_1 - c_3 \equiv 2(d_5 + d_7) \pmod 4.$$

Thus, $d_1+d_3$ must be odd since $d_1+d_3 \equiv d_5+d_7 \pmod 2$. Hence, from $b_1+b_3+c_1+c_3 \equiv 2(d_1+d_3) \equiv 2 \pmod 4$, we have $(b_1+b_3, c_1+c_3) \equiv (0, 2)$ or $(2, 0) \pmod 4$. We consider the case of $(b_1+b_3, c_1+c_3) \equiv (0, 2) \pmod 4$. From $\boldsymbol{b} \equiv (1, 0, 0, 0) \pmod 2$ and Lemma 6 (1), there exists $j \in \mathbb{Z}$ satisfying $D_4(\boldsymbol{b}) = 8j + 1$. On the other hand, from $\boldsymbol{c} \equiv (1, 0, 0, 0) \pmod 2$ and $c_1 + c_3 \equiv 2 \pmod 4$, we have $c_1 - c_3 \equiv 2 \pmod 4$. Thus, from Corollary 3, there exist $l' \in \mathbb{Z}$, $8l-3 \in P$ satisfying $(c_0-c_2)^2+(c_1-c_3)^2 = (8l'+1)(8l-3)$. Also, there exists $k' \in \mathbb{Z}$ satisfying $(c_0 + c_2)^2 - (c_1 + c_3)^2 = 8k'-3$. From Corollary 1, we have $D_4(\boldsymbol{c}) = (8k'-3)(8l'+1)(8l-3)$. Therefore, there exists $k \in \mathbb{Z}$ satisfying $D_4(\boldsymbol{b})D_4(\boldsymbol{c}) = (8k-3)(8l-3)$. In the same way, the case of $(b_1 + b_3, c_1 + c_3) \equiv (2, 0) \pmod 4$ can also be proved.

Next, suppose that $b_0b_2 + b_1b_3 \equiv 1 \pmod 2$. Then, exactly one of $b_0, b_1, b_2, b_3$ is even. From Remarks 1 and 2 (1), we may assume without loss of generality that $\boldsymbol{b} \equiv \boldsymbol{c} \equiv (0, 1, 1, 1) \pmod 2$. In the same way as in the above, we have $(b_1+b_3, c_1+c_3) \equiv (0, 2)$ or $(2, 0) \pmod 4$. We consider the case of $(b_1+b_3, c_1+c_3) \equiv (0, 2) \pmod 4$. From $\boldsymbol{c} \equiv (0, 1, 1, 1) \pmod 2$ and Lemma 6 (2), there exists $k' \in \mathbb{Z}$ satisfying $D_4(\boldsymbol{c}) = 8k'-3$. On the other hand, from $\boldsymbol{b} \equiv (0, 1, 1, 1) \pmod 2$ and $b_1 + b_3 \equiv 0 \pmod 4$, we have $b_1 - b_3 \equiv 2 \pmod 4$. Thus, from Corollary 3, there exist $l' \in \mathbb{Z}$, $8l-3 \in P$ satisfying $(b_0 - b_2)^2 + (b_1 - b_3)^2 = (8l'+1)(8l-3)$. Also, there exists $j \in \mathbb{Z}$ satisfying $(b_0 + b_2)^2 - (b_1 + b_3)^2 = 8j + 1$. From Corollary 1, we have $D_4(\boldsymbol{b}) = (8j+1)(8l'+1)(8l-3)$. Therefore, there exists $k \in \mathbb{Z}$ satisfying $D_4(\boldsymbol{b})D_4(\boldsymbol{c}) = (8k-3)(8l-3)$. In the same way, the case of $(b_1+b_3, c_1+c_3) \equiv (2, 0) \pmod 4$ can also be proved. □

**Lemma 12.** *Let* $b_0 + b_2 \not\equiv b_1 + b_3 \pmod 2$. *Then we have*

$$\beta\overline{\beta} - \gamma\overline{\gamma} \equiv 4(b_0b_2 + b_1b_3 + c_0c_2 + c_1c_3) \pmod{16}.$$

*Proof.* From Remark 2 (1), we have $d_0 + d_2 + d_4 + d_6 \not\equiv d_1 + d_3 + d_5 + d_7 \pmod 2$. Thus, $d_0^2 + d_2^2 + d_4^2 + d_6^2 + d_1^2 + d_3^2 + d_5^2 + d_7^2 \equiv 1 \pmod 2$. Also, since exactly one or three of $d_0 + d_2$, $d_4 + d_6$, $d_1 + d_3$, $d_5 + d_7$ are even, it holds that

$$\begin{aligned}&\{(d_0 + d_2)(d_1 + d_3) + (d_4 + d_6)(d_5 + d_7)\}^2 \\ &\quad \equiv \{(d_0 - d_2)(d_5 - d_7) - (d_4 - d_6)(d_1 - d_3)\}^2 \pmod 4.\end{aligned}$$

From the above and Lemmas 3, 4 and 5 (1), we have

$$\begin{aligned}\beta\overline{\beta} - \gamma\overline{\gamma} &\equiv 8(d_0^2 + d_2^2 + d_4^2 + d_6^2 + d_1^2 + d_3^2 + d_5^2 + d_7^2)(d_0d_2 + d_4d_6 + d_1d_3 + d_5d_7) \\ &\equiv 8(d_0d_2 + d_4d_6 + d_1d_3 + d_5d_7) \\ &\equiv 4(b_0b_2 + b_1b_3 + c_0c_2 + c_1c_3) \pmod{16}.\end{aligned}$$

□

**Lemma 13.** *Let* $b_0 + b_2 \not\equiv b_1 + b_3 \pmod 2$ *and* $\beta\overline{\beta} \equiv -3 \pmod 8$. *Then the following hold:*

(1) *If* $b_0b_2 + b_1b_3 + c_0c_2 + c_1c_3 \equiv 0 \pmod 4$, *then*

$$\beta\overline{\beta}\gamma\overline{\gamma} \in \{(8j + 1)(8m - 3)(8n - 3) \mid j \in \mathbb{Z},\ 8m - 3, 8n - 3 \in P,\ j \equiv m + n \pmod 2\};$$

(2) *If* $b_0b_2 + b_1b_3 + c_0c_2 + c_1c_3 \equiv 2 \pmod 4$, *then*

$$\beta\overline{\beta}\gamma\overline{\gamma} \in \{(8j + 1)(8m - 3)(8n - 3) \mid j \in \mathbb{Z},\ 8m - 3, 8n - 3 \in P,\ j \not\equiv m + n \pmod 2\}.$$

*Proof.* From Remark 2 (1) and Lemma 12, we have $\gamma\overline{\gamma} \equiv \beta\overline{\beta} \equiv -3 \pmod 8$. Therefore, from Corollary 3, there exist $m', n' \in \mathbb{Z}$, $8m - 3, 8n - 3 \in P$ satisfying

$$\beta\overline{\beta} = (8m' + 1)(8m - 3), \quad \gamma\overline{\gamma} = (8n' + 1)(8n - 3).$$

Let $j := 8m'n' + m' + n'$. Then, $\beta\overline{\beta}\gamma\overline{\gamma} = (8j + 1)(8m - 3)(8n - 3)$. From Lemma 12,

$$8(m + m' + n + n') \equiv \beta\overline{\beta} - \gamma\overline{\gamma} \equiv 4(b_0b_2 + b_1b_3 + c_0c_2 + c_1c_3) \pmod{16}.$$

Therefore, we have $2(j + m + n) \equiv 2(m' + n' + m + n) \equiv b_0b_2 + b_1b_3 + c_0c_2 + c_1c_3 \pmod 4$. □

*Proof of Lemma* 8. Let $D_{4\times4}(\boldsymbol{a}) = D_4(\boldsymbol{b})D_4(\boldsymbol{c})\beta\overline{\beta}\gamma\overline{\gamma} \in \mathbb{Z}_{\text{odd}}$. Then, $b_0 + b_2 \not\equiv b_1 + b_3 \pmod 2$ holds from $D_4(\boldsymbol{b}) \in \mathbb{Z}_{\text{odd}}$ and Corollary 1. Since $\beta\overline{\beta}$ is an odd number expressible in the form $x^2+y^2$, we have $\beta\overline{\beta} \equiv 1 \pmod 4$. Therefore, from Lemma 12,

$$\begin{aligned}\beta\overline{\beta}\gamma\overline{\gamma} &\equiv \beta\overline{\beta}\left\{\beta\overline{\beta} - 4(b_0b_2+b_1b_3+c_0c_2+c_1c_3)\right\}\\ &\equiv (\beta\overline{\beta})^2 - 4(b_0b_2+b_1b_3+c_0c_2+c_1c_3) \pmod{16}.\end{aligned}$$

From this and Lemma 10, we have

$$\begin{aligned}D_4(\boldsymbol{b})D_4(\boldsymbol{c})\beta\overline{\beta}\gamma\overline{\gamma} &\equiv \{1 - 4(b_0b_2+b_1b_3+c_0c_2+c_1c_3)\}\\ &\qquad \times \left\{(\beta\overline{\beta})^2 - 4(b_0b_2+b_1b_3+c_0c_2+c_1c_3)\right\}\\ &\equiv (\beta\overline{\beta})^2 - 8(b_0b_2+b_1b_3+c_0c_2+c_1c_3) \pmod{16}.\end{aligned}$$

Moreover, from Remark 2 (1), it follows that $D_4(\boldsymbol{b})D_4(\boldsymbol{c})\beta\overline{\beta}\gamma\overline{\gamma} \equiv (\beta\overline{\beta})^2 \pmod{16}$. Therefore, if $\beta\overline{\beta} \equiv 1 \pmod 8$, then $D_{4\times 4}(\boldsymbol{a}) \in \{16m+1 \mid m \in \mathbb{Z}\}$. If $\beta\overline{\beta} \equiv -3 \pmod 8$, then we have $D_{4\times 4}(\boldsymbol{a}) \in A'$ from Lemmas 11 and 13, where

$$\begin{aligned}A' := \{&(8j+1)(8k-3)(8l-3)(8m-3)(8n-3) \mid\\ &j,k \in \mathbb{Z},\ 8l-3, 8m-3, 8n-3 \in P,\ j \not\equiv k+l+m+n \pmod 2\}.\end{aligned}$$

Since $A' = A$, the lemma is proved. □

## 5. Even Values Must Be of the Stated Form

In this section, we prove that the even values must be of the stated form.

**Lemma 14.** *The following hold:*

(1) $S(\mathrm{C}_4^2) \cap 2\mathbb{Z} \subset 2^{15}\mathbb{Z}$;

(2) $S(\mathrm{C}_4^2) \cap 2^{15}\mathbb{Z}_{\mathrm{odd}} \subset \left\{2^{15}p(2m+1) \mid p \in P,\ m \in \mathbb{Z}\right\}$.

To prove Lemma 14, we use the following five lemmas.

**Lemma 15.** *The following hold:*

(1) *If* $b_0 \equiv b_1 \equiv b_2 \equiv b_3 \equiv 0 \pmod 2$, *then*

$$D_4(\boldsymbol{b})D_4(\boldsymbol{c}) \in \begin{cases} 2^8\mathbb{Z}_{\mathrm{odd}}, & b_0+b_1+b_2+b_3 \equiv c_0+c_1+c_2+c_3 \equiv 2 \pmod 4,\\ 2^{12}\mathbb{Z}, & \textit{otherwise}; \end{cases}$$

(2) *If* $b_0 \equiv b_1 \equiv b_2 \equiv b_3 \equiv 1 \pmod 2$, *then*

$$D_4(\boldsymbol{b})D_4(\boldsymbol{c}) \in \begin{cases} 2^8\mathbb{Z}_{\mathrm{odd}}, & b_0+b_1+b_2+b_3 \equiv c_0+c_1+c_2+c_3 \equiv 2 \pmod 4,\\ 2^{11}\mathbb{Z}, & \textit{otherwise}; \end{cases}$$

(3) *If* $b_0 \equiv b_2 \not\equiv b_1 \equiv b_3 \pmod 2$, *then*

$$D_4(\boldsymbol{b})D_4(\boldsymbol{c}) \in \begin{cases} 2^{10}\mathbb{Z}_{\text{odd}}, & b_0 - b_2 \equiv b_1 - b_3 \equiv c_0 - c_2 \equiv c_1 - c_3 \equiv 2 \pmod 4, \\ 2^{11}\mathbb{Z}, & \textit{otherwise}; \end{cases}$$

(4) *If* $b_0 + b_2 \equiv b_1 + b_3 \equiv 1 \pmod 2$, *then*

$$D_4(\boldsymbol{b})D_4(\boldsymbol{c}) \in \begin{cases} 2^{8}\mathbb{Z}_{\text{odd}}, & (b_0 + b_2)(b_1 + b_3) \equiv \pm 3,\ (c_0 + c_2)(c_1 + c_3) \equiv \pm 3 \pmod 8, \\ 2^{9}\mathbb{Z}, & \textit{otherwise}. \end{cases}$$

*Proof.* For any $k, l, m, n \in \mathbb{Z}$,

$$k + m \not\equiv l + n \pmod 2 \text{ if and only if } 2k + 2l + 2m + 2n \equiv 2 \pmod 4,$$
$$k - m \equiv l - n \equiv 1 \pmod 2 \text{ if and only if } 2k - 2m \equiv 2l - 2n \equiv 2 \pmod 4.$$

Therefore, from Remarks 1 and 2 (1) and Lemma 7, the lemma is proved. □

**Lemma 16.** *The following hold:*

(1) *If* $b_0 + b_2 \equiv b_1 + b_3 \equiv 0 \pmod 2$, *then*

$$\beta\overline{\beta}\gamma\overline{\gamma} \in \begin{cases} 2^{4}\mathbb{Z}_{\text{odd}}, & b_0 + b_1 + b_2 + b_3 \not\equiv c_0 + c_1 + c_2 + c_3 \pmod 4, \\ 2^{8}\mathbb{Z}, & b_0 + b_1 + b_2 + b_3 \equiv c_0 + c_1 + c_2 + c_3 \pmod 4; \end{cases}$$

(2) *If* $b_0 + b_2 \equiv b_1 + b_3 \equiv 1 \pmod 2$, *then*

$$\beta\overline{\beta}\gamma\overline{\gamma} \in \begin{cases} 2^{7}\mathbb{Z}_{\text{odd}}, & d \equiv 2 \pmod 4, \\ 2^{8}\mathbb{Z}, & d \equiv 0 \pmod 4, \end{cases}$$

*where*

$$d := \{(d_0 + d_2)(d_5 + d_7) + (d_4 + d_6)(d_1 + d_3)\} \times \{(d_0 - d_2)(d_1 - d_3) + (d_4 - d_6)(d_5 - d_7)\}.$$

*Proof.* We prove (1). Let $b_0 + b_2 \equiv b_1 + b_3 \equiv 0 \pmod 2$. Then, from Remark 2 (1), we have $d_0 + d_2 + d_4 + d_6 \equiv d_1 + d_3 + d_5 + d_7 \equiv 0 \pmod 2$. Also, from Remark 2 (1) and (2), $b_0 + b_1 + b_2 + b_3 \equiv c_0 + c_1 + c_2 + c_3 \pmod 4$ if and only if

$$2(d_0 + d_2) \equiv b_0 + b_2 + c_0 + c_2 \equiv b_1 + b_3 + c_1 + c_3 \equiv 2(d_1 + d_3) \pmod 4.$$

Therefore, if $b_0 + b_1 + b_2 + b_3 \not\equiv c_0 + c_1 + c_2 + c_3 \pmod 4$, then $d_0 + d_2 \equiv d_4 + d_6 \not\equiv d_1 + d_3 \equiv d_5 + d_7 \pmod 2$. Thus, from Lemma 1, we have $\beta\overline{\beta}\gamma\overline{\gamma} \in 2^4\mathbb{Z}_{\text{odd}}$. If

$b_0+b_1+b_2+b_3 \equiv c_0+c_1+c_2+c_3 \pmod 4$, then $d_0+d_2 \equiv d_4+d_6 \equiv d_1+d_3 \equiv d_5+d_7 \pmod 2$. Thus, from Lemma 1, we have $\beta\overline{\beta}\gamma\overline{\gamma} \in 2^8\mathbb{Z}$.

We prove (2). Let $b_0 + b_2 \equiv b_1 + b_3 \equiv 1 \pmod 2$. Then, from Remark 2 (1), we have $d_0 + d_2 + d_4 + d_6 \equiv d_1 + d_3 + d_5 + d_7 \equiv 1 \pmod 2$. From Remark 3, we may assume without loss of generality that $d_0 + d_2 \equiv 0$, $d_4 + d_6 \equiv 1 \pmod 2$. We divide the proof into the following two cases:

(i) $d_0 + d_2 \equiv d_1 + d_3 \equiv 0$, $d_4 + d_6 \equiv d_5 + d_7 \equiv 1 \pmod 2$;

(ii) $d_0 + d_2 \equiv d_5 + d_7 \equiv 0$, $d_4 + d_6 \equiv d_1 + d_3 \equiv 1 \pmod 2$.

We remark that if (i), then $d \equiv d_0 + d_2 + d_1 + d_3 \pmod 4$ holds, and if (ii), then $d \equiv d_0 - d_2 + d_5 - d_7 \pmod 4$ holds. First, suppose that (i) holds. Then, from Lemma 1, we have $\gamma\overline{\gamma} \in 2^2\mathbb{Z}_{\text{odd}}$. Also, from

$$\begin{aligned}(d_0 + d_2 + d_1 + d_3, d_0 + d_2 - d_1 - d_3) &\equiv (0, 0) \text{ or } (2, 2) \pmod 4,\\ (d_4 + d_6 + d_5 + d_7, d_4 + d_6 - d_5 - d_7) &\equiv (0, 2) \text{ or } (2, 0) \pmod 4\end{aligned}$$

and Lemma 1, we have

$$\beta\overline{\beta} \in \begin{cases} 2^5\mathbb{Z}_{\text{odd}}, & d \equiv d_0 + d_2 + d_1 + d_3 \equiv 2 \pmod 4,\\ 2^6\mathbb{Z}, & d \equiv d_0 + d_2 + d_1 + d_3 \equiv 0 \pmod 4.\end{cases}$$

Next, suppose that (ii) holds. Then, from Lemma 1, we have $\beta\overline{\beta} \in 2^2\mathbb{Z}_{\text{odd}}$. Also, from

$$\begin{aligned}(d_0 - d_2 - d_5 + d_7, d_0 - d_2 + d_5 - d_7) &\equiv (0, 0) \text{ or } (2, 2) \pmod 4,\\ (d_4 - d_6 + d_1 - d_3, d_4 - d_6 - d_1 + d_3) &\equiv (0, 2) \text{ or } (2, 0) \pmod 4\end{aligned}$$

and Lemma 1, we have

$$\gamma\overline{\gamma} \in \begin{cases} 2^5\mathbb{Z}_{\text{odd}}, & d \equiv d_0 - d_2 + d_5 - d_7 \equiv 2 \pmod 4,\\ 2^6\mathbb{Z}, & d \equiv d_0 - d_2 + d_5 - d_7 \equiv 0 \pmod 4.\end{cases}$$

This completes the proof. □

**Lemma 17.** *The following hold:*

(1) *If* $b_0 \equiv b_1 \equiv b_2 \equiv b_3 \equiv 1 \pmod 2$ *and* $D_4(\boldsymbol{b})D_4(\boldsymbol{c}) \in 2^{11}\mathbb{Z}_{\text{odd}}$, *then*

$$b_0b_2 + b_1b_3 + c_0c_2 + c_1c_3 \equiv 2 \pmod 4;$$

(2) *If* $b_0 \equiv b_2 \not\equiv b_1 \equiv b_3 \pmod 2$, $D_4(\boldsymbol{b})D_4(\boldsymbol{c}) \in 2^{11}\mathbb{Z}_{\text{odd}}$ *and* $\beta\overline{\beta}\gamma\overline{\gamma} \in 2^4\mathbb{Z}_{\text{odd}}$, *then*

$$b_0b_2 + b_1b_3 + c_0c_2 + c_1c_3 \equiv 2 \pmod 4.$$

*Proof.* We prove (1). Let $b_0 \equiv b_1 \equiv b_2 \equiv b_3 \equiv 1 \pmod 2$ and $D_4(\boldsymbol{b})D_4(\boldsymbol{c}) \in 2^{11}\mathbb{Z}_{\mathrm{odd}}$. Then, from Remark 2 (1) and Lemma 7, one of the following cases holds:

(i) $D_4(\boldsymbol{b}) \in 2^4\mathbb{Z}_{\mathrm{odd}}$ and $D_4(\boldsymbol{c}) \in 2^7\mathbb{Z}_{\mathrm{odd}}$;

(ii) $D_4(\boldsymbol{b}) \in 2^7\mathbb{Z}_{\mathrm{odd}}$ and $D_4(\boldsymbol{c}) \in 2^4\mathbb{Z}_{\mathrm{odd}}$.

If (i), then $b_0 + b_2 \not\equiv b_1 + b_3 \pmod 4$ and $c_0 + c_2 \equiv c_1 + c_3 \equiv 0 \pmod 4$ hold. Therefore, $(b_0b_2 + b_1b_3, c_0c_2 + c_1c_3) \equiv (0, 2) \pmod 4$ holds. Thus, we have $b_0b_2 + b_1b_3 + c_0c_2 + c_1c_3 \equiv 2 \pmod 4$. In the same way, the case (ii) can also be proved. We prove (2). Let $b_0 \equiv b_2 \not\equiv b_1 \equiv b_3 \pmod 2$, $D_4(\boldsymbol{b})D_4(\boldsymbol{c}) \in 2^{11}\mathbb{Z}_{\mathrm{odd}}$ and $\beta\overline{\beta}\gamma\overline{\gamma} \in 2^4\mathbb{Z}_{\mathrm{odd}}$. Then, from Remarks 1 and 2 (1), we may assume without loss of generality that $\boldsymbol{b} \equiv \boldsymbol{c} \equiv (0, 1, 0, 1) \pmod 2$. From Lemma 7, either one of the following cases holds:

(iii) $D_4(\boldsymbol{b}) \in 2^5\mathbb{Z}_{\mathrm{odd}}$ and $D_4(\boldsymbol{c}) \in 2^6\mathbb{Z}_{\mathrm{odd}}$;

(iv) $D_4(\boldsymbol{b}) \in 2^6\mathbb{Z}_{\mathrm{odd}}$ and $D_4(\boldsymbol{c}) \in 2^5\mathbb{Z}_{\mathrm{odd}}$.

If (iii), then $b_0 - b_2 \equiv b_1 - b_3 \equiv 2 \pmod 4$ and $c_0 \equiv c_2 \equiv 0$ or $2 \pmod 4$ hold. Therefore, $b_0b_2 + b_1b_3 \equiv -1 \pmod 4$ and $b_0 + b_1 + b_2 + b_3 \equiv 2 \pmod 4$ hold. Also, since $b_0 + b_1 + b_2 + b_3 \not\equiv c_0 + c_1 + c_2 + c_3 \pmod 4$ from Lemma 16, it follows that $c_0c_2 + c_1c_3 \equiv -1 \pmod 4$ holds. Thus, we have $b_0b_2 + b_1b_3 + c_0c_2 + c_1c_3 \equiv 2 \pmod 4$. In the same way, the case (iv) can also be proved. □

**Lemma 18.** *Suppose that* $b_0 + b_2 \equiv b_1 + b_3 \equiv 0 \pmod 2$, $\beta\overline{\beta}\gamma\overline{\gamma} \in 2^4\mathbb{Z}_{\mathrm{odd}}$ *and* $b_0b_2 + b_1b_3 + c_0c_2 + c_1c_3 \equiv 2 \pmod 4$. *Then* $\beta\overline{\beta}\gamma\overline{\gamma} \in \left\{2^4p(2m+1) \mid p \in P,\ m \in \mathbb{Z}\right\}$.

*Proof.* Since $b_0 + b_2 \equiv b_1 + b_3 \equiv 0 \pmod 2$ and $\beta\overline{\beta}\gamma\overline{\gamma} \in 2^4\mathbb{Z}_{\mathrm{odd}}$, from Remark 2 (1), we have

$$d_0 + d_2 + d_4 + d_6 \equiv d_1 + d_3 + d_5 + d_7 \equiv 0 \pmod 2.$$

On the other hand, from Lemma 16, we have $b_0 + b_1 + b_2 + b_3 \not\equiv c_0 + c_1 + c_2 + c_3 \pmod 4$. From this and Remark 2 (1) and (2), we have

$$2d_0 + 2d_2 \equiv b_0 + b_2 + c_0 + c_2 \not\equiv b_1 + b_3 + c_1 + c_3 \equiv 2d_1 + 2d_3 \pmod 4.$$

Thus, $d_0 + d_2 \not\equiv d_1 + d_3 \pmod 2$. From the above, we have $d_0 + d_2 \equiv d_4 + d_6 \not\equiv d_1 + d_3 \equiv d_5 + d_7 \pmod 2$. Hence,

$$\begin{aligned}
&\left\{(d_0 + d_2)(d_1 + d_3) + (d_4 + d_6)(d_5 + d_7)\right\}^2 \\
&\qquad - \left\{(d_0 - d_2)(d_5 - d_7) - (d_4 - d_6)(d_1 - d_3)\right\}^2 \\
&\quad \equiv 4(d_0d_2 + d_4d_6 + d_1d_3 + d_5d_7) \pmod 8, \\
&(d_0^2 + d_2^2 + d_4^2 + d_6^2 + d_1^2 + d_3^2 + d_5^2 + d_7^2)(d_0d_2 + d_4d_6 + d_1d_3 + d_5d_7) \equiv 0 \pmod 4.
\end{aligned}$$

Therefore, from Lemmas 3, 4 and 5 (1),

$$\begin{aligned}\beta\overline{\beta}-\gamma\overline{\gamma} &\equiv 8(d_0^2+d_2^2+d_4^2+d_6^2+d_1^2+d_3^2+d_5^2+d_7^2)(d_0d_2+d_4d_6+d_1d_3+d_5d_7)\\ &\quad +16(d_0d_2+d_4d_6+d_1d_3+d_5d_7)\\ &\equiv 16(d_0d_2+d_4d_6+d_1d_3+d_5d_7)\\ &\equiv 8(b_0b_2+b_1b_3+c_0c_2+c_1c_3) \pmod{32}.\end{aligned}$$

Since $b_0b_2+b_1b_3+c_0c_2+c_1c_3 \equiv 2 \pmod 4$, we have $\beta\overline{\beta}-\gamma\overline{\gamma} \equiv 16 \pmod{32}$. Note that $\beta\overline{\beta} \equiv 4$ or $-12 \pmod{32}$ since $\beta\overline{\beta} \equiv 4 \pmod{16}$ from Lemma 3. If $\beta\overline{\beta} \equiv 4 \pmod{32}$, then $\gamma\overline{\gamma} \equiv 4-16 \equiv -12 \pmod{32}$. This implies that $\gamma\overline{\gamma}$ has at least one prime factor of the form $8k-3$. If $\beta\overline{\beta} \equiv -12 \pmod{32}$, then $\beta\overline{\beta}$ has at least one prime factor of the form $8k-3$. □

**Lemma 19.** *Suppose that* $b_0+b_2 \equiv b_1+b_3 \equiv 1 \pmod 2$*,* $D_4(\boldsymbol{b})D_4(\boldsymbol{c}) \in 2^8\mathbb{Z}_{\mathrm{odd}}$ *and* $\beta\overline{\beta}\gamma\overline{\gamma} \in 2^7\mathbb{Z}_{\mathrm{odd}}$*. Then* $D_4(\boldsymbol{b})D_4(\boldsymbol{c})\beta\overline{\beta}\gamma\overline{\gamma} \in \left\{2^{15}p(2m+1) \mid p \in P,\ m \in \mathbb{Z}\right\}$*.*

*Proof.* From Lemmas 15 and 16, we have

$$(b_0+b_2)(b_1+b_3) \equiv \pm 3, \quad (c_0+c_2)(c_1+c_3) \equiv \pm 3 \pmod 8, \quad d \equiv 2 \pmod 4,$$

where

$$\begin{aligned} d &:= \{(d_0+d_2)(d_5+d_7)+(d_4+d_6)(d_1+d_3)\}\\ &\quad \times \{(d_0-d_2)(d_1-d_3)+(d_4-d_6)(d_5-d_7)\}.\end{aligned}$$

We divide the proof into the following cases:

(i) $(b_0b_3+b_2b_1, c_0c_3+c_2c_1) \equiv (0,0),\ (0,2)$ or $(2,0) \pmod 4$;

(ii) $(b_0b_3+b_2b_1, c_0c_3+c_2c_1) \equiv (2,2) \pmod 4$;

(iii) $(b_0b_1+b_2b_3, c_0c_1+c_2c_3) \equiv (0,0),\ (0,2)$ or $(2,0) \pmod 4$;

(iv) $(b_0b_1+b_2b_3, c_0c_1+c_2c_3) \equiv (2,2) \pmod 4$.

First, we consider the case (i). If $b_0b_3+b_2b_1 \equiv 0 \pmod 4$, then

$$(b_0-b_2)(b_1-b_3) = (b_0+b_2)(b_1+b_3) - 2(b_0b_3+b_2b_1) \equiv \pm 3 \pmod 8.$$

Thus, $(b_0-b_2)^2+(b_1-b_3)^2 \equiv -6 \pmod{16}$. It implies that $(b_0-b_2)^2+(b_1-b_3)^2$ has at least one prime factor of the form $8k-3$. That is, $D_4(\boldsymbol{b})$ has at least one prime factor of the form $8k-3$. In the same way, we can prove that $D_4(\boldsymbol{c})$ has at least one prime factor of the form $8k-3$ when $c_0c_3+c_2c_1 \equiv 0 \pmod 4$. Therefore, if (i), then

$$D_4(\boldsymbol{b})D_4(\boldsymbol{c}) \in \left\{2^8p(2m+1) \mid p \in P,\ m \in \mathbb{Z}\right\}.$$

We can obtain the same conclusion for the case (iii). Next, we consider the case (ii). From $b_0+b_2 \equiv b_1+b_3 \equiv 1 \pmod 2$ and Remark 2 (1), we have $d_0+d_2+d_4+d_6 \equiv d_1+d_3+d_5+d_7 \equiv 1 \pmod 2$. From Remark 3, we may assume without loss of generality that $d_0+d_2 \equiv 0$, $d_4+d_6 \equiv 1 \pmod 2$. Then, either one of the following cases holds:

(ii-1) $d_0+d_2 \equiv d_1+d_3 \equiv 0, \quad d_4+d_6 \equiv d_5+d_7 \equiv 1 \pmod 2$;

(ii-2) $d_0+d_2 \equiv d_5+d_7 \equiv 0, \quad d_4+d_6 \equiv d_1+d_3 \equiv 1 \pmod 2$.

Suppose that (ii) and (ii-1) hold. Then, from Lemma 5 (2), it follows that

$$\begin{aligned}(d_0-d_2)+(d_1-d_3) &\equiv (d_0-d_2)(d_5-d_7)+(d_4-d_6)(d_1-d_3)\\ &\equiv (d_0+d_2)(d_5+d_7)+(d_4+d_6)(d_1+d_3)\\ &\quad +2(d_0d_7+d_2d_5+d_4d_3+d_6d_1)\\ &\equiv d+(b_0b_3+b_2b_1)-(c_0c_3+c_2c_1)\\ &\equiv 2 \pmod 4.\end{aligned}$$

Therefore, from Lemma 3, we have $\gamma\overline{\gamma} \equiv 4-16 \equiv -12 \pmod{32}$. Thus, $\gamma\overline{\gamma}$ has at least one prime factor of the form $8k-3$. Suppose that (ii) and (ii-2) hold. Then, from Lemma 5 (3), it follows that

$$\begin{aligned}(d_0+d_2)+(d_5+d_7) &\equiv (d_0+d_2)(d_1+d_3)+(d_4+d_6)(d_5+d_7)\\ &\equiv (d_0-d_2)(d_1-d_3)+(d_4-d_6)(d_5-d_7)\\ &\quad +2(d_0d_3+d_2d_1+d_4d_7+d_6d_5)\\ &\equiv d+(b_0b_3+b_2b_1)+(c_0c_3+c_2c_1)\\ &\equiv 2 \pmod 4.\end{aligned}$$

Therefore, from Lemma 3, we have $\beta\overline{\beta} \equiv 4-16 \equiv -12 \pmod{32}$. Thus, $\beta\overline{\beta}$ has at least one prime factor of the form $8k-3$. From the above, if (ii), then

$$\beta\overline{\beta}\gamma\overline{\gamma} \in \left\{2^7 p(2m+1) \mid p \in P,\ m \in \mathbb{Z}\right\}.$$

We can obtain the same conclusion for the case (iv) by using Lemma 5 (4) and (5). □

*Proof of Lemma* 14. We prove (1). Let $D_{4\times4}(\boldsymbol{a}) = D_4(\boldsymbol{b})D_4(\boldsymbol{c})\beta\overline{\beta}\gamma\overline{\gamma} \in 2\mathbb{Z}$. Then, from Lemma 2, we have $D_4(\boldsymbol{b}) \in 2\mathbb{Z}$. Thus, $b_0+b_2 \equiv b_1+b_3 \pmod 2$ from Corollary 1. Therefore, from Lemmas 15 and 16, we obtain (1). We prove (2). Let $D_{4\times4}(\boldsymbol{a}) \in 2^{15}\mathbb{Z}_{\mathrm{odd}}$. Then, from Lemmas 15 and 16, one of the following cases holds:

(i) $b_0 \equiv b_1 \equiv b_2 \equiv b_3 \equiv 1 \pmod 2$, $D_4(\boldsymbol{b})D_4(\boldsymbol{c}) \in 2^{11}\mathbb{Z}_{\mathrm{odd}}$ and $\beta\overline{\beta}\gamma\overline{\gamma} \in 2^4\mathbb{Z}_{\mathrm{odd}}$;

(ii) $b_0 \equiv b_2 \not\equiv b_1 \equiv b_3 \pmod 2$, $D_4(\boldsymbol{b})D_4(\boldsymbol{c}) \in 2^{11}\mathbb{Z}_{\text{odd}}$ and $\beta\overline{\beta}\gamma\overline{\gamma} \in 2^4\mathbb{Z}_{\text{odd}}$;

(iii) $b_0 + b_2 \equiv b_1 + b_3 \equiv 1 \pmod 2$, $D_4(\boldsymbol{b})D_4(\boldsymbol{c}) \in 2^8\mathbb{Z}_{\text{odd}}$ and $\beta\overline{\beta}\gamma\overline{\gamma} \in 2^7\mathbb{Z}_{\text{odd}}$.

Therefore, from Lemmas 17-19, we obtain (2). □

## 6. Achieving the Values

In this section, we complete the proof of Theorem 1. Lemmas 8 and 14 imply that $S\left(\mathrm{C}_4^2\right)$ does not include every integer that is not mentioned in Lemmas 20–22.

**Lemma 20.** *For any $m \in \mathbb{Z}$, the following are elements of $S(\mathrm{C}_4^2)$:*

(1) $16m + 1$;

(2) $2^{16}(4m + 1)$;

(3) $2^{16}(4m - 1)$;

(4) $2^{16}(2m)$.

**Lemma 21.** *For any $k \in \mathbb{Z}$, $16l-3$, $16m-3$, $16n-3$, $16l+5$, $16m+5$, $16n+5 \in P$, the following are elements of $S(\mathrm{C}_4^2)$:*

(1) $(16k - 3)(16l + 5)(16m - 3)(16n - 3)$;

(2) $(16k - 3)(16l + 5)(16m + 5)(16n + 5)$;

(3) $(16k + 5)(16l - 3)(16m - 3)(16n - 3)$;

(4) $(16k + 5)(16l - 3)(16m + 5)(16n + 5)$.

**Lemma 22.** *For any $m \in \mathbb{Z}$ and $p \in P$, we have $2^{15}p(2m + 1) \in S(\mathrm{C}_4^2)$.*

*Proof of Lemma* 20. We obtain (1) from

$$\begin{aligned} D_{4\times4}(m+1, m, \ldots, m) &= D_4(4m+1, 4m, 4m, 4m)D_4(1, 0, 0, 0)^3 \\ &= (8m+1)^2 - (8m)^2 \\ &= 16m + 1. \end{aligned}$$

We obtain (2) from

$$\begin{aligned} &D_{4\times4}(m+2, m, m, m, m, m, m+1, m, m+1, m, \ldots, m) \\ &\quad = D_4(4m+3, 4m, 4m+1, 4m)D_4(3, 0, -1, 0) \\ &\qquad \times D_4\left(1, 0, \sqrt{-1}, 0\right) D_4\left(1, 0, -\sqrt{-1}, 0\right) \\ &\quad = \left\{(8m+4)^2 - (8m)^2\right\} \cdot 2^2 \cdot 2^2 \cdot 4^2 \cdot \left(1+\sqrt{-1}\right)^4 \left(1-\sqrt{-1}\right)^4 \\ &\quad = 2^{16}(4m+1). \end{aligned}$$

We obtain (3) from

$$\begin{aligned}
&D_{4\times 4}(m+1,m,m,m-1,m,m-1,m,m,m,m,m,m-1,m,m-1,m-1,m)\\
&\quad = D_4(4m+1,4m-2,4m-1,4m-2)D_4(1,2,1,-2)\\
&\qquad \times D_4\left(1,0,\sqrt{-1},0\right) D_4\left(1,0,-\sqrt{-1},0\right)\\
&\quad = \left\{(8m)^2-(8m-4)^2\right\}\cdot 2^2\cdot 2^2\cdot 4^2\cdot\left(1+\sqrt{-1}\right)^4\left(1-\sqrt{-1}\right)^4\\
&\quad = 2^{16}(4m-1).
\end{aligned}$$

We obtain (4) from

$$\begin{aligned}
&D_{4\times 4}(m+1,m,m,m,m,m,m,m,m+1,m-1,m,m,m,m-1,m,m)\\
&\quad = D_4(4m+2,4m-2,4m,4m)D_4(2,0,0,0)\\
&\qquad \times D_4\left(0,1+\sqrt{-1},0,0\right) D_4(0,1-\sqrt{-1},0,0)\\
&\quad = \left\{(8m+2)^2-(8m-2)^2\right\}\cdot 2^3\cdot 2^2\cdot 2^2\cdot(1+\sqrt{-1})^4(1-\sqrt{-1})^4\\
&\quad = 2^{16}(2m).
\end{aligned}$$

□

**Remark 5.** When

$$\begin{aligned}
&d_0 = 2t-2v, && d_1 = 2t+2w+1, && d_2 = 2t+2v+2e, && d_3 = 2t-2w,\\
&d_4 = 2u+2w+1, && d_5 = 2u+2v+1, && d_6 = 2u-2w, && d_7 = 2u-2v,
\end{aligned}$$

where $e \in \{0,1\}$, from Lemma 1, we have

$$\beta\overline{\beta}\gamma\overline{\gamma} = \left\{(8t+2e+1)^2+(8u+2)^2\right\}\left\{(8v+2e+1)^2+(8w+2)^2\right\}.$$

*Proof of Lemma* 21. We remark that for any $16m-8e+5, 16n-8e+5 \in P$ with $e \in \{0,1\}$, there exist $t,u,v,w \in \mathbb{Z}$ satisfying

$$\begin{aligned}
16m-8e+5 &= (8t+2e+1)^2+(8u+2)^2,\\
16n-8e+5 &= (8v+2e+1)^2+(8w+2)^2.
\end{aligned} \tag{$*$}$$

We prove (1) and (2). For any $16l+5 \in P$, we can take $r,s \in \mathbb{Z}$ satisfying

$$16l+5 = (8r+1)^2+(8s+2)^2.$$

Also, we take $t,u,v,w$ satisfying $(*)$. Let

$$\begin{aligned}
&a_0 = k-r+t-v, && a_1 = k-s+t+w, && a_2 = k+r+t+v+e,\\
&a_3 = k+s+t-w, && a_4 = -k+r+u+w+1, && a_5 = -k+s+u+v+1,\\
&a_6 = -k-r+u-w, && a_7 = -k-s+u-v, && a_8 = k-r-t+v,\\
&a_9 = k-s-t-w-1, && a_{10} = k+r-t-v-e, && a_{11} = k+s-t+w,\\
&a_{12} = -k+r-u-w, && a_{13} = -k+s-u-v, && a_{14} = -k-r-u+w,\\
&a_{15} = -k-s-u+v.
\end{aligned}$$

Then, from Remark 5, we have

$$\begin{aligned} D_{4\times 4}(\boldsymbol{a}) &= D_4(1,0,0,0)D_4(4k-4r-1,4k-4s-2,4k+4r,4k+4s)\beta\overline{\beta}\gamma\overline{\gamma} \\ &= \left\{(8k-1)^2-(8k-2)^2\right\}\left\{(8r+1)^2+(8s+2)^2\right\} \\ &\quad \times \left\{(8t+2e+1)^2+(8u+2)^2\right\}\left\{(8v+2e+1)^2+(8w+2)^2\right\} \\ &= (16k-3)(16l+5)(16m-8e+5)(16n-8e+5). \end{aligned}$$

We prove (3) and (4). For any $16l-3 \in P$, we can take $r, s \in \mathbb{Z}$ satisfying

$$16l-3 = (8r+3)^2+(8s+2)^2.$$

Also, we take $t, u, v, w$ satisfying $(*)$. Let

$$\begin{aligned} &a_0 = k+r+t-v+1, && a_1 = k+s+t+w+1, && a_2 = k-r+t+v+e, \\ &a_3 = k-s+t-w, && a_4 = -k-r+u+w, && a_5 = -k-s+u+v, \\ &a_6 = -k+r+u-w, && a_7 = -k+s+u-v, && a_8 = k+r-t+v+1, \\ &a_9 = k+s-t-w, && a_{10} = k-r-t-v-e, && a_{11} = k-s-t+w, \\ &a_{12} = -k-r-u-w-1, && a_{13} = -k-s-u-v-1, && a_{14} = -k+r-u+w, \\ &a_{15} = -k+s-u+v. \end{aligned}$$

Then, from Remark 5, we have

$$\begin{aligned} D_{4\times 4}(\boldsymbol{a}) &= D_4(1,0,0,0)D_4(4k+4r+3,4k+4s+2,4k-4r,4k-4s)\beta\overline{\beta}\gamma\overline{\gamma} \\ &= \left\{(8k+3)^2-(8k+2)^2\right\}\left\{(8r+3)^2+(8s+2)^2\right\} \\ &\quad \times \left\{(8t+2e+1)^2+(8u+2)^2\right\}\left\{(8v+2e+1)^2+(8w+2)^2\right\} \\ &= (16k+5)(16l-3)(16m-8e+5)(16n-8e+5). \end{aligned}$$ □

*Proof of Lemma* 22. For any $p \in P$, there exist $r, s \in \mathbb{Z}$ satisfying $2p = (8r+3)^2 + (8s+1)^2$. Let

$$\begin{aligned} &a_0 = m+r+1, && a_1 = m+r+1, && a_2 = m+r+1, && a_3 = m+r, \\ &a_4 = m+s, && a_5 = m+s, && a_6 = m+s+1, && a_7 = m+s, \\ &a_8 = m-r+1, && a_9 = m-r, && a_{10} = m-r, && a_{11} = m-r-1, \\ &a_{12} = m-s, && a_{13} = m-s, && a_{14} = m-s, && a_{15} = m-s. \end{aligned}$$

Then, from Lemma 1, we have $\beta\overline{\beta}\gamma\overline{\gamma} = 2^3\left\{(8r+3)^2+(8s+1)^2\right\} = 2^4 p$. Hence, we have

$$\begin{aligned} D_{4\times 4}(\boldsymbol{a}) &= D_4(4m+2,4m+1,4m+2,4m-1)D_4(2,1,0,-1)\beta\overline{\beta}\gamma\overline{\gamma} \\ &= \left\{(8m+4)^2-(8m)^2\right\}\cdot 2^2\cdot 2^2\cdot(2^2+2^2)\cdot 2^4 p \\ &= 2^{15}p(4m+1). \end{aligned}$$

On the other hand, let

$$\begin{array}{llll} a_0 = m + r, & a_1 = m + r, & a_2 = m + r + 1, & a_3 = m + r, \\ a_4 = m + s, & a_5 = m + s, & a_6 = m + s, & a_7 = m + s - 1, \\ a_8 = m - r, & a_9 = m - r - 1, & a_{10} = m - r, & a_{11} = m - r - 1, \\ a_{12} = m - s, & a_{13} = m - s, & a_{14} = m - s - 1, & a_{15} = m - s - 1. \end{array}$$

Then, from Lemma 1, we have $\beta\overline{\beta}\gamma\overline{\gamma} = 2^3\left\{(8r+3)^2 + (8s+1)^2\right\} = 2^4 p$. Hence, we have

$$\begin{aligned} D_{4\times 4}(\boldsymbol{a}) &= D_4(4m, 4m-1, 4m, 4m-3)D_4(0, -1, 2, 1)\beta\overline{\beta}\gamma\overline{\gamma} \\ &= \left\{(8m)^2 - (8m-4)^2\right\} \cdot 2^2 \cdot 2^2 \cdot (2^2 + 2^2) \cdot 2^4 p \\ &= 2^{15} p(4m-1). \end{aligned}$$

□

From Lemmas 8, 14 and 20–22, Theorem 1 is proved.

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