MEAN VALUES OF THE PRODUCT OF AN INTEGER AND ITS MODULAR INVERSE

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Abstract
In this note, we extend a recent result of Chan (2016), that deals with mean values of the product of an integer and its multiplicative inverse modulo a prime number, to such products modulo a composite number.

1. Introduction

Let $k \geq 2$ be any integer. Chan [3] studies the sum $S_k(p) := \sum_{\substack{a_1=1 \\ a_1 \cdots a_k \equiv 1 \pmod{p}}}^{p-1} \cdots \sum_{a_k=1}^{p-1} a_1 \cdots a_k$, where p is a prime number, and shows that

$$S_k(p) = 2^{-k} p^k (p-1)^{k-1} + O\left(p^{\vartheta_k} (\log p)^k\right) \tag{1}$$

where $\vartheta_2 = \frac{5}{2}$ and $\vartheta_k = \frac{3k}{2}$ if $k \geq 3$. As can be seen in [3, Theorem 3], this result also holds for the more general sum

$$S_k(p,m) := \sum_{\substack{a_1=1 \\ a_1 \cdots a_k \equiv m \pmod{p}}}^{p-1} \cdots \sum_{a_k=1}^{p-1} a_1 \cdots a_k$$

where m is any positive integer satisfying $(m,p) = 1$. The main aim of this work is to generalize the estimate (1) to any composite modulus $n \geq 2$. We therefore investigate the sum

$$S_k(n,m) := \sum_{\substack{a_1=1 \\ a_1 \cdots a_k \equiv m \pmod{n}}}^{n} \cdots \sum_{a_k=1}^{n} a_1 \cdots a_k, \tag{2}$$

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where m is any positive integer satisfying $(m,n)=1$, and with the convention that the sum $S_k(n,m)$ vanishes whenever there exists $j\in\{1,\dots,k\}$ such that $(a_j,n)>1$. We will use the notation $f(n)=M(n)+O^\star(R(n))$, meaning $|f(n)-M(n)|\le R(n)$ for all integers $n\ge 2$.

Theorem 1. *For $k\ge 3$, $n\ge 2$, and $m\ge 1$ such that $(m,n)=1$,*

$$S_k(n,m)=2^{-k}n^k\varphi(n)^{k-1}+O^\star\left((2\sqrt{3})^k n^{3k/2}(\log n)^k\right).$$

When $k\ge 4$ and $n\ge 3$ is odd, more recent results for mean values of odd Dirichlet characters enable us to slightly improve the above estimate.

Theorem 2. *For $k\ge 4$, $n\ge 3$ odd, and $m\ge 1$ such that $(m,n)=1$,*

$$S_k(n,m)=2^{-k}n^k\varphi(n)^{k-1}+O_k\left(n^{\frac{3}{2}(k-2)}\varphi(n)^3(\log n)^{k-4}(\log\log n)^2\right).$$

The implied constant in the error term depends on k.

The case $k=2$ needs specific estimates involving additive characters and exponential sums. On the other hand, as in Theorem 1, we are able to give a fully explicit error term in this case.

Theorem 3. *For all $n\ge 2$ and $m\ge 1$ such that $(m,n)=1$,*

$$S_2(n,m)=4^{-1}n^2\varphi(n)+O^\star\left(n^2\,\sigma_{1/2}(n)\,(\tau(n)\log en)^2+\tfrac{1}{4}n\varphi(n)\right)$$

where $\sigma_{1/2}(n):=\sum_{d\mid n}\sqrt{d}$ and $\tau(n)=\sum_{d\mid n}1$.

To study the sharpness of the previous result, it can be helpful to investigate the mean square of its error term. When n is odd, we can prove the next result.

Theorem 4. *For all $n\ge 3$ odd and all $\varepsilon>0$ small, we have*

$$\sum_{\substack{m=1\\(m,n)=1}}^{n}\left|S_2(n,m)-4^{-1}n^2\varphi(n)\right|^2=\frac{5C_n}{144}\left(n\varphi(n)\right)^3+O\left(n^{5+\varepsilon}\right)$$

where the constant C_n is given in Equation (4) below. In particular, for all sufficiently large odd integers n, there exists $m\in\{1,\dots,n\}$ such that $(m,n)=1$ and

$$\left|S_2(n,m)-4^{-1}n^2\varphi(n)\right|\gg n^{3/2}\varphi(n).$$

Note that $n^{3/2}\varphi(n)\gg n^{5/2}(\log\log n)^{-1}$ while $n^2\,\sigma_{1/2}(n)\,(\tau(n)\log en)^2\ll n^{5/2+\varepsilon}$. Hence, when n is odd, the error term in Theorem 3 is sharp apart from the n^{ε}-factor.

2. Notation

In what follows, $n \geq 2$, $k \geq 2$, and $m \geq 1$ are integers, and we always assume that $(m,n) = 1$. For all $x \in \mathbb{R}$, we set $e_n(x) := e^{\frac{2i\pi x}{n}}$, and $\|x\|$ is the distance to the nearest integer. As for arithmetic functions, μ is the Möbius function, φ is the Euler totient function and, for all $h \in \mathbb{R}$, let $\sigma_h(n) := \sum_{d \mid n} d^h$. It is customary to set $\tau := \sigma_0$. The main sum $S_k(n,m)$ studied here is defined in Equation (2).

3. Proofs of Theorems 1 and 2

Proof of Theorem 1. By the orthogonality relations of the Dirichlet characters, we derive

$$\begin{aligned} S_k(n,m) &= \frac{1}{\varphi(n)} \sum_{a_1=1}^{n} \cdots \sum_{a_k=1}^{n} a_1 \cdots a_k \sum_{\chi \,(\mathrm{mod}\, n)} \overline{\chi}(m) \chi(a_1) \cdots \chi(a_k) \\ &= \frac{1}{\varphi(n)} \sum_{\chi \,(\mathrm{mod}\, n)} \overline{\chi}(m) \left(\sum_{a=1}^{n} a\chi(a) \right)^k \\ &= \frac{1}{\varphi(n)} \left(\sum_{\substack{a=1 \\ (a,n)=1}}^{n} a \right)^k + \frac{1}{\varphi(n)} \sum_{\substack{\chi \,(\mathrm{mod}\, n) \\ \chi \neq \chi_0}} \overline{\chi}(m) \left(\sum_{a=1}^{n} a\chi(a) \right)^k \\ &= 2^{-k} n^k \varphi(n)^{k-1} + R_k(n,m) \end{aligned}$$

with

$$R_k(n,m) := \frac{1}{\varphi(n)} \sum_{\substack{\chi \,(\mathrm{mod}\, n) \\ \chi \neq \chi_0}} \overline{\chi}(m) \left(\sum_{a=1}^{n} a\chi(a) \right)^k,$$

and using Abel summation [2, Corollary 1.1] and the Pólya-Vinogradov inequality [2, Theorem 6.10], we get

$$\left| \sum_{a \leq n} a\chi(a) \right| \leq 2n \max_{A \leq n} \left| \sum_{a \leq A} \chi(a) \right| \leq 2\sqrt{3}\, n^{3/2} \log n \tag{3}$$

so that $|R_k(n,m)| \leq (2\sqrt{3})^k n^{3k/2} (\log n)^k$. □

For the proof of Theorem 2, we assume $k \geq 4$ and suppose that $n \geq 3$ is an odd integer. The proof rests on the following result, which is [4, Lemma 9].

Lemma 1. *Let $q \in \mathbb{Z}_{\geq 3}$ be odd and set*

$$C_q := \prod_{p^\alpha \| q} \left(1 + \frac{2p^3 + p^2 - 1}{(p^2+1)(p^2+p+1)} - \frac{1}{p^{3(\alpha-1)}(p^2+p+1)} \right). \tag{4}$$

Then, for all $\ell \in \mathbb{Z}_{\geq 0}$

$$\sum_{\substack{\chi \,(\mathrm{mod}\, q)\\ \chi(-1)=-1}} \overline{\chi}\left(2^{\ell}\right)\left|\sum_{a=1}^{q} a\chi(a)\right|^{4} = C_q\,\frac{(3\ell+5)q^3\varphi(q)^4}{72\times 2^{\ell+1}} + O_{\ell}\left(q^{6+\varepsilon}\right).$$

We will make use of this result in the following weaker form.

Corollary 1. *Let $n \in \mathbb{Z}_{\geq 3}$ be odd. Then*

$$\sum_{\substack{\chi \,(\mathrm{mod}\, n)\\ \chi(-1)=-1}} \left|\sum_{a=1}^{n} a\chi(a)\right|^{4} \ll n^3\varphi(n)^4(\log\log n)^2. \tag{5}$$

Proof. Using Lemma 1 with $q = n$ odd and $\ell = 0$, we derive

$$\sum_{\substack{\chi \,(\mathrm{mod}\, q)\\ \chi(-1)=-1}} \left|\sum_{a=1}^{n} a\chi(a)\right|^{4} = C_n\,\frac{5n^3\varphi(n)^4}{144} + O\left(n^{6+\varepsilon}\right)$$

where C_n is defined in Equation (4) above. Now

$$\prod_{p|n}\left(1+\frac{4}{5p}\right) \leq C_n \leq \prod_{p|n}\left(1+\frac{2}{p}\right),$$

so that C_n is unbounded and, using [2, Exercise 61], we get

$$C_n \leq \exp\left(2\sum_{p|n}\frac{1}{p}\right) \leq \exp\left(2\log\log\log n + 6\right) \leq \left(e^3\log\log n\right)^2,$$

completing the proof. □

Remark 1. Note that bounding the inner sum of the left-hand side of the estimate (5) using the bound (3) would only give

$$\sum_{\substack{\chi \,(\mathrm{mod}\, n)\\ \chi(-1)=-1}} \left|\sum_{a=1}^{n} a\chi(a)\right|^{4} \ll n^6\varphi(n)(\log n)^4.$$

Proof of Theorem 2. First notice that, for all $\chi \neq \chi_0$, we have the identity $\sum_{a=1}^{q} a\chi(a) = -\chi(-1)\sum_{a=1}^{q} a\chi(a)$ (cf. the proof of [1, Theorem 12.20] which does not require q to be odd), so that, for all even Dirichlet characters $\chi \neq \chi_0$, we have

$$\sum_{a=1}^{q} a\chi(a) = 0. \tag{6}$$

Hence $S_k(n,m) = 2^{-k}n^k\varphi(n)^{k-1} + R_k^{\mathrm{odd}}(n,m)$, where

$$R_k^{\mathrm{odd}}(n,m) := \frac{1}{\varphi(n)} \sum_{\substack{\chi \,(\mathrm{mod}\, n)\\ \chi(-1)=-1}} \overline{\chi}(m) \left(\sum_{a=1}^{n} a\chi(a) \right)^k .$$

Now using (3) and Corollary 1, we derive

$$\begin{aligned}\sum_{\substack{\chi \,(\mathrm{mod}\, n)\\ \chi(-1)=-1}} \left| \sum_{a=1}^{n} a\chi(a) \right|^k &= \sum_{\substack{\chi \,(\mathrm{mod}\, n)\\ \chi(-1)=-1}} \left| \sum_{a=1}^{n} a\chi(a) \right|^{k-4} \left| \sum_{a=1}^{n} a\chi(a) \right|^4 \\ &\ll n^{\frac{3}{2}(k-4)}(\log n)^{k-4} \times n^3\varphi(n)^4(\log\log n)^2 \\ &\ll n^{\frac{3}{2}(k-2)}\varphi(n)^4(\log n)^{k-4}(\log\log n)^2\end{aligned}$$

implying the asserted estimate. □

4. Proof of Theorem 3

4.1. Technical Lemmas

Lemma 2. *For all $n,k,m \in \mathbb{Z}_{\geq 1}$ such that $(m,n)=1$,*

$$\left| \sum_{\substack{a=1\\ (a,n)=1}}^{n} a\, e_n\left(km\overline{a}\right) \right| \leq 2n^{3/2}\tau(n)(n,k)^{1/2}\log(en).$$

Proof. From Weil's bound applied to Kloosterman sums and a completion device, it is known [2, (6.21) p. 454] that, for all positive integers $t \leq n$, we have

$$\left| \sum_{\substack{a\leq t\\ (a,n)=1}} e_n\left(km\overline{a}\right) \right| \leq n^{1/2}\tau(n)(n,mk)^{1/2}\log(en)$$

so that, by Abel summation, we derive

$$\begin{aligned}\left| \sum_{\substack{a=1\\ (a,n)=1}}^{n} a\, e_n\left(km\overline{a}\right) \right| &\leq 2n \max_{t\leq n} \left| \sum_{\substack{a\leq t\\ (a,n)=1}} e_n\left(km\overline{a}\right) \right| \\ &\leq 2n^{3/2}\tau(n)(n,mk)^{1/2}\log(en)\end{aligned}$$

and we conclude the proof, noticing that $(n,mk)=(n,k)$ since $(n,m)=1$. □

The next lemma is well-known, but we provide here a proof for the sake of completeness.

Lemma 3. *Let $q \in \mathbb{Z}_{\geq 2}$ and $k \in \mathbb{Z}_{\geq 1}$. If $q \nmid k$, then*

$$\sum_{b=1}^{q-1} b\, e_q(-kb) = \frac{-q}{1 - e_q(-k)}.$$

Proof. If $z^q = 1$, $z \neq 1$, then $\sum_{b=0}^{q-1} z^b = (1 - z^q)/(1 - z) = 0$. Hence

$$\begin{aligned}\left(\sum_{b=1}^{q-1} b z^{b-1}\right)(1 - z) &= z + 2z^2 + \cdots + (q-1)z^{q-1} - z^2 - 2z^3 - \cdots - (q-1)z^q \\ &= z + z^2 + \cdots + z^{q-1} - (q-1)z^q = -1 - (q-1) = -q.\end{aligned}$$

□

Lemma 4. *For all $n \in \mathbb{Z}_{\geq 2}$ and $k \in \mathbb{Z}_{\geq 1}$,*

$$\sum_{\substack{b=1 \\ (b,n)=1}}^{n} b\, e_n(-kb) = -n \sum_{\substack{d \mid n \\ \frac{n}{d} \nmid k}} \frac{\mu(d)}{1 - e_{n/d}(-k)} + \frac{n}{2} \sum_{\substack{d \mid n \\ d < n \\ \frac{n}{d} \mid k}} \mu(d) \left(\frac{n}{d} - 1\right).$$

Proof. We have

$$\begin{aligned}\sum_{\substack{b=1 \\ (b,n)=1}}^{n} b\, e_n(-kb) &= \sum_{d \mid n} d\mu(d) \sum_{b \leq n/d} b\, e_n(-kdb) = \sum_{d \mid n} d\mu(d) \sum_{b \leq n/d} b e_{n/d}(-kb) \\ &= \sum_{d \mid n} d\mu(d) \left(\frac{n}{d} + \sum_{b \leq \frac{n}{d} - 1} b\, e_{n/d}(-kb)\right) \\ &= n \underbrace{\sum_{d \mid n} \mu(d)}_{=0,\text{ since } n \geq 2} + \left(\sum_{\substack{d \mid n \\ \frac{n}{d} \nmid k}} + \sum_{\substack{d \mid n \\ \frac{n}{d} \mid k}}\right) d\mu(d) \sum_{b \leq \frac{n}{d} - 1} b\, e_{n/d}(-kb) \\ &= \sum_{\substack{d \mid n \\ \frac{n}{d} \nmid k}} d\mu(d) \frac{-n/d}{1 - e_{n/d}(-k)} + \sum_{\substack{d \mid n \\ \frac{n}{d} \mid k}} d\mu(d) \sum_{b \leq \frac{n}{d} - 1} b \\ &= -n \sum_{\substack{d \mid n \\ \frac{n}{d} \nmid k}} \frac{\mu(d)}{1 - e_{n/d}(-k)} + \frac{n}{2} \sum_{\substack{d \mid n \\ d < n \\ \frac{n}{d} \mid k}} \mu(d) \left(\frac{n}{d} - 1\right)\end{aligned}$$

where we used Vinogradov's lemma [2, Corollary 4.6] in the 1st equality of the 1st line, and Lemma 3 in the 1st sum of the penultimate line. □

We will make use of the following principle.

Lemma 5. *Let $f : \mathbb{Z}_{\geq 1} \to \mathbb{C}$ be any arithmetic function. Then, for any $n \in \mathbb{Z}_{\geq 1}$,*

$$\sum_{k=1}^{n} f(k) = \sum_{d|n} \sum_{\substack{h=1 \\ \left(\frac{n}{d}, h\right)=1}}^{n/d} f(hd).$$

Proof. Set $d = (n, k)$ and sum over all values of $d \mid n$, assuming by convention that the inner sum vanishes if $d \nmid k$, yielding

$$\sum_{k=1}^{n} f(k) = \sum_{d|n} \sum_{\substack{k=1 \\ d|k \\ \left(\frac{n}{d}, \frac{k}{d}\right)=1}}^{n} f(k).$$

The change $k = hd$ then gives the asserted result. □

Lemma 6. *Let $q \in \mathbb{Z}_{\geq 2}$ and $\ell \in \mathbb{Z}_{\geq 1}$. Then*

$$\sum_{\substack{a \pmod q \\ q \nmid a\ell}} \frac{1}{|1 - e_q(\pm a\ell)|} \leq \frac{q}{2} \log\left(\frac{eq}{2(\ell, q)}\right) < \frac{q}{2} \log(eq).$$

Proof. The inequality $|e(x) - e(y)| \geq 4\|x - y\|$ [2, Exercise 114] and periodicity yield

$$\sum_{\substack{a \pmod q \\ q \nmid a\ell}} \frac{1}{|1 - e_q(\pm a\ell)|} \leq \frac{1}{4} \sum_{\substack{a=1 \\ q \nmid a\ell}}^{q-1} \frac{1}{\|a\ell/q\|} = \frac{1}{4} \sum_{j=1}^{q-1} \frac{1}{\|j/q\|} \sum_{\substack{a=1 \\ a\ell \equiv j \pmod q}}^{q-1} 1.$$

For fixed ℓ, j, and q, the linear congruence $a\ell \equiv j \pmod q$ has solutions in a if and only if $(\ell, q) \mid j$, and in this case there are (ℓ, q) solutions $\pmod q$. Hence, with the notation $q/(\ell, q) := r$,

$$\begin{aligned}
\sum_{\substack{a \pmod q \\ q \nmid a\ell}} \frac{1}{|1 - e_q(\pm a\ell)|} &\leq \frac{(\ell, q)}{4} \sum_{\substack{j=1 \\ (\ell,q)|j}}^{q-1} \frac{1}{\|j/q\|} = \frac{(\ell, q)}{4} \sum_{t=1}^{r-1} \frac{1}{\|t/r\|} \\
&= \frac{(\ell, q)}{4} \left(\sum_{t \leq r/2} \frac{r}{t} + \sum_{r/2 < t \leq r-1} \frac{r}{r - t} \right) \\
&\leq \frac{r(\ell, q)}{2} \sum_{t \leq r/2} \frac{1}{t} \leq \frac{q}{2} \log\left(er/2\right) < \frac{q}{2} \log(eq),
\end{aligned}$$

as required. □

Lemma 7. *Let $n \in \mathbb{Z}_{\geq 2}$, and d be a divisor of n. Then*

$$\sum_{\substack{k=1\\ \frac{n}{d}\nmid k}}^{n-1} \frac{(n,k)^{1/2}}{\left|1-e_{n/d}(-k)\right|} \leq \tfrac{1}{2}\, n^{1/2}\, \sigma_{1/2}(n) \log(en).$$

Proof. Using Lemma 5 we first derive

$$\begin{aligned}\sum_{\substack{k=1\\ \frac{n}{d}\nmid k}}^{n-1} \frac{(n,k)^{1/2}}{\left|1-e_{n/d}(-k)\right|} &= \sum_{\substack{\delta\mid n\\ \delta<n}} \delta^{1/2} \sum_{\substack{h=1\\ \left(\frac{n}{\delta},h\right)=1\\ \frac{n}{d}\nmid h\delta}}^{\frac{n}{\delta}-1} \frac{1}{\left|1-e_{n/d}(-h\delta)\right|}\\ &= \sum_{\substack{\delta\mid n\\ \delta<n}} \delta^{1/2} \sum_{\substack{h=1\\ \left(\frac{n}{\delta},h\right)=1\\ \frac{n}{\delta}\nmid hd}}^{\frac{n}{\delta}-1} \frac{1}{\left|1-e_{n/\delta}(-hd)\right|}.\end{aligned}$$

Notice that $\frac{n}{\delta} \geq 2$ since $\delta \mid n$ and $\delta < n$. Now Lemma 6 with $q = \frac{n}{\delta}$ and $\ell = d$ yields

$$\sum_{\substack{k=1\\ \frac{n}{d}\nmid k}}^{n-1} \frac{(n,k)^{1/2}}{\left|1-e_{n/d}(-k)\right|} \leq \frac{n}{2} \sum_{\delta\mid n} \delta^{-1/2} \log\left(\frac{en}{\delta}\right) \leq \tfrac{1}{2}\, n^{1/2}\, \sigma_{1/2}(n) \log(en).$$

□

4.2. Finalizing the Proof of Theorem 3

The orthogonality relations for additive characters yield

$$\begin{aligned}S_2(n,m) &= \frac{1}{n} \sum_{\substack{a=1\\ (a,n)=1}}^{n} a \sum_{\substack{b=1\\ (b,n)=1}}^{n} b \sum_{k=1}^{n} e_n\left(k(m\overline{a}-b)\right)\\ &= \frac{1}{n} \sum_{k=1}^{n} \sum_{\substack{a=1\\ (a,n)=1}}^{n} a\, e_n\left(km\overline{a}\right) \sum_{\substack{b=1\\ (b,n)=1}}^{n} b\, e_n\left(-kb\right).\end{aligned}$$

We start by isolating the term corresponding to $k = n$, so that

$$\begin{aligned}S_2(n,m) &= \tfrac{1}{4} n \left(\varphi(n)\right)^2 + \frac{1}{n} \sum_{k=1}^{n-1} \sum_{\substack{a=1\\ (a,n)=1}}^{n} a\, e_n\left(km\overline{a}\right) \sum_{\substack{b=1\\ (b,n)=1}}^{n} b\, e_n\left(-kb\right)\\ &:= \tfrac{1}{4} n \left(\varphi(n)\right)^2 + R_2(n,m). \qquad (7)\end{aligned}$$

Applying Lemma 4, we first derive

$$R_2(n,m) = \frac{1}{n}\sum_{k=1}^{n-1}\sum_{\substack{a=1\\(a,n)=1}}^{n} a\, e_n(km\overline{a}) \left(-n \sum_{\substack{d|n\\ \frac{n}{d}\nmid k}} \frac{\mu(d)}{1-e_{n/d}(-k)} + \frac{n}{2}\sum_{\substack{d|n\\ d<n\\ \frac{n}{d}|k}} \mu(d)\left(\frac{n}{d}-1\right)\right)$$

$$\begin{aligned} &= -\sum_{d|n}\mu(d) \sum_{\substack{k=1\\ \frac{n}{d}\nmid k}}^{n-1} \frac{1}{1-e_{n/d}(-k)} \sum_{\substack{a=1\\(a,n)=1}}^{n} a\, e_n(km\overline{a}) \\ &\qquad + \frac{1}{2}\sum_{\substack{d|n\\ d<n}}\mu(d) \left(\frac{n}{d}-1\right) \sum_{\substack{k=1\\ \frac{n}{d}|k}}^{n-1} \sum_{\substack{a=1\\(a,n)=1}}^{n} a\, e_n(km\overline{a}) \\ &:= R_{21}(n,m) + R_{22}(n,m). \end{aligned}$$

Lemmas 2 and 7 yield

$$\begin{aligned} |R_{21}(n,m)| &\le \sum_{d|n} \sum_{\substack{k=1\\ \frac{n}{d}\nmid k}}^{n-1} \frac{1}{\left|1-e_{n/d}(-k)\right|} \left| \sum_{\substack{a=1\\(a,n)=1}}^{n} a\, e_n(km\overline{a}) \right| \\ &\le 2n^{3/2}\tau(n)\, \log(en) \sum_{d|n} \sum_{\substack{k=1\\ \frac{n}{d}\nmid k}}^{n-1} \frac{(n,k)^{1/2}}{\left|1-e_{n/d}(-k)\right|} \\ &\le n^2\, \sigma_{1/2}(n)\, \tau(n) (\log en)^2 \sum_{d|n} 1 \\ &= n^2\, \sigma_{1/2}(n)\, (\tau(n)\log en)^2. \end{aligned} \tag{8}$$

Now setting $k = \frac{hn}{d}$ in the sum over k in $R_{22}(n,m)$ yields

$$\begin{aligned} R_{22}(n,m) &= \frac{1}{2}\sum_{\substack{d|n\\ d<n}}\mu(d) \left(\frac{n}{d}-1\right) \sum_{h=1}^{d-1} \sum_{\substack{a=1\\(a,n)=1}}^{n} a\, e_d(hm\overline{a}) \\ &= \frac{1}{2}\sum_{\substack{d|n\\ 1<d<n}}\mu(d) \left(\frac{n}{d}-1\right) \sum_{h=1}^{d-1} \sum_{\substack{a=1\\(a,n)=1}}^{n} a\, e_d(hm\overline{a}). \end{aligned}$$

It should be pointed out that $d \mid n$ and $d > 1$ imply that $d \nmid \overline{a}$. Now inverting the summations, we derive, for all $d \mid n$ such that $1 < d < n$,

$$\begin{aligned}\sum_{h=1}^{d-1} \sum_{\substack{a=1\\(a,n)=1}}^{n} a\, e_d\,(hm\overline{a}) = \sum_{\substack{a=1\\(a,n)=1}}^{n} a \sum_{h=1}^{d-1} e_d\,(hm\overline{a}) &= \sum_{\substack{a=1\\(a,n)=1}}^{n} a\, e_d\,(m\overline{a}) \times \frac{1 - e_d\,((d-1)m\overline{a})}{1 - e_d\,(m\overline{a})} \\ &= \sum_{\substack{a=1\\(a,n)=1}}^{n} a\, e_d\,(m\overline{a}) \times \frac{1 - e_d\,(-m\overline{a})}{1 - e_d\,(m\overline{a})} \\ &= - \sum_{\substack{a=1\\(a,n)=1}}^{n} a\, e_d\,(m\overline{a})\; e_d\,(-m\overline{a}) \\ &= - \sum_{\substack{a=1\\(a,n)=1}}^{n} a = -\frac{n\varphi(n)}{2},\end{aligned}$$

so that, since $n \geq 2$,

$$R_{22}(n,m) = -\frac{n\varphi(n)}{4} \sum_{\substack{d \mid n\\1<d<n}} \mu(d)\; \left(\frac{n}{d} - 1\right) = \frac{n^2\varphi(n)}{4} - \frac{n\varphi(n)^2}{4} - \frac{n\varphi(n)}{4}, \qquad (9)$$

where we used the well-known Dirichlet convolution identities $\sum_{d \mid n} \mu(d) = 0$ if $n > 1$ and $\sum_{d \mid n} \frac{\mu(d)}{d} = \frac{\varphi(n)}{n}$. The result follows by inserting the estimates (8) and (9) in Equation (7). □

5. Proof of Theorem 4

Let $n \geq 3$ be odd. Using orthogonality relations for Dirichlet characters, we get by the proof of Theorem 1

$$\begin{aligned}&\sum_{\substack{m=1\\(m,n)=1}}^{n} \left|S_2(n,m) - 4^{-1}n^2\varphi(n)\right|^2 \\ &= \frac{1}{\varphi(n)^2} \sum_{\substack{m=1\\(m,n)=1}}^{n} \left| \sum_{\substack{\chi \,(\mathrm{mod}\, n)\\ \chi \neq \chi_0}} \overline{\chi}(m) \left(\sum_{a=1}^{n} a\chi(a) \right)^2 \right|^2 \\ &= \frac{1}{\varphi(n)^2} \sum_{\substack{\chi_1 \,(\mathrm{mod}\, n)\\ \chi_1 \neq \chi_0}} \sum_{\substack{\chi_2 \,(\mathrm{mod}\, n)\\ \chi_2 \neq \chi_0}} \left(\sum_{a=1}^{n} a\chi_1(a) \right)^2 \left(\sum_{a=1}^{n} a\overline{\chi_2}(a) \right)^2 \underbrace{\sum_{\substack{m=1\\(m,n)=1}}^{n} \chi_1(m)\, \overline{\chi_2}(m)}_{=\varphi(n) \text{ if } \chi_1 = \chi_2,\ 0 \text{ if not}}\end{aligned}$$

$$= \frac{1}{\varphi(n)} \sum_{\substack{\chi \pmod n \\ \chi \neq \chi_0}} \left| \sum_{a=1}^{n} a\chi(a) \right|^4 = \frac{1}{\varphi(n)} \sum_{\substack{\chi \pmod n \\ \chi(-1)=-1}} \left| \sum_{a=1}^{n} a\chi(a) \right|^4$$

by the remark (6), and the proof follows by using Lemma 1 with $q = n$ and $\ell = 0$. $\square$

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